Journal of the Egyptian Mathematical Society (2012) **20**, 57–63

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On permutation graphs

M.A. Seoud *, A.E.A. Mahran

Department of Mathematics, Faculty of science, Ain Shams university, Abbassia, Cairo, Egypt

Received 14 April 2011; revised 30 January 2012
Available online 2 November 2012

KEYWORDS
Permutation graph;
Graph labeling

Abstract We give an upper bound of the number of edges of a permutation graph. We introduce some necessary conditions for a graph to be a permutation graph, and we discuss the independence of these necessary conditions. We show that they are altogether not sufficient for a graph to be a permutation graph.

1. Introduction

Hegde and Shetty [1] define a simple graph G with n vertices to be a permutation graph if its vertices can be labeled with distinct integers 1, 2, …, n such that when each edge uv, where $u > v$, is labeled with $^u P_v$, the induced edge labels will be distinct. They prove that K_n is a permutation graph if and only if $n \leqslant 5$. Throughout this paper, we use the basic notations and conventions in graph theory as in [2], and in number theory as in [3] and [4]. We use $|A|$ to denote the size of the set A, i.e., its number of its elements, $\prod_{i=1}^{i=n} A_i$ to denote the cartesian product of the sets $A_1, A_2, \ldots, A_n$, and $A - B$ to denote the usual difference between the sets A and B. All graphs here are simple, i.e., containing no loops or multiple edges.

2. Some necessary conditions

2.1. Properties of permutation graphs

Definition 2.1.1 [5]. A simple graph G with n vertices to be a permutation graph if its vertices can be labeled with distinct integers $1, 2, \ldots, n$ such that when each edge uv, where $u > v$, is labeled with $^u P_v$, the induced edge labels will be distinct. A graph which is not a permutation graph is said to be a non-permutation graph.

Definition 2.1.2. A maximal permutation graph of n vertices is a permutation graph such that adding any new edge yields a non-permutation graph.

Remark 1. The maximal permutation graph is not unique (i.e., there are many non-isomorphic maximal permutation graphs of the same number of vertices). So, we denote them by $R^k(n)$, where the first graph is denoted by $R^1(n)$, and so on …

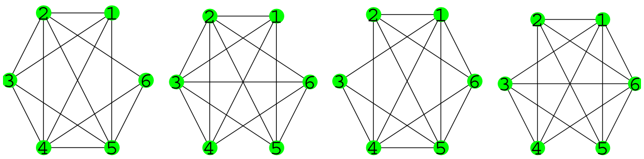
Example 2.1.1. The following graphs are the maximal permutation graphs of six vertices.

* Corresponding author.
E-mail addresses: m.a.seoud@hotmail.com (M.A. Seoud), a.e.a.mahran@gmail.com (A.E.A. Mahran).

Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

1110-256X © 2012 Egyptian Mathematical Society. Production and hosting by Elsevier B.V. Open access under CC BY-NC-ND license.
http://dx.doi.org/10.1016/j.joems.2012.08.008



Considering the four previous graphs without labeling, we see that the first graph is isomorphic to the third and the second graph is isomorphic to the fourth.

Definition 2.1.3 [3]. A positive integer $n \neq 1$ is said to be prime if it has no divisors other than 1 and n, while it is said to be composite if it is not prime.

The following lemmas are immediate.

Now, we give two theorems. The first gives an upper bound for the number of edges of a permutation graph of n vertices, which is easily calculated, and the second gives a formula of the exact number of edges of a maximal permutation graph of n vertices, but it takes very long time to be calculated.

Theorem 2.1.1. *If G is a graph of $n \geqslant 10$ vertices, with a number of edges more than*

$$\bar{\lambda}(n) = \frac{n(n-1)}{2} - 2\left(\left\lfloor \frac{-1+\sqrt{(1+4n)}}{2} \right\rfloor\right)$$

Then G is a non-permutation graph.

