Journal of the Egyptian Mathematical Society (2012) 20, 53–56

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On the cohomology of Banach A_∞-module over admissible Banach A_∞-algebra

Y. Gh. Gouda

Dept. of Mathematics, Faculty of Science at Aswan, South Valley University, Aswan, Egypt

Received 20 June 2010; revised 10 September 2011
Available online 31 October 2012

KEYWORDS

Banach A_∞-module–admissible Banach A_∞-algebra; Cohomology group

Abstract In this paper we are concerned with Banach A_∞-module M over admissible Banach A_∞-algebra A. We give some properties of admissible modules and algebras. We study the cohomology of the complex $C_\infty(A,M)$. We show that the vanishing of cohomology of this complex in certain dimensions implies to the existence of the A_∞-module structure.

1. Introduction

Kadeishvili [3] defined A_∞-module over A_∞-algebra, homotopy between two morphisms and homotopy equivalence between two modules. In [4] Simirnov and others described the cohomology of Banach and siminormed algebras using A_∞-structures of Stasheff [1]. They also proved that If A is allowable differential Banach algebra, then its homology $H_*(A)$ has the structure of a graded A_∞-algebra. Lodoshkii [6] has studied over where algebras over field. Lapin [8] has studied multiplicative-structure in term of spectral sequences. The present work is concerned with the Banach A_∞-module over admissible Banach A_∞-algebra and its cohomology group. First of all we recall some necessary definitions and facts about admissible Banach module, admissible Banach algebra and there properties useful in the sequel. The main references are [4,5,7,2].

Definition 1. For a given Banach algebra A, a differential Banach module M over A is a pair (M,d), where $M = \{M_n\}$, $n \in Z$ is a family of Banach modules over Banach algebra A, equipped with a differential $d = \{d_n\}: M_n \to M_{n-1}$ such that $d^2 = 0$.

Definition 2. A Banach module M over Banach algebra A is called admissible if there exists a family of continuous operators $d = \{S_n\}: M_{n-1} \to M_n$, satisfying the relation $d \circ s \circ d = d$.

Proposition 3. *The tensor product of admissible Banach modules is admissible.*

Proof 1. Suppose admissible Banach modules (M', d', S'), (M'', d'', S''). Define the operator $S: M' \otimes M'' \to M' \otimes M''$ such that

$$S = S' \otimes 1 + 1 \otimes S'' - (d' \circ S' + S' \circ d') \otimes S''$$

The direct calculation shows that $d \circ s \circ d = d$. □

E-mail address: yasiengouda@yahoo.com
Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

1110-256X
http://dx.doi.org/10.1016/j.joems.2012.08.006

Definition 4. Let A be Banach algebra, then the complex $BA = ((BA)_n, \delta)$, is called B-construction over A, where $(BA)_n = A^{\otimes n}$, if $n \geqslant 1$, $(BA)_n = 0$, *if* $n < 1$, $A^{\otimes n} = A \otimes \ldots \otimes A$ (n times), and differential δ is given by:

$$\delta(a_1 \otimes \ldots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^{i+1} [a_1 \otimes \ldots \otimes \pi(a_i \otimes a_{i+1}) \otimes \ldots \otimes a_n].$$

Definition 5. Banach algebra is admissible if the complex BA is admissible.

Note that :

- Any finite Banach algebra is admissible.
- An example of infinite admissible Banach algebra is $A = \ell_1$, the space of all absolute convergent series $\sum_{n=1}^{\infty} a_n$ with the multiplication

$$\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n = \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} a_n \cdot b_{n-k} \right).$$

Definition 6. A differential admissible Banach algebra is the triple (M, d, π), such that (M, d) is admissible Banach module M over Banach algebra A and $\pi: A \otimes A \to A$ such that $\pi(1 \otimes \pi) = \pi(\pi \otimes 1)$.

Note that [5] if A is admissible Banach algebra, then its homology $H_*(A)$ is graded Banach algebra.

Definition 7. Two maps f, $g: A \to B$ of differential admissible Banach algebras are called homotopic (denoted $f \simeq g$), if there exists a map $h: A \to B$ of dimension 1, satisfying the relation $dh + hd = g - f$.

It is easy to see that the homotopy relation is an equivalence relation. The differential Banach algebras A and B are called homotopy equivalent (Denoted by $A \simeq B$) if there are chain maps $f: A \to B$, $g: B \to A$ such that $g \circ f = id_A$, $f \circ g = i\, d_B$.

