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ORIGINAL ARTICLE

On existence solutions and solution sets of differential equations and differential inclusions with delay in Banach spaces

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Abstract Let $C_E([-d,0])$ (resp. $C_{B(0,T)}([-d,0])$ be the Banach space of continuous functions from $[-d,0]$ into a Banach space E (resp. into $B(0,T)$), where $B(0,T)=\{x\in E: \|x\|\leqslant T\}$ and let $\mathcal{C}\in C_E([-d,0])$. In this paper we prove an existence theorem for the differential equation with delay

$$(P)\begin{cases}\dot{x}(t)=f^d(t,\bar{\theta}_t x), & t\in[0,T],\\ x=\mathcal{C}, & \text{on } [-d,0],\end{cases}$$

where $\bar{\theta}_t : C_{B(0,T)}([-d,t])\to C_E([-d,0])$ is such that $\bar{\theta}_t x(s)=x(t+s)$ for all $s\in[-d,0]$ and for all $x\in C_{B(0,T)}([-d,t])$ while f^d is a function from $[0,T]\times C_{B(0,T)}([-d,0])$ into E. By using $(\mathcal{R}_E,\mathcal{N},p)$– measure of noncompactness and under a generalization of the compactness assumptions, we prove an existence theorem and give some topological properties of solution sets of the problem

$$(Q)\begin{cases}\dot{x}(t)\in A(t)x(t)+F^d(t,\theta_t x), & t\in[0,T],\\ x=\mathcal{C}, & \text{on } [-d,0],\end{cases}$$

where $F^d:[0,T]\times C_E([-d,0])\to P_{fc}(E)$, $P_{fc}(E)$ is the set of all nonempty closed convex subsets of E while $\theta_t : C_E([-d,t])\to C_E([-d,0])$ defined by $\theta_t x(s)=x(t+s)\ \forall\, x\in C_E([-d,t])$, $\forall s\in[-d,0]$ and $\{A(t): 0\leqslant t\leqslant b\}$ is a family of densely defined closed linear operators generating a continuous evolution operator $\mathcal{S}(t,s)$.

1. Introduction

Put $B(0,T)=\{x\in E: \|x\|\leqslant T\}$, $\overline{C}=C_{B(0,T)}([-d,0])$ and $C_0=C_E([-d,0])$, where $C_{B(0,T)}([-d,0])$ is the Banach space of continuous functions from $[-d,0]$ into $B(0,T)$ and $C_E([-d,0])$ is the Banach space of continuous functions from

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$[-d,0]$ into a Banach space E. Let $f^d : [0,T] \times \overline{C} \to E$ be a Carathéodory function. For any $t \in [0,T]$, let $\bar{\theta}_t : C_{B(0,T)}([-d,t]) \to C_0$ defined by $\bar{\theta}_t x(s) = x(t+s)$ for all $s \in [-d,0]$ and $x \in C_{B(0,T)}([-d,t])$. Assume that F^d is a multivalued function from $[0,T] \times C_0$ into the set, $P_{fc}(E)$, of all nonempty closed convex subsets of E and $\{A(t) : 0 \leqslant t \leqslant T\}$ be a family of densely defined closed linear operators generating a continuous evolution operator $\mathcal{N}(t,s)$. For each $\mathcal{C} \in \overline{C}$, in Section 3 we deal with the existence solutions to the differential equations with delay of the form

$$(P)\begin{cases} \dot{x}(t) = f^d(t, \bar{\theta}_t x) & t \in [0,T] \\ x = \mathcal{C}, & \text{on } [-d,0], \end{cases}$$

we have a finite delay and use a measure of noncompactness thus we improvement Theorem 9 in [1] and that with a generalization of Theorem 2 in [2].

Moreover in Section 4 we consider the differential inclusion

$$(Q)\begin{cases} \dot{x}(t) \in A(t)x(t) + F^d(t, \theta_t x), & \mathrm{t} \in [0,T] \\ x = \mathcal{C} & \text{on } [-d,0], \end{cases}$$

where $\theta_t : C_E([-d,t]) \to C_0$ defined by $\theta_t x(s) = x(t+s)$ for all $s \in [-d,0]$ and for all $x \in C_E([-d,t])$. Our purpose in this section is to prove an existence theorem for integral solution of the problem (Q) and we give some topological properties for the solution set, $S(\mathcal{C})$, of the integral solutions for (Q), also we have an important consequence of Theorem 4.1 in the abstract control problems. In this section we have a generalization to the existence theorems of Deimling [3], Ibrahim and Gomaa [4], Kisielewicz [5] and Papageorgiou [6,7]. As $A(t) \neq 0$ the results extend that of [6,8–10].

2. Preliminaries

Let E be a Banach space and let us denote by $P(E)$ the collection of all nonempty subsets of E. Let $\mathcal{B}_E$ be the family of all nonempty bounded subsets of E and let $\mathcal{R}_E$ be the family of all nonempty and relatively weakly compact subsets of E.

Definition 2.1. A nonempty family $\mathcal{K} \subset \mathcal{R}_E$ is said to be a kernel if it satisfies the following conditions:

(i) $A \in \mathcal{K} \Rightarrow conv\ A \in \mathcal{K}$,
(ii) $B \neq \emptyset,\ B \subset A \Rightarrow A \in \mathcal{K}$,
(iii) A subfamily of all weakly compact sets in $\mathcal{K}$ is closed in the family of all bounded and closed subsets of E with the topology generated by the Hausdorff distance.

Definition 2.2. A function $\gamma : \mathcal{B}_E \to [0,\infty)$ is said to be a measure of noncompactness with the kernel $\mathcal{K}$ if it is subject to the conditions:

(i) $\gamma(A) = 0 \Longleftrightarrow A \in \mathcal{K}$,
(ii) $\gamma(A) = \gamma(\overline{A})$, where $\overline{A}$ is weak closure of the set A,
(iii) $\gamma(conv\ A) = \gamma(A)$,
(iv) $A, B \in \mathcal{B}_E,\ B \subset A \Rightarrow \gamma(B) \leqslant \gamma(A)$ [11,12].

Denote by $\mathcal{N}$ a basis of neighbourhoods of zero in a locally convex space composed of closed convex sets and $\mathcal{N}' = \{rN : N \in \mathcal{N}, r > 0\}$. The following two definitions can be found in [13,14].

Definition 2.3. A function $p : \mathcal{N}' \to [0,\infty)$ is said to be p–function if it satisfies the following conditions:

(i) $X, Y \in \mathcal{N}',\ X \subset Y \to p(X) \leqslant p(Y)$,
(ii) for each $\varepsilon > 0$ there exists X$\in \mathcal{N}'$ such that $p(X) < \varepsilon$,
(iii) $p(X) > 0$ whenever $X \notin \mathcal{K}$.

Definition 2.4. A function $\gamma : \mathcal{B}_E \to [0,\infty)$ is said to be $(\mathcal{K}, \mathcal{N}, p)$– measure of noncompactness if and only if

$$\gamma(U) = inf\{\varepsilon > 0 : \exists A \in \mathcal{K}, X \in \mathcal{N}', U \subset A + X, p(X) \leqslant \varepsilon\},$$

for each $U \in \mathcal{B}_E$.