Proof. Since ${}^{i(i+1)}P_1 = {}^{i+1}P_2$, it follows that for every $i(i+1) \leqslant n$, the two edges connecting the vertices having the labels $1, i(i+1)$ and $i+1, 2$ have the same edge label, so, they cannot co-exist in a permutation graph. Also, we have ${}^{i(i+1)}P_{i^2-1} = {}^{i(i+1)-1}P_{i^2}$ for every $i(i+1) \leqslant n$. So, the two edges connecting the vertices having the labels $i(i+1)$, i^2-1 and $i(i+1)-1$, i^2 have the same edge label, so, they cannot co-exist in a permutation graph. Hence, for every $i(i+1) \leqslant n$ we have two repetitions in the edge labels. To get an upper bound for the number of edges of a permutation graph we should eliminate two edges from the complete graph of n vertices for every i satisfying $i(i+1) \leqslant n$, it remains to calculate the number of i's satisfying $i(i+1) \leqslant n$. The condition ${}^{i(i+1)}P_{i^2-1} = {}^{i(i+1)-1}P_{i^2}$ requires $i \geqslant 2$. The solution of the inequality $i^2 + i - n \leqslant 0$ is normally $\frac{-1-\sqrt{(1+4n)}}{2} \leqslant i \leqslant \frac{-1+\sqrt{(1+4n)}}{2}$. However, since we also need $i \geqslant 2$, and i has to be an integer, we immediately obtain the solution $2 \leqslant i \leqslant \left\lfloor \frac{-1+\sqrt{(1+4n)}}{2} \right\rfloor$, which is valid for $n \geqslant 6$. So, $\bar{\lambda}(n) = \frac{n(n-1)}{2} - 2\left(\left\lfloor \frac{-1+\sqrt{(1+4n)}}{2} \right\rfloor - 1\right)$. To improve the upper bound $\bar{\lambda}(n)$ we eliminate "-1" and put the condition $n \geqslant 10$. Hence for $n \geqslant 10$ we have

$$\bar{\lambda}(n) = \frac{n(n-1)}{2} - 2\left(\left\lfloor \frac{-1+\sqrt{(1+4n)}}{2} \right\rfloor\right)$$

The following theorem gives the exact number of edges of a maximal permutation graph.

Theorem 2.1.2. *The maximal permutation graph of n vertices has $\lambda(n)$ edges, where*

$$\lambda(n) = \frac{n(n-1)}{2} + \sum_{s=4}^{n}\sum_{r=1}^{s-3}\left\lfloor -\frac{\sum_{k=r+1}^{s-2}\sum_{m=k+1}^{s-1}\delta({}^sP_r, {}^mP_k)}{n} \right\rfloor$$

such that $\delta(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{Otherwise} \end{cases}$

Proof. Let $\mu(n) = \lambda(n) - \lambda(n-1)$ which is the number of edges having non-repeated labels incident to the vertex having the label n in a maximal permutation graph of n vertices. Then

$$\lambda(n) = \sum_{s=2}^{n}\mu(s)$$

To obtain a formula of $\mu(n)$ consider the edge labels

${}^2P_1$	${}^3P_1$	${}^4P_1$	...	${}^{n-1}P_1$	${}^nP_1$
	${}^3P_2$	${}^4P_2$	...	${}^{n-1}P_2$	${}^nP_2$
		${}^4P_3$	...	${}^{n-1}P_3$	${}^nP_3$
				.	.
				.	.
				${}^{n-1}P_{n-2}$	${}^nP_{n-2}$
					${}^nP_{n-1}$

If ${}^sP_r$ is a repetition of a permutation which has appeared in an earlier column, then the number of such repetitions is

$$\sum_{k=r+1}^{s-2}\sum_{m=k+1}^{s-1}\delta({}^sP_r, {}^mP_k)$$

which is less than n, since ${}^3P_r \neq {}^2P_1$ for every r. So,

$$0 \leqslant \sum_{k=r+1}^{s-2}\sum_{m=k+1}^{s-1}\delta({}^sP_r, {}^mP_k) < n$$

Thus

$$0 \leqslant \frac{\sum_{k=r+1}^{s-2}\sum_{m=k+1}^{s-1}\delta({}^sP_r, {}^mP_k)}{n} < 1$$

Hence

$$\left\lfloor -\frac{\sum_{k=r+1}^{s-2}\sum_{m=k+1}^{s-1}\delta({}^sP_r, {}^mP_k)}{n} \right\rfloor = \begin{cases} -1, & \text{if } {}^sP_r \text{ repeated} \\ 0, & \text{otherwise} \end{cases}$$

So

$$\mu(s) = s - 1 + \sum_{r=1}^{s-3}\left\lfloor -\frac{\sum_{k=r+1}^{s-2}\sum_{m=k+1}^{s-1}\delta({}^sP_r, {}^mP_k)}{n} \right\rfloor$$

Finally we get

$$\lambda(n) = \frac{n(n-1)}{2} + \sum_{s=4}^{n}\sum_{r=1}^{s-3}\left\lfloor -\frac{\sum_{k=r+1}^{s-2}\sum_{m=k+1}^{s-1}\delta({}^sP_r, {}^mP_k)}{n} \right\rfloor \quad \square$$

Corollary 2.1.3 (Condition 1). *If G is a graph of n vertices, which has number of edges more than $\lambda(n)$, then G is a non-permutation graph, where*

$$\lambda(n) = \frac{n(n-1)}{2} + \sum\nolimits_{s=4}^{n} \sum\nolimits_{r=1}^{s-3} \left\lfloor - \frac{\sum_{k=r+1}^{s-2} \sum_{m=k+1}^{s-1} \delta({}^{s}P_r, {}^{m}P_k)}{n} \right\rfloor$$

Note 1. Since $\lambda(n) < \overline{\lambda(n)}$, it follows that the previous corollary is valid also for $\overline{\lambda(n)}$.