The differential Banach module is contractible if it is homotopy equivalent to zero.

Some properties of admissible Banach algebra [4]:

Proposition 8. *Let A be admissible Banach complex, then the homology $H_*(A)$, which is Banach complex with zero differential, satisfies the following isomorphism $H_*(A) \simeq A$.*

Proposition 9. *For the admissible Banach complexesA and B, then the following holds $H_*(A \otimes B) \simeq H_*(A) \otimes H_*(B)$.*

Proposition 10. *Let A be admissible Banach complex, then the homology $H_*(A)$ is graded Banach algebra and the following multiplication holds $\pi_* = H_*(A) \otimes H_*(A) \to H_*(A)$, where $\pi_* = \eta \circ \pi \circ (\xi \otimes \xi)$, $\xi: H_*(A) \to A$, $\eta: A \to H_*(A)$, $\pi: A \otimes A \to A$.*

2. Banach A_∞-module over admissible Banach A_∞-algebra and Hochschild cohomology

Definition 11. A Banach A_∞-algebra (A, π_i, d) is a Banach graded module (A, d) with the multiplication $\pi_i: A^{\otimes i+2} \to A$, such that:

$$\sum_{i=0}^{n} (-1)^{\varepsilon} \pi_i (1 \otimes \ldots \otimes \pi_{n-i} \otimes \ldots \otimes 1) = 0, \ \varepsilon = nk + ik + k$$

Definition 12. For given a Banach A_∞-algebra (A, π_i), a Banach A_∞-module (M, P_i) over Banach A_∞-algebra (A, π_i) is graded module M, equipped with multiplication $\{P_i: A^{\otimes i} \otimes M \to M\}$, $i \geqslant 1$ such that: $P_i((: A^{\otimes i} \otimes M))_q \subset^{M}_{q+i-1}$ $\prime$, $i \geqslant 1$ and

$$\sum_j (-1)^{k(j+1)+i} P_{i-j+1}(1 \otimes \ldots \otimes 1 \otimes \pi_j \otimes \ldots \otimes 1) + \sum (-1)^{(i-+1)(j-1)} P_{i-j}(1 \otimes \ldots \otimes 1 \otimes P_j) = 0 \tag{1}$$

The relation (1) is called Stasheff relation for Banach A_∞-module over Banach A_∞-algebra [1].

The morphism between Banach A_∞-modules (M, P_i) and (M', P'_i) over Banach A_∞-algebras (A, π_i) and (A', π'_i), respectively, is a family of morphisms $\{f_i: A \to A'\}$, $\{g_i: A^{\otimes i} \otimes M \to M\}$, $i \geqslant 0$, where f_i are morphisms between A_∞-algebras and $g_i((A^{\otimes i} \otimes M))_q \subset M'_{q-i}$.

Satisfy the following identity:

$$\sum_j (-1)^{k(j-1)+i} g_{i-j+1}(1 \otimes \ldots \otimes 1 \otimes \pi_j \otimes \ldots \otimes 1) + \sum (-1)^{j(i-+1)(j-1)} g_{i-j}(1 \otimes \ldots \otimes 1 \otimes P_j) + \sum_{K_1 + .. + K_t = i+1} (-1)^{K_2 + K_4 + \ldots} P_t(f_{K_1} \otimes \ldots \otimes f_{K_{t-1}} \otimes g_{f_{K_t}}) = 0 \tag{2}$$

Definition 13. Banach A_∞-algebra A is called admissible if the Banach graded module is admissible.

Definition 14. [4] The B-constructor BM of Banach A_∞-module M over Banach A_∞-algebra A is given by the tensor product $A^{\otimes i} \otimes M$ such that:

$$\deg\,(a_1 \otimes \ldots \otimes a_k \otimes b) = \deg\,(a_1) + \ldots + \deg(a_k) + \deg(b) + k.$$

The B-constructor BM of Banach A_∞-module M over Banach A_∞-algebra A is given by the tensor product $A^{\otimes i} \otimes M$ such that:

$$\deg\,(a_1 \otimes \ldots \otimes a_k \otimes b) = \deg\,(a_1) + \ldots + \deg(a_k) + \deg(b) + k.$$