For any nonempty bounded subset Z of E we recall the definition of Kuratatowski measure, α, of noncompactness and the Haudorff measure, α^*, of noncompactness

$$\alpha(Z) = \inf\{\varepsilon > 0 : Z \text{ admits a finite number of sets with diameter} < \varepsilon\},$$
$$\alpha^*(Z) = \inf\{\varepsilon > 0 : Z \text{ admits a finite number of balls with radius} < \varepsilon\}.$$

For the properties of α and α^* we refer to [12,15] for instance. Each the Kuratowski measure of noncompactness and the Hausdorff measure of noncompactnessare is $(\mathcal{K}, \mathcal{N}, p)$– measure of noncompactness (see [13]).

In this paper we consider $I = [0,T]$, λ is the Lebesgue measure on I and $\mathcal{L}(E)$ is the algebra of all continuous, linear operators from E to E. For each $t \in I$, $\bar{\theta}_t$ is the function from $C_{B(0,\ T)}([-d,t])$ into C_0 defined by

$$\bar{\theta}_t x(s) = x(t+s) \quad \forall\ s \in [-d,0], \forall\ x \in \overline{C}.$$

and θ_t is that from $C_E([-d,t])$ into C_0 such that

$$\theta_t x(s) = x(t+s) \quad \forall\ s \in [-d,0], \forall\ x \in C_0.$$

If $\mathcal{Q} : I \to 2^E - \{\emptyset\}$ is measurable and integrable bounded with weakly compact values, then set of all integrable selections of $\mathcal{Q}, \nabla^1_{\mathcal{Q}}$, is weakly compact in the Banach space, $L^1(I,E)$, of Lebesque Bochner integrable functions $f : I \to E$ endowed with the usual norm [16].

Definition 2.5. If $\mathcal{S} : I \times I \to \mathcal{L}(E)$ such that $\mathcal{S}(t,0)x_0$ is a solution of the problem

$$(i)\begin{cases} \dot{x}(t) = A(t)x \\ x(0) = x_0 \end{cases}$$

where $\{A(t) : t \in I\}$ is a family of densely defined closed linear operators on E, then a continuous function $x : [-d,T] \to E$ is called an integral solution of the problem (Q) if

$$x = \mathcal{C} \text{ on } [-d,0] \text{ and } x(t) = \mathcal{S}(t,0)\mathcal{C}(0) + \int_0^t \mathcal{S}(t,s)f(s)ds \text{ for all } t \in I,$$

since $f(s) \in F^d(s, \theta_s x)$ and $f \in L^1(I,E)$.

A multivalued function $F : E \to P(E)$ is upper semicontinuous if for all open subset V of E $F^-(V) = \{x \in E : F(x) \subset V\}$ is open in E. Let E^* be the topological dual of the Banach space E and E_w be the Banach space E endowed with the weak topology. By taking $P_f(E)$ the collection of all nonempty closed subsets of E, so the multivalued function $F : E \to P_f(E)$ is $w - w$ sequentially upper semicontinuous if every weakly closed subset A of E $F^-(A)$ is weakly sequentially closed and it is upper hemi-continuous (resp. weakly upper hemi-continuous) if and only if for any $x^* \in E^*$, $c \in \mathbb{R}$ $\{x \in E : \sup_{y \in F(x)} (x^*, y) < c\}$ is open in E (resp. in E_w). Moreover F is weakly sequentially upper hemi-continuous if and only if for any $x^* \in E^*$ the function $h : E_w \to \mathbb{R}$ defined by $h(x) = \sup_{y \in F(x)} (x^*, y)$ is sequentially upper semicontinuous. For details and equivalent definitions see, e.g. [17,18,16].

The following lemmas will be crucial in the proof our existence results.

Lemma 2.6. [8]. *If $F : E \to P_{cwk}(E)$ is weakly sequentially upper hemi-continuous then, for any weakly compact subset $A \subseteq E$, $F(A)$ is weakly compact. Let Y be a Banach space and let $F : E \to P_{fc}(Y)$ be weakly sequentially upper hemi-continuous. If there exist $a \in L^1(I, \mathbb{R})$, $(x_n)_{n \in IN} \subset C(I, E)$ and $(y_n)_{n \in IN \cup \{0\}} \subset L^1(I, E)$ such that $\|F(x(t))\| \leqslant a(t)$ almost all $t \in I$ and for all $x \in C(I, E)$, $x_n(t) \to x_0(t)$ weakly a.e. on I, $y_n \to y_0$ weakly and $y_n(t) \in F(x_n(t))$ a.e. on I, then $y_0(t) \in F(x_0(t))$ a.e. on I.*

Lemma 2.7. [14]. *Let γ be a $(\mathcal{K}, \mathcal{N}, p)$–measure of noncompactness such that $p(\alpha X) = \alpha p(X)$ with $X \in \mathcal{N}\prime$, $\alpha \in \mathbb{R}$ and let $\mathcal{N}$ be composed of balanced sets. So, for each bounded subset U of E and for each $A \in \mathcal{L}(E)$, we have $\gamma(AU) \leqslant |A| \gamma(U)$.*

Lemma 2.8. [13]. *If γ is a $(\mathcal{K}, \mathcal{N}, p)$– measure of noncompactness such that $p(\alpha X) = \alpha p(X)$ with $X \in \mathcal{N}'$, $\alpha \in \mathbb{R}^+$ and for each $X, Y \in \mathcal{N}'$ we find $X + Y \in \mathcal{N}'$, then*

(M_1) $\gamma(U + V) \leqslant \gamma(U) + \gamma(V)$,
(M_2) $\gamma(\alpha U) = \alpha \gamma(U)$,
(M_3) $\gamma(U) = 0 \iff U$ *is relatively compact in* E,
(M_4) $\gamma(U \cup \{x\}) = \gamma(U)$, $x \in E$,
(M_5) $U \subseteq V \Rightarrow \gamma(U) \leqslant \gamma(V)$,
(M_6) $\gamma(\overline{conv} U) = \gamma(U)$.

Under the assumptions in Lemma 2.8 on the measure γ we state the following lemma.

Lemma 2.9. [19]. *Let $V \subseteq C(I, E)$ be a bounded equicontinuous for the strong topology and $V(J) = \{x(t) : x \in V,\ t \in J\}$, where J is a subinterval of I. Then, under the assumptions in* Lemma 2.8, *$\gamma(V(J)) = sup_{t \in J} \gamma(V(\{t\})) = \gamma(J(s))$ for some $s \in J$.*

3. Existence result for (P)

In the following theorem we have a finite delay and use a measure of noncompactness thus we improvement Theorem 9 in [1] and that with a generalization of Theorem 2 in [2].

Theorem 3.1. *Let $f^d : I \times \overline{C} \to E$ be a Carathéodory function and w be a Kamke function such that $w(t, \cdot)$ is nondecreasing for every $t \in [0, T]$. Suppose that the following conditions are satisfied:*

(1) $\forall (t, \varphi) \in I \times \overline{C}$, $\|f^d(t, \varphi)\| \leqslant \mu(t)$ for some $\mu \in L^1(I, \mathbb{R}^+)$,
(2) for each $\varepsilon > 0$ and any nonempty bounded subset Z of $\overline{C}$ there exists a closed subset I_ε of I with $\lambda(I - I_\varepsilon) < \varepsilon$ and $\gamma(f^d(J \times Z)) \leqslant \sup_{t \in J} w(t, \beta(Z))$ for any compact subset J of I_ε,
(3) for each $\varphi \in \overline{C}$, $f^d(I \times \{\varphi\})$ is separable.