Table 2.1.1 This table presents a comparison between the values of $\lambda(n)$ and $\overline{\lambda(n)}$.

n	$\lambda(n)$	$\bar{\lambda}(n)$	$\frac{\bar{\lambda}(n)-\lambda(n)}{\lambda(n)}$
10	41	41	0
11	51	51	0
12	60	60	0
13	72	72	0
14	85	85	0
15	97	99	0.02061856
16	112	114	0.01785714
17	128	130	0.01562500
18	145	147	0.01379310
19	163	165	0.01226994
20	180	182	0.01111111
21	200	202	0.01000000
22	221	223	0.00904977
23	243	245	0.00823045
24	264	268	0.01515152
25	288	292	0.01388889
26	313	317	0.01277955
27	339	343	0.01179941
28	366	370	0.01092896
29	394	398	0.01015228
30	421	425	0.00950119
31	451	455	0.00886918
32	482	486	0.00829876
33	514	518	0.00778210
34	547	551	0.00731261
35	581	585	0.00688468
36	616	620	0.00649351
37	652	656	0.00613497
38	689	693	0.00580552
39	727	731	0.00550206
40	766	770	0.00522193
41	806	810	0.00496278
42	845	849	0.00473373
43	887	891	0.00450958
44	930	934	0.00430108
45	974	978	0.00410678
46	1019	1023	0.00392542
47	1065	1069	0.00375587
48	1112	1116	0.00359712
49	1160	1164	0.00344828
50	1209	1213	0.00330852

The following table gives a comparison between $\lambda(n)$ and $\overline{\lambda(n)}$ (see Table 2.1.1)

Condition 2. If the minimum degree of a graph G of n vertices is greater than the largest minimum degree in all corresponding maximal permutation graphs $\delta(n)$, then the graph is a non-permutation graph.

Condition 3. If G is a graph on n vertices, whose number of vertices of degree $n-1$ is more than $t(n)$, where $t(n)$ is the maximum number of vertices of degree $n-1$ in all maximal permutation graphs of n vertices, then G is a non-permutation graph.

Lemma 2.1.4. *If m is a prime number, then $t(m) = t(m-1) + 1$, where $t(n)$ is the maximum number of vertices of degree $n-1$ in all maximal permutation graphs of n vertices, and $\delta(m) = \delta(m-1) + 1$, where $\delta(n)$ is the largest minimum degree in all maximal permutation graphs of n vertices.*

Proof. Since the labels of the edges incident to the vertex having the prime label m in any maximal permutation graph of m vertices is not repeated, it follows that all the edges incident to the vertex having the label m exist in any maximal permutation graph of m vertices. Hence, the vertex having the label m in any maximal permutation graph of m vertices has degree $m-1$, hence $t(m) = t(m-1) + 1$, and $\delta(m) = \delta(m-1) + 1$. □

Condition 4. If G is a graph of n vertices, and $K_{1+\chi(n)} \subseteq G$, where $\chi(n)$ is the order of the largest complete subgraph in all maximal permutation graphs on n vertices, then G is a non-permutation graph.

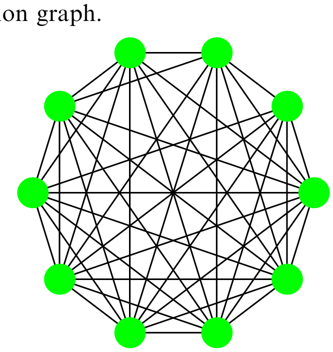


Table 2.2.1 The values in this table facilitates the calculations in the next examples.

n	$\lambda(n)$	$\delta(n)$	$t(n)$	$1+\chi(n)$
2	1	1	2	3
3	3	2	3	4
4	6	3	4	5
5	10	3	3	6
6	13	4	3	6
7	19	5	4	7
8	26	6	5	8
9	34	7	6	9
10	41	7	5	10
11	51	8	6	11
12	60	9	5	11

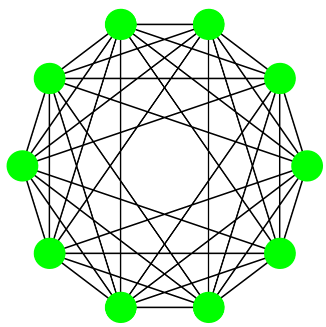
2.2. Independence of some necessary conditions

Now, we show that the four necessary conditions for permutation graphs are independent, and they are altogether not sufficient for a graph to be a permutation graph by using the following table, which we obtained by drawing all maximal permutation graphs on n vertices as we did Example 2.1.1 (see Table 2.2.1).