The complex $C = (Hom(BM, M), \delta)$ is the Hochschild complex for Banach A_∞-module M over Banach A_∞-algebra A and denoted by $C_\infty(A, M)$, such that :

$$\delta = \sum_i (-1)^{i+n+1} f(1 \otimes \ldots \otimes 1 \otimes \pi_j \otimes \ldots \otimes 1) + \sum_{i=1}^{n} (-1)^{i+1} \pi_i (1 \otimes \ldots \otimes 1 \otimes f) + \sum_{i=1}^{n} f(1 \otimes \ldots \otimes 1 \otimes \pi_j 1) \tag{3}$$

where

$$\delta : Hom^n(BM, M) \to Hom^{n-1}(BM, M),$$

$$f \in Hom^n(BM, M), \ f = \{f\} : (A^{\otimes i} \otimes M)_q \to M_{q+n+i-1} \to A_{i+n}.$$

Note that:

- The first summation in (3) is given in all possible place of π_i.

- If the module M is trivial, then the differential in coincides with an ordinary differential δ in Hochschild complex $C_\infty(A, M)$ for module over algebra.

Theorem 15. *The map δ in (3) is differential, that is $\delta(\delta f) = 0$.*

Proof 2.

$$\delta(\delta f) = \sum_i (-1)^{i+n+1} \delta f(1 \otimes \ldots \otimes 1 \otimes \pi_j \otimes \ldots \otimes 1) + \sum_{i=1}^{n} (-1)^{i+1} \pi_i(1 \otimes \ldots \otimes 1 \otimes \delta f) + \sum_{i=1}^{n} \delta f(1 \otimes \ldots \otimes 1 \otimes \pi_j 1) \quad \square \tag{4}$$

Definition 16. Note that the summation in (4) is given in all possible place of π_j and all components of a map f. The direct calculation of (4) using the relation (3) and the Stasheff relations for A_∞-algebra [1], show that $\delta(\delta f) = 0$ and hence the operator δ is differential.

Definition 17. The homology of Hochschild complex $C_\infty(A, M)$ is the Hochschild homology of Banach A_∞-module M over admissible Banach A_∞-algebra A and denoted by $H(C_\infty(A, M))$.

Theorem 18. *[5] Let A be admissible Banach A_∞-algebra, then the homology group $A_* = H_*(A)$ has graded A_∞-algebra's structure and the homotopy equivalent of A_∞-algebras $A \simeq A_*$ induces the homotopy equivalent of differential coalgebras $BA \simeq BA_*$.*

Definition 19. [2] On Hochschild complex $C(A, A)$ the following operators $\cup$, $\cup_1$ can be defined as follows :

$$\cup : (C(A,A) \otimes C(A,A))^i \to (C(A,A))^i$$
$$\cup_1 : (C(A,A) \otimes C(A,A))^i \to (C(A,A))^{i+1}$$

, such that:

$$f \cup g : \pi(f \otimes g)$$
$$f \cup_1 g : \sum_k f(1 \otimes \ldots \otimes 1 \otimes g \otimes 1 \otimes \ldots \otimes 1), \; f, g \in C^m(A, A)$$

Similarly for Hochschild complex $C_\infty(A, M)$ we have

$$\cup : C_\infty(A, M) \otimes C(A, A) \to (C_\infty(A, A)$$
$$\cup_1 : C_\infty(A, M) \otimes C(A, A) \to (C_\infty(A, A)$$

of degree (-1) such that:

$$f \cup g : \pi(f \otimes g) : (-1)^{(n+m-2)(m-1)} f(1 \otimes \ldots \otimes 1 \otimes g) \tag{5}$$

$$f \cup_1 g : \sum_i (-1)^{i(n-1)+m+n-3} f(1 \otimes \ldots \otimes 1 \otimes g \otimes \ldots \otimes 1) \tag{6}$$

Note that:

- If A_∞-module is trivial, then the action of $\cup_1$ on Hochschild complex coincides with the action of $\cup$ in Hochschild complex for module.
- For a given a Banach Hochschild complex $C_\infty(A, M)$ of Banach A_∞-module M over Banach A_∞-algebra the operations $\cup$, $\cup_1$ are easily defined.