Then, for any $\mathcal{C} \in \overline{C}$, the problem (P) has a solution.

Proof. For any arbitrary $n \in \mathbb{N}$ and for every $(t, x) \in \left[-d, \frac{T}{n}\right] \times B(0, T)$, set

$$f_1^n(t, x) = \begin{cases} \mathcal{C}(\mathrm{t}) & \text{if } t \in [-d, 0] \\ \mathcal{C}(0) + nt(x - \mathcal{C}(0)) & \text{if t } \in \left[0, \frac{T}{n}\right]. \end{cases}$$

Also let $h_1(t, x) = f^d\left(t, \bar{\theta}_{\frac{T}{n}}\left(f_1^n(\cdot, x)\right)\right)$ and thus $\|h_1(t, x)\| \leqslant \mu(t)$. Set

$$S = \left\{ x \in C\left(\left[0, \frac{T}{n}\right], B(0, T)\right) : \ \|x(t) - \mathcal{C}(0)\| \leqslant \int_0^{\frac{T}{n}} \mu(s)\ ds \right\},$$

then for each $n \in \mathbb{N}$ we can define an element x_m of S by

$$x_m(t) = \begin{cases} \mathcal{C}(0) & \text{if } 0 \leqslant t \leqslant \frac{T}{nm} \\ \mathcal{C}(0) + \int_0^{t - \frac{T}{nm}} h_1(s, x_m(s))\ ds & \text{if } \frac{T}{nm} \leqslant t \leqslant \frac{T}{n}, \end{cases}$$

and ϱ: $S \to S$ such that $\varrho(x)(t) = \mathcal{C}(0) + \int_0^t h_1(s, x(s))\ ds$, for all $t \in \left[0, \frac{T}{n}\right]$. Now

$$\lim_{n\to\infty} \|\varrho(x_m) - x_m\| = \lim_{n\to\infty} \sup_{t \in \left[0, \frac{T}{n}\right]} \|\varrho(x_m)(t) - x_m(t)\|$$
$$\leqslant \lim_{n\to\infty} \left(\sup_{t \in \left[0, \frac{T}{nm}\right]} \|\varrho(x_m)(t) - x_m(t)\| + \sup_{t \in \left[\frac{T}{nm}, \frac{T}{n}\right]} \|\varrho(x_m)(t) - x_m(t)\| \right) \to 0.$$

Let $K = \{x_m :\ m \in \mathbb{N}\}$, $L = \{\varrho(x_m) :\ m \in \mathbb{N}\}$. If Id is the identity function on A, then $\gamma\left((Id - \varrho)K\left(\left[0, \frac{T}{n}\right]\right)\right) = 0$ and from Lemma 2.9 $\gamma((Id - \varrho)K(t)) = 0$ also for each $t \in \left[0, \frac{T}{n}\right]$

$$\gamma(K(t)) \leqslant \gamma((Id - \varrho)K(t)) + \gamma(L(t)).$$

Moreover

$$\gamma(L(t)) \leqslant \gamma((Id - \varrho)K(t)) + \gamma(K(t)).$$

Thus we see that

$$\gamma(K(t)) = \gamma(L(t)).$$

Obvious the sets A and B are equicontinuous. Let $\chi(t) = \gamma(L(t))$, $t \in \left[0, \frac{T}{n}\right]$ and so $\chi(0) = 0$. For each $t, \tau \in \left[0, \frac{T}{n}\right]$ we have

$$\gamma(L(\tau)) \leqslant \gamma(L(t)) + \gamma(L(\tau) - L(t)),$$
$$\gamma(L(t)) \leqslant \gamma(L(\tau)) + \gamma(L(t) - L(\tau)).$$

Then

$$|\chi(\tau) - \chi(t)| \leqslant \gamma(B(0, 1)) \left(\int_t^\tau \mu(s)\ ds \right)$$

and so χ is absolutely continuous function that is it is differentiable a.e. on $\left[0, \frac{T}{n}\right]$. Let $(t, \tau) \in \left[0, \frac{T}{n}\right] \times \left[0, \frac{T}{n}\right]$ such that $t \leqslant \tau$.

Since χ is continuous and w is Carathéodory we can find a closed subset I_ε of $\left[0, \frac{T}{n}\right]$, $\delta > 0$, $\eta > 0$ $(\eta < \delta)$ and for s_1, $s_2 \in I_\varepsilon$; r_1, $r_2 \in [0, T]$ such that if $|s_1 - s_2| < \delta$, $|r_1 - r_2| < \delta$, then $|w(s_1,r_1) - w(s_2,r_2)| < \varepsilon$ and if $|s_1 - s_2| < \eta$, then $|\chi(s_1) - \chi(s_2)| < \frac{\delta}{2}$. Consider the partition $P = \{t_0, t_1, t_2, \ldots, t_m\}$ of $[t,\tau]$ such that $t_i - t_{i-1} < \eta$ for $i = 1, \ldots, n$. Let $A_i = \{x(s)$: $x \in K$, $s \in [t_{i-1}, t_i] \cap I_\varepsilon\}$. Moreover we can find a closed subset J_ε of $\left[0, \frac{T}{n}\right]$ such that $\lambda\left(\left[0, \frac{T}{n}\right] - J_\varepsilon\right) < \varepsilon$ and that for any compact subset $\mathcal{K}$ of J_ε and any bounded subset Z of E, $\gamma(h_1(\mathcal{K} \times Z)) \leqslant \sup_{s\in\mathcal{K}} w(s, \gamma(Z))$. Let $T_i = J_\varepsilon \cap [t_{i-1}, t_i] \cap I_\varepsilon$, $P = \sum_{i=1}^m T_i = [t, \tau] \cap J_\varepsilon \cap I_\varepsilon$ and $Q = [t,\tau] - P$. Thus $\int_t^\tau h_1(s, K(s))\, ds \subset \int_P h_1(s, K(s))\, ds + \int_Q h_1(s, K(s))$. In virtue of Lemma 2.9 and from the continuity of χ, we have $r_i \in [t_{i-1}, t_i] \cap I_\varepsilon$ such that

$$\gamma(A_i) = \sup\{\gamma(K(s)) :\ s \in [t_{i-1}, t_i] \cap I_\varepsilon\} = \chi(r_i).$$

Let Z be a bounded subset of E and $N = \left\{\bar{\theta}_{\frac{T}{n}}\left(f_1^n(\cdot, x)\right) : x \in Z\right\}$. Thus, for each t $\in \left[0, \frac{T}{n}\right]$, $\gamma(h_1(\{t\} \times Z)) = \gamma(f^d(\{t\} \times N))$. From Condition (2), there exists a closed subset J_ε of $\left[0, \frac{T}{n}\right]$ such that $\lambda\left(\left[0, \frac{T}{n}\right] - J_\varepsilon\right) < \varepsilon$ and that for any compact subset C of J_ε

$$\gamma(h_1(C \times Z)) = \gamma(f^d(C \times N)) \leqslant \sup_{s\in C} w(s, \beta(N)).$$

Now if $\overline{N} = \left\{\bar{\theta}_{\frac{T}{n}}\left(f_1^n(\cdot, x)\right) : x \in A_i\right\}$, then

$$\gamma(\mathcal{D}_1(C \times A_i)) = \gamma(F^d(C \times \overline{N})) \leqslant \sup_{s\in C} w(s, \beta(\overline{N})).$$