Example 2.2.1. Only Condition 1 proves that the following graph with $n = 10$ vertices is a non-permutation graph.

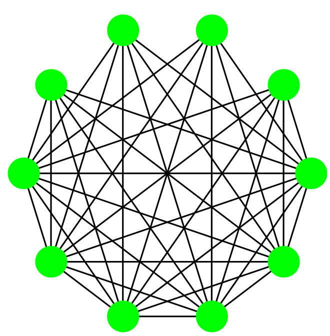
For Condition 2: the minimum degree of the graph equals $7 = \delta(10)$. For Condition 3: the number of vertices of degree $9 = 5 = t(10)$. For Condition 4: $K_{10} = K_{1+\chi(10)} \nsubseteq G$. But for Condition 1: the number of edges of $Gm = 42 > 41 = \lambda(10)$.

Example 2.2.2. Only Condition 3 proves that the following graph with $n = 10$ vertices is a non-permutation graph.



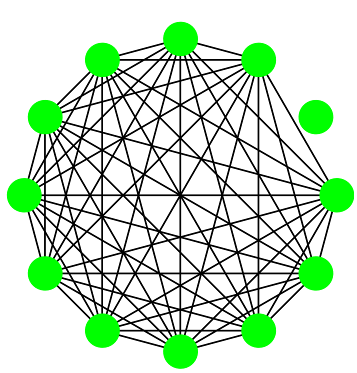
For Condition 1: The number of edges of G is $m = 40 < 41 = \lambda(10)$. For Condition 3: the number of vertices of degree $9 = 0 < 5 = t(10)$. For Condition 4: $K_{10} = K_{1+\chi(10)} \nsubseteq G$. But For Condition 2 : the minimum degree of the graph equals $8 > 7 = \delta(10)$.

Example 2.2.3. Only Condition 3 proves that the following graph $G = K_6 + \overline{K_4}$ with $n = 10$ is a non-permutation graph.



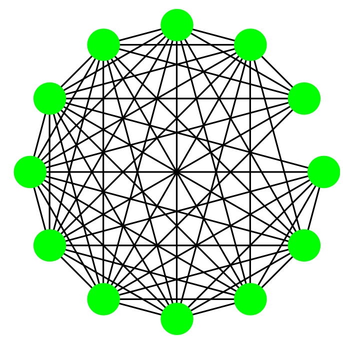
For Condition 1: The number of edges of G is $m = 39 < 41 = \lambda(10)$. For Condition 2: the minimum degree of the graph equals $6 < 7 = \delta(10)$. For Condition 4: $K_{10} = K_{1+\chi(10)} \nsubseteq G$. But for Condition 3 : the number of vertices of degree $9 = 6 > 5 = t(10)$.

Example 2.2.4. Only Condition 4 proves that the following graph $G = K_{11} \cup \overline{K_1}$ with $n = 12$ is a non-permutation graph.



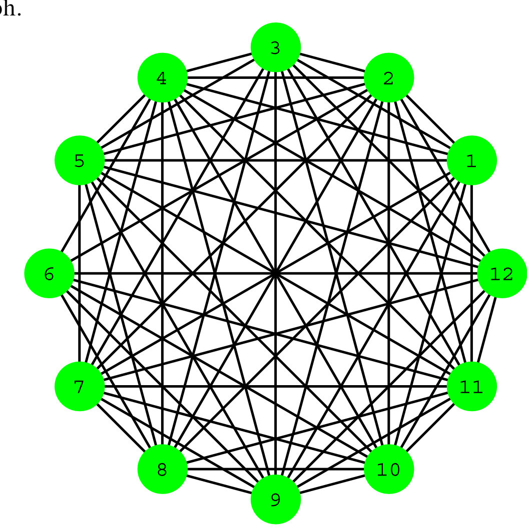
For Condition 1: The number of edges of G is $m = 55 < 60 = \lambda(12)$. For Condition 2: the minimum degree of the graph equals $0 < 9 = \delta(12)$. For Condition 3: the number of vertices of degree $11 = 0 < 5 = t(12)$. But For Condition 4: $K_{11} = K_{1+\chi(12)} \subseteq G$.

Example 2.2.5. Here we give an example of a non-permutation graph with $n = 12$ vertices, but the four conditions fail to decide that it is a non-permutation graph, i.e., they are altogether not sufficient for a graph to be a non-permutation graph.