Proposition 20. *Let the maps $f, g, h \in C_\infty(A, M)$, then the following holds*

$$(f \cup g) \cup_1 h = (-1)^m f \cup (g \cup_1 h) + (f \cup_1 h) \cup g \tag{7}$$

Proof 3. From relations (5) and (6) the left hand side of relation (7) is given by:

$$(f \cup g) \cup_1 h = \sum_{i,k} (-1)^{i(n-1)+m+k+n-1+m(k-1)} f(1 \otimes \ldots \otimes h \otimes \ldots \otimes g) + \sum_{i,k} (-1)^{i(n-1)+m+k+n-1+m(k-1)} f(1 \otimes \ldots \otimes g(1 \otimes \ldots \otimes h \otimes \ldots \otimes 1)) \quad \square$$

Definition 21. The RHS of relation (7) by means of (5) and (6) is given by:

$$(-1)^m f \cup (g \cup_1 h) = (-1)^m \sum_{i,k} (-1)^{(i+m)(n-1)+k+n-1+m(k-1)} f(1 \otimes \ldots \otimes g(1 \otimes \ldots \otimes h \otimes \ldots \otimes 1))$$

and

$$(f \cup_1 g) \cup h = \sum_{i,k} (-1)^{i(n-1)+m+k+n-1+m(k-1)} f(1 \otimes \ldots \otimes h \otimes \ldots \otimes g)$$

Hence $(f \cup g) \cup {}_1 h = (-1)^m f \cup (g \cup {}_1 h) + (f \cup {}_1 h) \cup g$.

The following assertion gives the relation between an operator and differential δ.

Theorem 22. *An operator $\cup_1$ satisfies the Leibniz condition:*

$$\delta(f \cup_1 g) = -\delta f \cup_1 g + (-1)^n f \cup_1 \delta g \tag{8}$$

Proof 4. The left hand side of (8) by considering the relation (6) can be written in the form:

$$\begin{aligned}\delta(f \cup_1 g) &= \sum (-1)^{i+n+m+1} (f \otimes_1 g)(1 \otimes \ldots \otimes \pi_i \otimes \ldots \otimes 1) \\ &\quad + (-1)^{n+m+1} \pi_i (1 \otimes \ldots \otimes f \otimes_1 g) + f \cup_1 g(1 \otimes \ldots \otimes \pi_i)) \\ &= \sum (-1)^{i+n+\ m+(n+m-3)(m-1)} f(1 \otimes \ldots \otimes g)(1 \otimes \ldots \otimes \pi_i \otimes \ldots \otimes 1) + (-1)^{(n+m)+(n+m-3)(m-1)} \pi_i (1 \otimes f(1 \otimes \ldots \otimes g) + (-1)^{(n+m-3)(m-1)} f(1 \otimes \ldots \otimes g(1 \otimes \ldots \otimes \pi_i).\end{aligned}$$

The summation in last relation is given in all possible place of i and all components of a maps f and g.

The first and second parts of the right hand side of (8) by considering the relation (6) are given by:

$$(-1)\delta(f \cup_1 g) = \sum (-1)^{(n+m-2)(m-1)+i+n-2} f(1 \otimes \ldots \otimes \ldots \otimes 1 \otimes \pi \otimes 1 \otimes \ldots \otimes g) + (-1) \times (-1)^{(n+m-1)(m-1)} f(1 \otimes \ldots \otimes \pi_i (1 \otimes g) + (-1)(-1)^{(n+m-2)(m-1)+n} \pi_i (1 \otimes f(1 \otimes g)). \tag{9}$$

and

$$(-1)\delta(f\cup_1 g) = (-1)^n(-1)^{(n+m-2)\ m}f(1\otimes\ldots\otimes\delta g)$$
$$= \sum(-1)^{(n+m-2)(m-1)+i+n-2}f(1\otimes\ldots\otimes\ldots$$
$$\otimes 1\otimes\pi\otimes 1\otimes\ldots\otimes g) + (-1)$$
$$\times(-1)^{(n+m-2)m+m+1}f(1\otimes\ldots\otimes\pi_i(1\otimes g))$$
$$+(-1)^n(-1)^{(n+m-2)\ m}f(1\otimes\ldots\otimes g(1$$
$$\otimes\ldots\otimes\pi_i)). \quad (10)$$

From (11) and (12) we have $\delta(f\cup_1 g) = -\delta\ f\cup_1 g + (-1)^n f\cup_1\delta g$.

Following [2], For a given module M over algebra A, the twisted cochain in Hochschild complex $C(A,M)$ (in the case of Banach module M over admissible Banach algebra A) is defined as follows: □

Definition 23. The twisted cochain in Hochschild complex $C(A,M)$ is an element $a = a^3 + a^4 + \cdots + a^i + \cdots$, where $a^i \in C^i(A,M)$, such that $\delta a = a\cup_1 a$, since $\cup_1$ is multiplication in the Hochschild complex for algebra. The set of Twisted cochains is denoted by $TW(A,M)$.