From the mean value theorem we obtain

$$\int_P h_1(s, N(s))\, ds \subset \sum_{i=1}^m \int_{T_i} h_1(s, N(s))\, ds \subset \sum_{i=1}^n \lambda(T_i)\overline{conv} h_1(T_i \times A_i)$$

Now, from Lemma 2.9 and Condition 2, we have

$$\gamma\left(\int_P h_1(s, N(s))\, ds\right) \leqslant \int_t^\tau w(s, \chi(s))\, ds + \varepsilon(\tau - t).$$

Invoking Lemma 2.7, we get

$$\gamma\left(\int_Q h_1(s, N(s))\, ds\right) \leqslant \gamma(B(0, 1)) \int_Q \mu(s)\, ds.$$

Therefore $\dot{\chi}(t) \leqslant w(t, \chi(t))$ a.e. on $\left[0, \frac{T}{n}\right]$ [19]. Since $\chi(0) = 0$ and w is a Kamke function, then $\chi \equiv 0$. Thus the closure of $(x_m)_{n\in\mathbb{N}}$ is compact and thus we can find a subsequence (x_{m_k}) of (x_m) which converges to a limit u_1 in $C\left(\left[0, \frac{T}{n}\right], E\right)$. Since $\|x_m - \varrho(x_m)\| \to 0$ and ϱ is continuous, then $x = \varrho(x)$. Therefore there is a continuous function u_1 such that $u_1 = \mathcal{C}$ on $[-d,0]$ and $u_1 = x$ on $\left[0, \frac{T}{n}\right]$ that is for each $t \in \left[0, \frac{T}{n}\right]$,

$$u_1(t) = \mathcal{C}(0) + \int_0^t h_1(s, u_1(s))ds = \mathcal{C}(0) + \int_0^t f^d\left(s, \bar{\theta}_{\frac{T}{n}}\left(f_0^n(s, u_1(s))\right)\right)ds.$$

Now for some $k \in \{1, 2, 3, \ldots, n\}$ and by taking $k' = k - 1$ we can assume that there exist a continuous function $h_{k'}$ and $u_{k'}$ such that $u_{k'} = \mathcal{C}$ on $[-d,0]$ and for each $t \in \left[0, \frac{k'T}{n}\right]$

$$u_{k'}(t) = \mathcal{C}(0) + \int_0^t h_{k'}(s, u_{k'}(s))ds = \mathcal{C}(0) + \int_0^t f^d\left(s, \bar{\theta}_{\frac{k'T}{n}} f_{k'}^n(s, u_{k'}(s))\right)ds$$

also let $g_k^n : \left[-d, \frac{kT}{n}\right] \times B(0, T) \to E$ be such that

$$f_k^n(t, x) = \begin{cases} u_{k'}(\mathrm{t}) & \text{if } t \in \left[-d, \frac{k'T}{n}\right] \\ u_{k'}\left(\frac{k'T}{n}\right) + n\left(t - \frac{k'T}{n}\right)\left(x - u_{k'}\left(\frac{k'T}{n}\right)\right) & \text{if t } \in \left[\frac{k'T}{n}, \frac{kT}{n}\right]. \end{cases}$$

Now for each $(t, x) \in \left[\frac{k'T}{n}, \frac{kT}{n}\right] \times B(0, T)$ set $h_k(t, x) = f^d\left(t, \bar{\theta}_{\frac{kT}{n}}\left(f_k^n(\cdot, x)\right)\right)$, then we have a continuous function u_k defined on $\left[\frac{k'T}{n}, \frac{kT}{n}\right]$ by

$$u_k(t) = u_{k'}\left(\frac{k'T}{n}\right) + \int_{\frac{k'T}{n}}^t h_k(s, h_k(s))ds.$$

Furthermore

$$u_{k'}\left(\frac{k'T}{n}\right) = \mathcal{C}(0) + \int_0^{\frac{k'T}{n}} h_{k'}(s, u_{k'}(s))ds.$$

Hence there exists

$$g_k(t, x_k(t)) = \begin{cases} h_{k'}(t, x_{k'}(t)) & \text{if } t \in \left[0, \frac{k'T}{n}\right] \\ h_k(t, x_k(t)) & \text{if t } \in \left[\frac{k'T}{n}, \frac{kT}{n}\right], \end{cases}$$

such that

$$u_k(t) = \mathcal{C}(0) + \int_0^t g_k(s, x_k(s))ds.$$

So, for any $n \in \mathbb{N}$, there exist v_n such that $v_n = \mathcal{C}$ on [-d,0] and for each $t \in I$, v_n is defined by

$$v_n(t) = \mathcal{C}(0) + \int_0^t h_n(s)ds,$$

where $\frac{k'T}{n} \leqslant t \leqslant \frac{kT}{n}$, $k \in \{1, 2, 3, \ldots, n\}$ and $h_n(t) = f^d\left(s, \bar{\theta}_{\frac{kT}{n}} f_k^n(., v_n(s))\right)$. Let t_1, $t_2 \in I$ and $t_1 < t_2$. Now

$$\|v_n(t_1) - v_n(t_2)\| \leqslant \int_{t_1}^{t_2} \left\|f^d\left(s, \bar{\theta}_{\frac{kT}{n}} f_k^n(., v_n(s))\right)\right\| ds \leqslant \int_{t_1}^{t_2} \mu(s)ds$$

since $v_n = \mathcal{C}$ on [-d,0], then L is equicontinuous in $C_{B(0,T)}[-d,T]$. Moreover the set $\beta(L(t)) = \beta(\{v_n(t) : n \in \mathbb{N}\})$ is such that $\beta(L(0)) = 0$ and, as in the proof of Theorem 9 in [1] and by using Lemmas 2.8 and 2.9, we get $\beta(L(t)) = 0$ for all $t \in I$. Thus by Ascoli's theorem we may assume that the sequence $\{v_n : n \in \mathbb{N}\}$ converges uniformly to a function $v \in C_{B(0,T)}([-d, T])$ such that $y = \mathcal{C}$ on $[-d,0]$. But $\beta(\{h_n(t) :\ n \in \mathbb{N}\}) = 0$ and so $\{h_n(t) :\ n \in \mathbb{N}\}$ is relatively compact. Create a multivalued function $\mathcal{F}(t) = \overline{conv}\{h_n(t) : n \in \mathbb{N}\}$. Thus $\mathcal{F}(t)$ is nonempty convex and compact, the set $\delta^1_{\mathcal{F}} = \{l \in L^1(I, E) : l(t) \in \mathcal{F}(t)\}$ is nonempty convex and weakly compact. By Eberlein–Šmulian Theorem there exists a subsequence (h_{n_k}) of (h_n) such that $h_{n_k} \to l$ weakly, $l \in \delta^1_{\mathcal{F}}$. Thus v_n tends weakly to $\mathcal{C}(0) + \int_0^t l(s)ds$. Moreover since, for each $n \in \mathbb{N}$, $v_n \in C_{B(0,T)}([-d, T])$, v_n converges uniformly to v on each compact subset of $[-d, T]$ and v is uniformly continuous on $[-d,0]$; also for each $t \in I$, there exists $n > \frac{T}{d}$ with $t \in \left[\frac{k'T}{n}, \frac{kT}{n}\right]$ for $k \in \{1, 2, 3, \ldots, n\}$ so,