For Condition 1: The number of edges of G is $m = 59 < 60 = \lambda(12)$. For Condition 2 : the minimum degree of the graph equals $7 < 9 = \delta(12)$. For Condition 3: the number of vertices of degree $11 = 4 < 5 = t(12)$. For Condition 4: $K_{11} = K_{1+\chi(12)} \nsubseteq G$.

It remains to show that the graph G is a non-permutation graph.



We notice that $K_{10} \subset G$, but among all the 64 maximal permutation graphs of 12 vertices the following graph is the only maximal permutation graph containing K_{10} as a subgraph.

So, our graph may be a subgraph only of this maximal permutation graph. But the number of vertices of degree greater than 9 in this maximal permutation graph is 9, and since the number of vertices of degree greater than 9 in our graph equals 10, it follows that our graph is not a subgraph of any maximal permutation graph of 12 vertices. Hence it is non-permutation.

2.3. Other necessary conditions

Definition 2.3.1. For Any two different positive integers $1 \leqslant i, j \leqslant n$, we define the set $V_p^n := \{(k,m) : {}^kP_m = p, \text{ and } 1 \leqslant m < k \leqslant n\}$.

For example, if $n = 6$, $V^6_{{}^6P_1} = V^6_{{}^3P_2} = V^6_6 = \{(6,1),(3,2)\}$, and $V^6_{{}^6P_3} = V^6_{{}^5P_4} = V^6_{120} = \{(6,3),(5,4)\}$.

Definition 2.3.2. A set theoretic operation

Let $A_1, A_2, \ldots, A_m$ be disjoint sets satisfying that for each i, $|A_i| > 1$, we define the operation $\overline{\prod_{i=1}^m A_i} := \{(\cup_{i=1}^m A_i) - \cup_{i=1}^m \{a_i\} : a_i \in A_i\}$.

Example 2.3.1. If $A_1 = \{a,b,c\}$, $A_2 = \{d,e\}$, $A_3 = \{f,g\}$, then $\overline{\prod_{i=1}^3 A_i} = \{\{b,c,e,g\},\{b,c,e,f\},\{b,c,d,g\},\{b,c,d,f\},\{a,c,e,g\},\{a,c,e,f\},\{a,c,d,g\},\{a,c,d,f\},\{a,b,e,g\},\{a,b,e,f\},\{a,b,d,g\},\{a,b,d,f\}\}$.

Lemma 2.3.1. *The operation* $\overline{\prod_{i=1}^m A_i}$ *is well-defined.*

Proof. The well definition of the operation $\overline{\prod_{i=1}^m A_i}$ is a direct consequence of the well definition of the union and difference. □

Remark 2. After calculating all the sets $V^n_{{}^iP_j}$ satisfying that $\left|V^n_{{}^iP_j}\right| > 1$, and calculating $\overline{\prod_{{}^2P_1 \leqslant {}^iP_j \leqslant {}^nP_{n-1}, |V^n_{{}^iP_j}|>1} V^n_{{}^iP_j}}$, we notice that, for each element $E \in \overline{\prod_{{}^2P_1 \leqslant {}^iP_j \leqslant {}^nP_{n-1}, |V^n_{{}^iP_j}|>1} V^n_{{}^iP_j}}$, we construct a maximal permutation graph by deleting all edges connecting the vertices having the labels k, m for all (k,m) in the set E from the complete graph of n vertices.

Now, we calculate the degree of the vertex labeled i in the maximal permutation graph associated with an element $E \in \overline{\prod_{{}^2P_1 \leqslant {}^iP_j \leqslant {}^nP_{n-1}, |V^n_{{}^iP_j}|>1} V^n_{{}^iP_j}}$ by the following lemma.

Lemma 2.3.2. *The degree of the vertex labeled t in the maximal permutation graph associated with an element* $E \in \overline{\prod_{{}^2P_1 \leqslant {}^iP_j \leqslant {}^nP_{n-1}, |V^n_{{}^iP_j}|>1} V^n_{{}^iP_j}}$ *is given by the following function* $F(t,E) = n - 1 - \sum_{(r,s)\in E} \theta_t(r,s)$, *where*

$$\theta_t(r,s) = \begin{cases} 1 & , \textit{if } r = t \textit{ or } s = t \\ 0 & , \textit{otherwise} \end{cases}$$

Definition 2.3.3. Let G be a given graph of n vertices. We define the following three sequences:

(1) The sequence of the distinct degrees of vertices in a maximal permutation graph arranged in an ascending order, we call it the maximal degree sequence. In fact we have many different sequences of such type due to the existence of non-isomorphic maximal permutation graphs. We denote such a sequence by $D_{R^k(n)} = (d_i^k)$, where d_i^k is the ith degree of vertices in the kth maximal permutation graph.
(2) The sequences $C_{R^k(n)} = (c_i^k)$, where c_i^k is defined to be the number of vertices of degree at most d_i^k in the kth maximal permutation graph, we call them the maximal permutation sequences.
(3) For the given graph G, the graph sequences $B_G^k = (b_i^k)$, where b_i^k is defined to be the number of vertices of degree at most d_i^k in G.