Definition 24. Two twisted cochains a and a' are equivalent if there exist an element $P = P^2 + P^3 + \ldots + P^i$, $P^i \in C^i(A,M)$ such that

$$a - a' = \delta P + P\cup_1 a + a'\cup_1(P\otimes P) + a'\cup_1(P\otimes P\otimes P) + \ldots$$

The set $TW(A,M)/\sim$, where $\sim$ is equivalent relation, is denoted by $D(A,M)$.

Theorem 25. *According to [2] the vanishing of Hochschild cohomology $H^n(A,A) = 0$ for $n > 0$, leads to the vanishing of the set $D(A,A)$.*

We define on the Hochschild complex $C_\infty(A,M)$, instead of Hochschild complex $C(A,M)$, the concept of twisted cochain for Banach module M over admissible Banach A_∞-algebra.

Definition 26. An element h in $C_\infty^{-2}(A,M)$ is twisted cochain, if :

1. $h_i = 0,\ if\ i < n+1.$
2. $\delta h = h\cup_1 h. \quad (11)$

Theorem 27. *where $\cup_1$ is defined above.*

The set of all twisted cochain in Hochschild complex $C_\infty(A,M)$ for Banach module over admissible Banach algebra, is denoted by $TW(C_\infty(A,M))$.

Definition 28. Two twisted cochain h and h' of Hochschild complex $C_\infty(A,M)$ of Banach module M over admissible Banach A_∞-algebra are equivalent $h\sim h'$, if there is an element $\ell \in C_\infty^{-1}(A,M)$, such that:

1. $\ell_1 = Id.$
2. $\delta\ell = \ell\cup_1 h + h'\cup_1\ell = 0.$

Theorem 29. *The relation defined in Definition 28 is equivalent relations.*

Proof 5. See [6]

The following theorems study the twisted cochain in Banach Hochschild complex $C_\infty(A,M)$ for Banach A_∞-module M over admissible Banach A_∞-algebra A and its relation with the cohomology of Banach Hochschild for these modules. The proof of these theorems analog to the cases of cohomology of pure A_∞-module over A_∞-algebra [see [6]]. □

Theorem 30. *Let $h \in TW(C_\infty(A,M))$ be an arbitrary twisted cochain and $\ell \in (C_\infty^{-1}(A,M))$, such that $\ell_1 = Id$, $\ell_i = 0$, for $i > n+1$, then there exist twisted cochain $\bar{h}$ such that:*

1. $h_i = \bar{h}_i,\ i < K+1,\ k > n$
2. $\bar{h}_{k+1} = h_{k+1} + (\delta f)_{k+1},\ \bar{h}\sim h. \quad (12)$

Theorem 31. *Let $H^{-2}(C_\infty(A,M)) = 0$, then $D(A,M) = 0$.*

References

[1] J.D. Stasheff, Homotopy associatively of H-space, 2 transfer, Math. Soc. 108 (2) (1963) 275–313.
[2] T.V. Kadeishvili, The – algebra structure and the Hochschild and Harrison cohomologies, Trudy Tbiliss. Mater. Inst. Razmadze Akad. Nauk Gruzin. SSR 91 (1988) 19–27.
[3] T.V. Kadeishvili, On the homology theory of fiber pace, YMH T.35 (3) (1980) 183–188.
[4] V.A. Smirnov, S.V. Kuznetsova, I.V. Mayorova, Description of the cohomology of Banach algebras and locally convex algebras in the language of – structures, Izvestiya. Math. 62 (4) (1998) 155–172.
[5] V.A. Smirnov, Simplicial and Operad Methods in Homotopy Theory, Factorial Press, 2002.
[6] Y. Gh. Gouda, S.A. Omran, On the cohomology with inner symmetry of a – infinity algebra, Int. J. Algebra 5 (5) (2011) 223–231.
[7] S.V. Lapin, D∞-differential E_∞-algebras and spectral sequences of fibrations, Sbornik Math. 198 (10) (2007) 1379–1406.
[8] S.V. Lapin, Multiplicative $A\infty$-structure in term of spectral sequences, Fundamentalnaya I prikladnia matematika 14 (6) (2008) 141–175 (in Russian).