$$\begin{aligned} \left\|\bar{\theta}_{\frac{kT}{n}} f_k^n(., v_n(t)) - \bar{\theta}_t v\right\| \leqslant & \sup_{s\in\left[-d, -\frac{T}{n}\right]} \left[\left\|v_n\left(\frac{kT}{n} + s\right) - v\left(\frac{kT}{n} + s\right)\right\|\right. \\ & \left. + \left\|v\left(\frac{kT}{n} + s\right) - v(t + s)\right\|\right] \\ & + \sup_{s\in\left[-\frac{T}{n}, 0\right]} \left[\left(T\left\|\left(v_n(t) - v_n\left(\frac{k'T}{n}\right)\right)\right\|\right.\right. \\ & \left.\left. + \left\|v_n\left(\frac{k'T}{n}\right) - v\left(\frac{kT}{n} + s\right)\right\| + \left\|v\left(\frac{kT}{n} + s\right) - v(t + s)\right\|\right)\right] \\ & \to 0 \text{ as } n \to \infty. \end{aligned}$$

From Lemma 2.6 the proof is therefore complete. □

4. Existence Results for (Q)

Assume that F: $I \times C_0 \to P_{fc}(E)$, where $P_{fc}(E)$ is the family of nonempty closed convex subsets of E. We say that F satisfies (A) if:

- (A_1) $\|F^d(t,\mathcal{C})\| \leqslant c(t)(1+\|\mathcal{C}(0)\|$ for each $\mathcal{C} \in C_0$ and for some $c \in L^1(I,\mathbb{R})$ a.e. on I.
- (A_2) for each $\varepsilon > 0$, there exists a closed subset I_ε of I with λ $(I - I_\varepsilon) < \varepsilon$ such that for any nonempty bounded subset Z of $C_E([-d,0])$ and for each closed subset $J \subseteq I_\varepsilon$, one has $$\gamma(F^d(J\times Z)) \leqslant \sup_{t\in J} w(t,\beta(Z)).$$
- (A_3) $F(\cdot,\mathcal{C})$ has a measurable selection, for each $\mathcal{C} \in C_E([-d,0])$.
- (A_4) for each $t \in I$, $F^d(t,\cdot)$ is weakly sequentially upper hemicontinuous.

In this section our purpose is to prove an existence theorem for integral solution of the problem (Q), and give some topological properties for the solution set of all integral solutions for (Q). The problem (Q) was investigated, without delay, by many authors [3,10,8,9,20] for instance. In this case when $A(t)=0$, $\mathcal{S}(t,s)=id$ and a mild solution is a Carathéodory one, we have a generalization to the existence theorems of Deimling [3], Ibrahim and Gomaa [4], Kisielewicz [5] and Papageorgiou [6,7]. As $A(t)\neq 0$ the following results extend that of [6,8–10].

Theorem 4.1. *Let $\{A(t):t\in I\}$ be a generator of a fundamental solution $\mathcal{N}: I\times I \to \mathcal{L}(E)$ such that $\mathcal{N}(t,t)=id$, $t\in I$, id is the identity function on E; $\mathcal{N}(t,s)\mathcal{N}(s,r)=\mathcal{N}(t,r)$, $t,s,r\in I$; $\|\mathcal{N}(t,s)\| \leqslant C<\infty$, $t,s\in I$; $\mathcal{N}$ is continuous; $\mathcal{N}(\cdot,s)$ is uniformly continuous, for each $s\in I$. Moreover let $F: I\times C_0 \to P_{fc}(E)$ satisfy (A) and w be a Kamke function such that $w(t,\cdot)$ is nondecreasing for every $t\in[0,T]$. Then, for each $x_0\in E$, the solution set of integral solutions $S(x_0)$ of (Q) is nonempty.*

Proof. Let $S^1_{F^d(.,v(.))} = \{f\in L^1(I,E): f(s)\in F^d(s,\theta_s v)\}$. If v is an integral of (Q), then $v=\mathcal{C}$ on $[-d,0]$ and $v(t)=\mathcal{N}(t,0)\mathcal{C}(0)+\int_0^t \mathcal{N}(t,s)f(s)ds$ for all $t\in I$ and $f\in S^1_{F^d(.,v(.))}$. So, for each $t\in I$,

$$\|v(t)\| \leqslant \|\mathcal{N}(t,0)\|\|\mathcal{C}(0)\| + \int_0^t \|\mathcal{N}(t,s)\|\|f(s)\|\ ds$$
$$\leqslant C\left(\|\mathcal{C}(0)\| + \|c\| + \int_0^t c(s)\|v(s)\|\ ds\right).$$

Put $C_1 = (C\|\mathcal{C}(0)\| + T\|c\|)e^{C\|c\|}$, from Bellman's inequality, $\|v(t)\| \leqslant C_1$. Put $\varphi(t) = c(t)(1+C_1)$. So we may assume $\|F^d(t,v(t))\| \leqslant \varphi(t)$ a.e. on I since, otherwise we can replace F^d by G^d which is defined by

$$G^d(t,x(t)) = \begin{cases} F^d(t,x(t)) & \text{if } x\in B(0,C_1)\\ F^d(t,\frac{C_1.x(t)}{\|x\|}) & \text{if } x\notin B(0,C_1).\end{cases}$$

For each $(t,x)\in\left[-d,\frac{T}{n}\right]\times E$ with a natural number $n\in\mathbb{N}$ set

$$g_1^n(t,x) = \begin{cases} \mathcal{C}(\mathrm{t}) & \text{if } t\in[-d,0]\\ \mathcal{C}(0)+nt(x-\mathcal{C}(0)) & \text{if t } \in\left[0,\frac{T}{n}\right].\end{cases}$$

Also let $\mathcal{D}_1(t,x) = F^d\left(t,\theta_{\frac{T}{n}}\left(g_1^n(\cdot,x)\right)\right)$. Now there exists a multivalued function $R: B(0,C_1)\to 2^{C\left(\left[0,\frac{T}{n}\right],E\right)}$ which is defined by the following formula:

$$(Rx)(t) = \mathcal{N}(t,0)\mathcal{C}(0) + \int_0^t \mathcal{N}(t,s)F^d\left(t,\theta_{\frac{T}{n}}\left(g_1^n(.,x(s))\right)\right)ds.$$

From Lemma 2.6 there exists an integrable selection $\sigma$of $\mathcal{D}_1(\cdot,v(\cdot))$ for each $v\in C\left(\left[0,\frac{T}{n}\right],E\right)$. So for each $x\in B(0,C_1)$ we have $Rx\neq\emptyset$. Since $\mathcal{N}$ is continuous we can define a function $\psi: L^1\left(\left[0,\frac{T}{n}\right],E\right)\to C\left(\left[0,\frac{T}{n}\right],E\right)$ by the formula

$$\psi(f)(t) = \mathcal{N}(t,0)\mathcal{C}(0) + \int_0^t \mathcal{N}(t,s)f(s)\ ds.$$