Example 2.3.2. For $n = 6$, $V^6_{{}^6P_1} = V^6_{{}^3P_2} = V^6_6 = \{(6,1),(3,2)\}$, and $V^6_{{}^6P_3} = V^6_{{}^5P_4} = V^6_{120} = \{(6,3),(5,4)\}$. So, $\overline{\prod_{{}^2P_1 \leqslant {}^iP_j \leqslant {}^6P_5, |V^6_{{}^iP_j}| > 1} V^6_{{}^iP_j}} = \{\{(6,1),(6,3)\},\{(6,1),(5,4)\},\{(3,2),(6,3)\},\{(3,2),(5,4)\}\}$.

The corresponding graphs are $R^1(6)$, $R^2(6)$, $R^3(6)$, $R^4(6)$ are shown in Example 2.1.1, their distinct sequences are $D_{R^2(6)} = D_{R^4(6)} = \{4,5\}, C_{R^2(6)} = C_{R^4(6)} = \{4,6\}$, $D_{R^1(6)} = D_{R^3(6)} = \{3,4,5\}, C_{R^1(6)} = C_{R^3(6)} = \{1,3,6\}$, and hence $\delta(6) = 4, t(6) = 3$.

Theorem 2.3.3 (Condition 5). *Let G be a simple graph for which there exists* i_0^k *such that* $b^k_{i_0^k} < c^k_{i_0^k}$; *for every k, then G is a non-permutation graph.*

Proof. For a fixed k, suppose that there exists i_0^k such that $b_{i_0^k} < c_{i_0^k}$, i.e., the number of vertices of degree at most $d_{i_0^k}$ in G is less than the number of vertices of degree at most $d_{i_0^k}$ in the corresponding kth maximal permutation graph, which is equal to the number of the labels of those vertices in the corresponding kth maximal permutation graph. Then, to distribute these labels on the vertices of G we must put them on vertices of degrees at most $d_{i_0^k}$, this implies that there exists at least one label, say r_0^k, on a vertex of degree at most $d_{i_0^k}$ in the corresponding kth maximal permutation graph must be given to a vertex of degree more than $d_{i_0^k}$ in G, say v_0^k, then there exist three vertices w_0^k, u_0^k, z_0^k, where w_0^k is adjacent with v_0^k, and has label, say m_0^k, also, the two vertices u_0^k and z_0^k are adjacent having labels, say s_0^k and t_0^k, satisfying $r_0^k * m_0^k = s_0^k * t_0^k$. Hence the graph G is not a subgraph of the kth maximal permutation graph. Since it happens for each k, it follows that G is not a subgraph of any maximal permutation graph. Hence G is a non-permutation graph.

Now, we'll show that Condition 5 is stronger than Conditions 1–3 in the sense that every non-permutation graph by these conditions is a non-permutation graph by Condition 5. □

Corollary 2.3.4. *If G is a graph of n vertices and m edges such that,* $m > \lambda(n)$, *then for each k, there exists* i_0^k *such that* $b^k_{i_0^k} < c^k_{i_0^k}$.

Proof. Suppose that m, the number of edges of G, is equal to $1 + \lambda(n)$, and by deleting an edge we get a permutation graph, i.e., G becomes a maximal permutation graph. Suppose that the edge which causes G to be a non-permutation graph with respect to the kth maximal permutation graph is one of g_k, h_k, connecting the vertices v_r^k, w_s^k and y_t^k, z_u^k respectively, having the labels r, s, t, u, such that ${}^rP_s = {}^tP_u$, then the degree of

v_r^k after removing the edge g_k is $\rho(r) = d_{i_r^k}$ and before removing g_k the degree is $d_{i_r^k} + 1$, then the number of vertices of degree at most $d_{i_r^k}$ in G is less than that number in the corresponding kth maximal permutation graph, i.e., there exists $i_0^k = i_r^k$ such that $b_{i_0^k}^k < c_{i_0^k}^k$. $\square$