If we set $V=\left\{f\in L^1\left(\left[0,\frac{T}{n}\right],E\right): \|f\|\leqslant\varphi(t) \text{ a. e. on } \left[0,\frac{T}{n}\right]\right\}$, then V is uniformly integrable in $L^1\left(\left[0,\frac{T}{n}\right],E\right)$ and, since $\mathcal{N}(\cdot,s)$ is uniformly continuous, $\psi(V)=\{x\in C\left(\left[0,\frac{T}{n}\right],E\right): x(t)=\mathcal{N}(0,t)\mathcal{C}(0)+\int_0^t\mathcal{N}(t,s)f(s)ds,\ f\in V\}$ is nonempty equicontinuous subset of $C\left(\left[0,\frac{T}{n}\right],E\right)$ and so, $\overline{conv}\psi(V)$ is nonempty convex closed bounded and equicontinuous subset of $C\left(\left[0,\frac{T}{n}\right],E\right)$. Now we can assume that there exist $(x_m,y_m)\in$ Graph R such that $x_m\to x$, $y_m\to y$ in C $\left(\left[0,\frac{T}{n}\right],E\right)$, $y_m: I\to C\left(\left[0,\frac{T}{n}\right],E\right)$ is given by

$$y_m(t) = \mathcal{N}(t,0)\mathcal{C}(0) + \int_0^t \mathcal{N}(t,s)z_m(s)\ ds$$

where $z_m\in L^1\left(\left[0,\frac{T}{n}\right],E\right), z_m(s)\in\mathcal{D}_1(s,x_m(s))$ and

$$z_m(t) = \begin{cases} \mathcal{N}(t,0)\mathcal{C}(0) & \text{if } 0\leqslant t\leqslant\frac{T}{nm}\\ \mathcal{N}(t,0)\mathcal{C}(0)+\int_0^{t-\frac{T}{nm}}\mathcal{N}(t,s)z_m\ (s)ds & \text{if } \frac{T}{nm}\leqslant t\leqslant\frac{T}{n}.\end{cases}$$

Thus

$$\lim_{m\to\infty}\|\psi(z_m)-z_m\| = \lim_{m\to\infty}\sup_{t\in\left[0,\frac{T}{n}\right]}\|\psi(z_m)(t)-z_m(t)\|$$
$$\leqslant \lim_{m\to\infty}\left(\sup_{t\in\left[0,\frac{T}{nm}\right]}\|\psi(z_m)(t)-z_m(t)\| + \sup_{t\in\left[\frac{T}{nm},\frac{T}{n}\right]}\|\psi(x_n)(t)-z_m(t)\|\right)$$
$$\leqslant \lim_{m\to\infty}\left(\sup_{t\in\left[0,\frac{T}{nm}\right]}\int_0^t C\varphi(s)\ ds + \sup_{t\in\left[\frac{T}{nm},\frac{T}{n}\right]}\int_{t-\frac{T}{nm}}^t C\varphi(s)\right) = 0.$$

Let $H:=\{z_m: m\in\mathbb{N}\}$ and $G:=\{\psi(z_m): m\in\mathbb{N}\}$. Thus H and G are equicontinuous. Let $\chi(t):=\gamma(H(t))$, $t\in\left[0,\frac{T}{n}\right]$. Then $\chi(0)=0$. We Since $\|z_m-\psi(z_m)\|\to 0$ as $m\to\infty$ so, from Lemma 2.9, $\gamma((Id-\psi)H)=0$ and

$$\gamma(\{z_m:\ m\in\mathbb{N}\}) = \gamma(\{\psi(z_m):\ m\in\mathbb{N}\}).$$

Since for all t, $\tau\in\left[0,\frac{T}{n}\right]$,

$$\gamma\{\psi(z_m)(\tau): m\in\mathbb{N}\} \leqslant \gamma\{\psi(z_m)(t): m\in\mathbb{N}\} + \gamma\{\psi(z_m)(\tau) - \psi(z_m)(t): m\in\mathbb{N}\}$$

and

$$\gamma\{\psi(z_m)(t): m\in\mathbb{N}\} \leqslant \gamma\{\psi(z_m)(\tau): m\in\mathbb{N}\} + \gamma\{\psi(z_m)(t) - \psi(z_m)(\tau): m\in\mathbb{N}\}$$

From Lemma 2.7 $|\chi(\tau)-\chi(t)| \leqslant \gamma(B(0,1))\int_t^\tau \mathrm{C}\ \varphi(s)\ ds$. Therefore χ is absolutely continuous function and thus it is differentiable a.e. on $\left[0,\frac{T}{n}\right]$. Let $(t,\tau)\in\left[0,\frac{T}{n}\right]\times\left[0,\frac{T}{n}\right]$ with $t\leqslant\tau$. Since χ

is continuous and w is Carathéodory we can find a closed subset I_ε of $\left[0,\frac{T}{n}\right]$, $\delta>0$, $\eta>0$ $(\eta<\delta)$ and for $s_1,s_2\in I_\varepsilon$; r_1, $r_2\in\left[0,\frac{2T}{n}\right]$ such that if $|s_1-s_2|<\delta$, $|r_1-r_2|<\delta$, then $|w(s_1,r_1)-w(s_2,r_2)|<\varepsilon$ and if $|s_1-s_2|<\eta$, then $|\chi(s_1)-\chi(s_2)|<\frac{\delta}{2}$. Consider the following partition, to $[t,\tau]$, $t=t_0<t_1<\cdots<t_r=\tau$ such that $t_i-t_{i-1}<\eta$ for $i=1,\ldots,r$. Let $A_i=\{x(s):\ x\in H,\ s\in[t_{i-1},t_i]\cap I_\varepsilon\}$. Let Z be a bounded subset of E and $A=\left\{\theta_{\frac{T}{n}}\left(g_1^n(\cdot,x)\right):x\in Z\right\}$. Thus, for each t$\in\left[0,\frac{T}{n}\right]$, $\gamma(\mathcal{D}_1(\{t\}\times Z))=\gamma(F^d(\{t\}\times A))$. From Condition (A_2),there exists a closed subset J_ε of $\left[0,\frac{T}{n}\right]$ such that $\lambda(I-J_\varepsilon)<\varepsilon$ and that for any compact subset C of J_ε

$$\gamma(\mathcal{D}_1(C\times Z))=\gamma(F^d(C\times A))\leqslant\sup_{s\in C}w(s,\beta(A)).$$

Now if $\overline{A}=\left\{\theta_{\frac{T}{n}}\left(g_1^n(\cdot,x)\right):x\in A_i\right\}$, then

$$\gamma(\mathcal{D}_1(C\times A_i))=\gamma(F^d(C\times\overline{A}))\leqslant\sup_{s\in C}w(s,\beta(\overline{A})).$$

Let $T_i=J_\varepsilon\cap[t_{i-1},t_i]\cap I_\varepsilon$, $P=\sum_{i=1}^m T_i=[t,\tau]\cap J_\varepsilon\cap I_\varepsilon$ and $Q=[t,\tau]-P$. Thus $\int_t^\tau\mathcal{D}_1(s,H(s))\,ds\subset\int_P\mathcal{D}_1(s,H(s))\,ds+\int_Q\mathcal{D}_1(s,H(s))$. In virtue of Lemma 2.6 and from the continuity of χ we have $r_i\in[t_{i-1},t_i]\cap I_\varepsilon$ such that

$$\gamma(A_i)=\sup\{\gamma(A(s)):\ s\in[t_{i-1},t_i]\cap I_\varepsilon\}=\chi(r_i).$$

and by the mean value theorem we obtain

$$\int_P\mathcal{D}_1(s,H(s))\,ds\subset\sum_{i=1}^m\int_{T_i}\mathcal{D}_1(s,H(s))\,ds\subset\sum_{i=1}^n\lambda(T_i)\overline{conv}\mathcal{D}_1(T_i\times A_i)$$

Now from the fact that $w(t,\cdot)$ is nondecreasing for every $t\in[0,T]$ we have

$$\begin{aligned}\gamma\left(\int_P\mathcal{D}_1(s,H(s))\,ds\right)&\leqslant\sum_{i=1}^m\lambda(T_i)\sup_{s_i\in T_i}w(s_i,\gamma(A_i))\\&\leqslant\sum_{i=1}^m\int_{T_i}w(s,\chi(s))\,ds+\varepsilon\lambda(T_i)\\&\leqslant\int_t^\tau w(s,\chi(s))\,ds+\varepsilon(\tau-t).\end{aligned}$$

and

$$\gamma\left(\int_Q\mathcal{D}_1(s,H(s))\,ds\right)\leqslant\gamma(B(0,1))\int_Q\varphi(s)\,ds.$$

Also we have

$$\gamma(\psi(H)(\tau))\leqslant\gamma(\psi(H)(t))+\gamma\left(\int_t^\tau\mathcal{D}_1(s,H(s))\,ds\right).$$

Therefore

$$\chi(\tau)-\chi(t)\leqslant\gamma\left(\int_t^\tau\mathcal{D}_1(s,H(s))\,ds\right)\leqslant\int_t^\tau w(s,\chi(s))\,ds.$$

So $\dot\chi(t)\leqslant w(t,\chi(t))$ a.e. on $\left[0,\frac{T}{n}\right]$, but $\chi(0)=0$ and w is a Kamke function, then $\chi\equiv0$. Consequently the weak closure of $(z_m)_{m\in\mathbb{N}}$ is weakly compact and so [26] we can suppose that there exist $l_1(s)\in F^d\left(s,\theta_{\frac{T}{n}}(\mathcal{D}_1(\cdot,x_1(t)))\right)$ a.e. on $\left[0,\frac{T}{n}\right]$ and the sequence $(z_m)_{m\in\mathbb{N}}$ converges to a continuous function x_1 such that $x_1=\mathcal{C}$ on $[-d,0]$ and for each $t\in\left[0,\frac{T}{n}\right]$

$$x_1(t)=\mathcal{N}(t,0)\mathcal{C}(0)+\int_0^t\mathcal{N}(t,s)l_1(s)ds.$$

Now, by the mathematical induction for some $k\in\{1,2,3,\ldots,n\}$, we can assume that there exist

$$l_{k'}(s)\in F^d\left(t,\theta_{\frac{k'T}{n}}f_{k'}^n(.,x_{k'}(s))\right)\text{ a.e. on }I;l_{k'}\in L^1\left(\left[0,\frac{k'T}{n}\right],E\right)$$

since the function $h_{k'}$ is such that $h_{k'}=\mathcal{C}$ on $[-d,0]$ and for each $t\in\left[0,\frac{k'T}{n}\right]$

$$x_{k'}(t)=\mathcal{N}(t,0)\mathcal{C}(0)+\int_0^t\mathcal{N}(t,s)l_{k'}(s)ds.$$

Also let $g_k^n:\left[-d,\frac{kT}{n}\right]\times E\to E$ be such that

$$g_k^n(t,x)=\begin{cases}x_{k'}(\mathrm{t}) & \text{if } t\in\left[-d,\frac{k'T}{n}\right]\\ x_{k'}\left(\frac{k'T}{n}\right)+n\left(t-\frac{k'T}{n}\right)\left(x-x_{k'}\left(\frac{k'T}{n}\right)\right) & \text{if t }\in\left[\frac{k'T}{n},\frac{kT}{n}\right].\end{cases}$$

Arguing as in above, for the multivalued function $F_k:\left[\frac{k'T}{n},\frac{kT}{n}\right]\times E\to P_{fc}(E)$ which is defined by $F_k(t,x)=F^d\left(t,\theta_{\frac{kT}{n}}\left(g_k^n(.,x)\right)\right)$, we have a continuous function x_k defined on $\left[\frac{k'T}{n},\frac{kT}{n}\right]$ by

$$x_k(t)=\mathcal{N}\left(t,\frac{k'T}{n}\right)x_{k'}\left(\frac{k'T}{n}\right)+\int_{\frac{k'T}{n}}^t\mathcal{N}(t,s)l_k(s)ds$$

where $l_k(s)\in F^d\left(s,\theta_{\frac{kT}{n}}\left(g_k^n(\cdot,x_k(s))\right)\right)$ a.e on $\left[\frac{k'T}{n},\frac{kT}{n}\right]$ and $l_k\in L^1\left(\left[\frac{k'T}{n},\frac{kT}{n}\right],E\right)$. Moreovere for each $t\in\left[\frac{k'T}{n},\frac{kT}{n}\right]$ we have

$$x_{k'}\left(\frac{k'T}{n}\right)=\mathcal{N}\left(\frac{k'T}{n},0\right)\mathcal{C}(0)+\int_0^{\frac{k'T}{n}}\mathcal{N}\left(\frac{k'T}{n},s\right)l_{k'}(s)ds$$

hence

$$\begin{aligned}x_k(t)&=\mathcal{N}\left(t,\frac{k'T}{n}\right)\mathcal{N}\left(\frac{k'T}{n},0\right)\mathcal{C}(0)+\int_0^{\frac{k'T}{n}}\mathcal{N}\left(t,\frac{k'T}{n}\right)\mathcal{N}\left(\frac{k'T}{n},s\right)l_{k'}(s)ds\\&\quad+\int_{\frac{k'T}{n}}^t\mathcal{N}(t,s)l_k(s)ds=\mathcal{N}(t,0)\mathcal{C}(0)+\int_0^{\frac{k'T}{n}}\mathcal{N}(t,s)l_{k'}(s)ds+\int_{\frac{k'T}{n}}^t\mathcal{N}(t,s)l_k(s)ds\\&=\mathcal{N}(t,0)\mathcal{C}(0)+\int_0^t\mathcal{N}(t,s)g_k(s)ds,\end{aligned}$$

where

$$g_k(t)=\begin{cases}l_{k'}(\mathrm{t}) & \text{if } t\in\left[0,\frac{k'T}{n}\right]\\ l_k(t) & \text{if t }\in\left[\frac{k'T}{n},\frac{kT}{n}\right]\end{cases}.$$