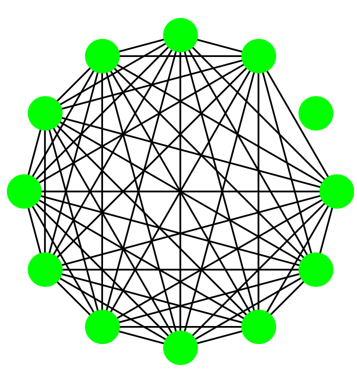
Corollary 2.3.5. *If the minimum degree of the graph is greater than $\delta(n)$, which is defined in Condition 2, then for every k, there exists i_0 such that $b_{i_0}^k < c_{i_0}^k$.*

Proof. Since the minimum degree of the graph is greater than $\delta(n)$, the largest minimum degree in all corresponding maximal permutation graphs, then for every k, the number of vertices of degree at most d_1^k in the graph equals zero which is less than the number of vertices of degree at most d_1^k in the corresponding kth maximal permutation graph, and it is clear that $i_0 = 1$, which satisfies for every k, $0 = b_{i_0}^k < c_{i_0}^k$. $\square$

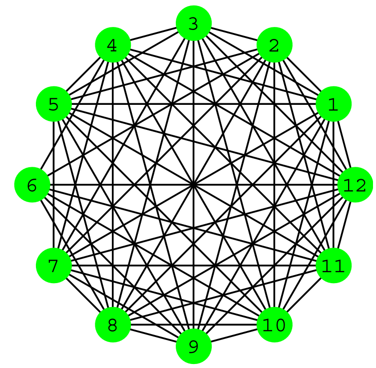
Corollary 2.3.6. *If a graph G of n vertices has a number of vertices of degree $n - 1$ more than $t(n)$ – which is defined in Condition 3 –, then there exists i_0 such that $b_{i_0}^k < c_{i_0}^k$, for each k.*

Proof. Suppose that the number of vertices of degree $n - 1$ in G is greater than $t(n)$, which is defined in Condition 3, then the number of vertices of degree less than $n - 1$ in G is $n - t(n)$, which is less than this number in all corresponding maximal permutation graphs. Then there exists i_0 (the second to the last term) such that $b_{i_0}^k < c_{i_0}^k$, for every k.

Here we give two examples to show that Conditions 4 and 5 are independent, in the sense that none of them implies the other.

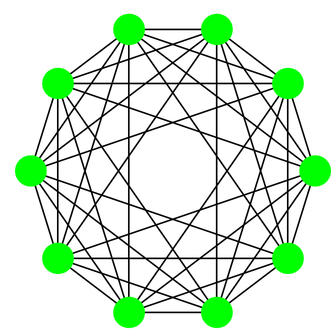


Example 2.3.3. Condition 4 proves that the following graph $G = K_{11} \cup K_1$ with $n = 12$ vertices is a non-permutation graph, while Condition 5 fails to decide that it is a non-permutation graph.



For Condition 5: the following graph is a maximal permutation graph having the sequences $D_{R^k(12)} = \{7, 10, 11\}$, $C_{R^k(12)} = \{1, 9, 12\}$.

Also, the corresponding graph sequence is $B_G^k = \{1, 12, 12\}$. Hence there exists k satisfies that $b_i^k \geqslant c_i^k$, for every i. But For Condition 4: $K_{1+\chi(12)} = K_{11} \subseteqq G$.



Example 2.3.4. Condition 5 proves that the following graph with $n = 10$ vertices is a non-permutation graph, while Condition 4 fails to decide that it is a non-permutation graph.

For Condition 4: $K_{10} = K_{1+\chi(10)} \nsubseteqq G$. But for Condition 5: from Table 2.3.1, the corresponding distinct sequences are $D_{R^1(10)} = \{7, 8, 9\}$, $C_{R^1(10)} = \{2, 6, 10\}$.

$D_{R^2(10)} = \{6, 7, 8, 9\}$, $C_{R^2(10)} = \{1, 2, 5, 10\}$.
$D_{R^3(10)} = \{5, 8, 9\}$, $C_{R^3(10)} = \{1, 5, 10\}$.
$D_{R^4(10)} = \{7, 8, 9\}$, $C_{R^4(10)} = \{1, 7, 10\}$.
$D_{R^5(10)} = \{7, 8, 9\}$, $C_{R^5(10)} = \{3, 5, 10\}$.
$D_{R^6(10)} = \{6, 8, 9\}$, $C_{R^6(10)} = \{1, 6, 10\}$.

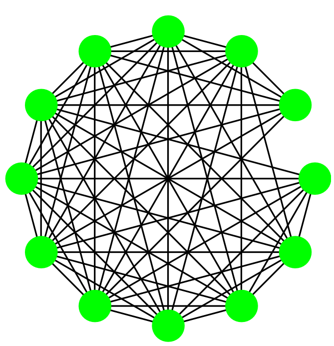
Also, their corresponding graph sequences are $B_G^1 = \{0, 10, 10\}$, $B_G^2 = \{0, 0, 10, 10\}$, $B_G^3 = \{0, 10, 10\}$, $B_G^4 = \{0, 10, 10\}$, $B_G^5 = \{0, 10, 10\}$, $B_G^6 = \{0, 10, 10\}$.

Here we give an example of a non-permutation graph with $n = 12$ vertices, but Conditions 4 and 5 fail to decide that it is a non-permutation graph, i.e., they are altogether not sufficient for a graph to be a non-permutation graph.

Table 2.3.1 This table gives the distinct sequences of all maximal permutation graphs of n vertices.

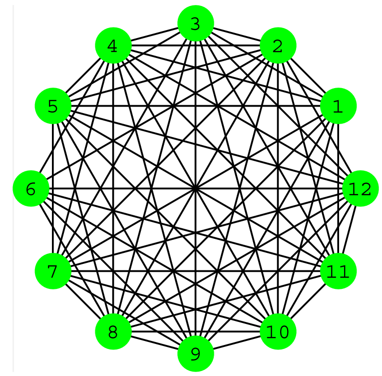
n	$1+\chi(n)$	$D_{R^i(n)}$	$C_{R^i(n)}$
6	6	$\{4,5\}$	$\{4,6\}$
		$\{3,4,5\}$	$\{1,3,6\}$
7	7	$\{5,6\}$	$\{4,7\}$
		$\{4,5,6\}$	$\{1,3,7\}$
8	8	$\{6,7\}$	$\{4,8\}$
		$\{5,6,7\}$	$\{1,3,8\}$
9	9	$\{7,8\}$	$\{4,9\}$
		$\{6,7,8\}$	$\{1,3,9\}$
10	10	$\{7,8,9\}$	$\{2,6,10\}$
		$\{6,7,8,9\}$	$\{1,2,5,10\}$
		$\{5,8,9\}$	$\{1,5,10\}$
		$\{7,8,9\}$	$\{1,7,10\}$
		$\{7,8,9\}$	$\{3,5,10\}$
10		$\{6,8,9\}$	$\{1,6,10\}$
11	11	$\{8,9,10\}$	$\{2,6,11\}$
		$\{7,8,9,10\}$	$\{1,2,5,11\}$
		$\{6,9,10\}$	$\{1,5,11\}$
		$\{8,9,10\}$	$\{1,7,11\}$
		$\{8,9,10\}$	$\{3,5,11\}$
		$\{7,9,10\}$	$\{1,6,11\}$
12	11	$\{9,10,11\}$	$\{4,8,12\}$
		$\{8,9,10,11\}$	$\{1,3,8,12\}$
		$\{7,10,11\}$	$\{1,9,12\}$
		$\{9,10,11\}$	$\{3,9,12\}$
		$\{9,10,11\}$	$\{2,10,12\}$
		$\{8,9,10,11\}$	$\{1,2,9,12\}$
		$\{8,9,10,11\}$	$\{1,4,7,12\}$
		$\{7,9,10,11\}$	$\{1,2,8,12\}$
		$\{9,10,11\}$	$\{1,11,12\}$
		$\{7,9,10,11\}$	$\{1,3,7,12\}$
12	11	$\{9,10,11\}$	$\{5,7,12\}$

In the following table we give the distinct sequences of all maximal permutation graphs of n vertices. □



Example 2.3.5.

For Condition 4: $K_{1+\chi(12)} = K_{11} \nsubseteq G$. For Condition 5: from Table 2.3.1, the following graph is a maximal permutation graph having the sequences $D_{R^k(12)} = \{7,9,10,11\}$, $C_{R^k(12)} = \{1,2,8,12\}$.



Also, the corresponding graph sequence is $B^k_G = \{2,2,8,12\}$. Hence there exists k satisfies that $b^k_i \geqslant c^k_i$, for every i. But from Example 2.2.5. G is a non-permutation graph.

References

[1] S.M. Hegde, S. Shetty, Combinatorial labelings of graphs, Appl. Math. ENotes 6 (2006) 251–258.
[2] F. Harary, Graph Theory, Addison-Wesely, Reading, Massachusetts, 1969.
[3] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, fifth ed., Clarendon Press, Oxford, 2002.
[4] J.E. Shockley, Introduction to Number Theory, Holt, Rinehart and Winston, Inc., Virginia, 1967.
[5] J.A. Gallian, A Dynamic Survey of Graph Labeling, Electronic Journal of Combinatorics 16 (2009) #DS6.