Consequently, for all $n\in\mathbb{N}$, we have a continuous function v_n such that $v_n=\mathcal{C}$ on $[-d,0]$ and for each $t\in I$ is defined by

$$v_n(t)=\mathcal{N}(t,0)\mathcal{C}(0)+\int_0^t\mathcal{N}(t,s)q_n(s)ds,$$

with

$$q_n(t)\in F^d\left(t,\theta_{\frac{kT}{n}}g_k^n(.,v_n(t))\right)\text{ a.e. on }I.$$

where $t\in\left[\frac{k'T}{n},\frac{kT}{n}\right]\subset I$ and $k\in\{1,2,3,\ldots,n\}$. Let $t_1,t_2\in I$ with $t_1<t_2$. Then

$$\|v_n(t_1) - v_n(t_2)\| \leqslant \|\mathcal{N}(t_1,0) - \mathcal{N}(t_2,0)\|\|\mathcal{C}(0)\| + \int_0^{t_1} \|\mathcal{N}(t_1,s) - \mathcal{N}(t_2,s)\|\|q_n(s)\|ds + \int_{t_1}^{t_2} \|\mathcal{N}(t_2,s)\|\|q_n(s)\|ds \leqslant \|\mathcal{N}(t_1,0) - \mathcal{N}(t_2,0)\|\|\mathcal{C}(0)\| + \int_0^{t_1} \|\mathcal{N}(t_1,s) - \mathcal{N}(t_2,s)\|\|\varphi(s)\|ds + C\int_{t_1}^{t_2} \varphi(s)ds$$

while $v_n = \mathcal{C}$ on $[-d,0]$, this shows that L is equicontinuous in $C_E[-d,T]$. Put $\gamma(L(t)) = \gamma(\{v_n(t) : n \in \mathbb{N}\})$, by the same as above we get $\gamma\ (L(t)) = 0$ for all $t \in I$. Thus by Ascoli's theorem the sequence $\{v_n\ : n \in \mathbb{N}\}$ converges uniformly to a function $v \in C_E([-d,T])$ such that $y = \mathcal{C}$ on $[-d,0]$. As in the proof of Theorem 3.1 we obtain $\gamma(\{q_n(t) :\ n \in \mathbb{N}\}) = 0$ and so $\{q_n(t) :\ n \in \mathbb{N}\}$ is relatively compact. Create a new multivalued function $\mathcal{Q}(t) = \overline{conv}\{q_n(t) : n \in \mathbb{N}\}$. Thus $\mathcal{Q}(t)$ is nonempty convex and compact, the set $\delta^1_{\mathcal{Q}} = \{l \in L^1(I,E) : l(t) \in \mathcal{Q}(t)\}$ is nonempty convex and weakly compact. By Eberlein–Šmulian Theorem there exists a subsequence (q_{n_k}) of (q_n) such that $q_{n_k} \to l$ weakly, $l \in \delta^1_{\mathcal{Q}}$. Thus v_n tends weakly to $\mathcal{N}(t,0)\mathcal{C}(0) + \int_0^t \mathcal{N}(t,s)l(s)ds$. Moreover since, for each $n \in \mathbb{N}$, $v_n \in C_E([-d,T])$, v_n converges uniformly to v on each compact subset of $[-d,T]$ and v is uniformly continuous on $[-d,0]$; also for each $t \in I$, there exists $n > \frac{T}{d}$ with $t \in \left[\frac{k'T}{n}, \frac{kT}{n}\right]$ for $k \in \{1,2,3,\ldots,n\}$ so,

$$\left\|\theta_{\frac{kT}{n}} g^n_k(.,v_n(t)) - \theta_t v\right\| \leqslant \sup_{s\in\left[-d,-\frac{T}{n}\right]} \left[\left\|v_n\left(\frac{kT}{n}+s\right) - v\left(\frac{kT}{n}+s\right)\right\| + \left\|v\left(\frac{kT}{n}+s\right) - v(t+s)\right\|\right] + \sup_{s\in\left[-\frac{T}{n},0\right]} \left[\left(T\left\|\left(v_n(t) - v_n\left(\frac{k'T}{n}\right)\right)\right\| + \left\|v_n\left(\frac{k'T}{n}\right) - v\left(\frac{kT}{n}+s\right)\right\| + \left\|v\left(\frac{kT}{n}+s\right) - v(t+s)\right\|\right)\right] \to 0 \text{ as } n\to\infty$$

Therefore from Lemma 2.6 the solution set $S(\mathcal{C})$ of integral solutions of (Q) is nonempty. □

Theorem 4.2. *Under the assumptions of* Theorem 4.1 *the solution set $S(\mathcal{C})$ of the problem (Q) is compact.*

Proof. If $\{v_n : n \in \mathbb{N}\}$ is a sequence of $S(\mathcal{C})$, then arguing as in the proof of Theorem 4.1 we can show that, for each $t \in I$, $\gamma(\{v_n(t) : n \in \mathbb{N}\}) = 0$. Thus this sequence has a convergent subsequence and so $S(\mathcal{C})$ is compact. □

Now we consider the multivalued functions $S{:}C_0 \to 2^{C([-d,T],E)}$ such that, for each $\mathcal{C} \in C_0$, we have $S(\mathcal{C})$ is the solution set of problem (Q), $S_t{:}C_0 \to 2^E$ with $S_t(\mathcal{C}) = \{v(t) : v \in S(\mathcal{C})\}$ and $S_{\mathcal{C}} : I \to 2^E$ defined by $S_{\mathcal{C}}(t) = \{v(t) : v \in S(\mathcal{C})\}$.

Theorem 4.3. *The multivalued function S is upper semicontinuous moreover, both S_t and $S_{\mathcal{C}}$ is upper semicontinuous and has compact values. Further, the set $\cup_{t\in I} S_{\mathcal{C}}(t)$ is compact in E.*

Proof. To show that S is upper semicontinuous, for each closed subset Z of $C_E([-d,T])$, we claim that the set $\mathcal{A} = \{\mathcal{C} \in C_E([-d,0]) : S_{\mathcal{C}} \cap Z \neq \emptyset\}$ is closed in $C_E([-d,0])$. Let $\{\mathcal{C}_n : n \in \mathbb{N}\} \subset \mathcal{A}$ such that $\mathcal{C}_n \to \mathcal{C}$. Then $S_{\mathcal{C}_n} \cap Z \neq \emptyset$ and hence there exists $v_n \in S_{\mathcal{C}_n} \cap Z$, where

$$v_n(t) = \mathcal{N}(t,0)\mathcal{C}_n(0) + \int_0^t \mathcal{N}(t,s)q_n(s)ds,$$

with $q_n(s) \in F^d(s,\theta_s v_n)$ a.e. on I and $q_n(\cdot) \in L^1(I,E)$. Now, for each $t \in I$, we have

$$\gamma(\{v_n(t) : n \in \mathbb{N}\}) \leqslant C\gamma(\{\mathcal{C}_n(0) : n \in \mathbb{N}\}) + C\gamma\left(\left\{\int_0^t q_n(s)ds : n \in \mathbb{N}\right\}\right).$$

But $\gamma(\{\mathcal{C}_n(0) : n \in \mathbb{N}\}) = 0$, where $\mathcal{C}_n \to \mathcal{C}$. Thus

$$\gamma(\{v_n(t) : n \in \mathbb{N}\}) \leqslant C\gamma\left(\left\{\int_0^t q_n(s)ds : n \in \mathbb{N}\right\}\right).$$

As in Theorem 4.1 we have $\gamma(\{v_n(t) : n \in \mathbb{N}\}) = 0$, but the sequence $\{v_n(t) : n \in \mathbb{N}\}$ is equicontinuous, so from Arzela–Ascoli theorem we can find a subsequence (v_{n_k}) converges to v_0 in $C_E([-d,b])$. Let $v_{n_k}(t) = \mathcal{N}(t,0)\mathcal{C}_{n_k}(0) + \int_0^t \mathcal{N}(t,s)q_{n_k}(s)ds$, where $q_{n_k}(s) \in F^d(s,\theta_s v_{n_k})$ a.e. on I and $q_{n_k}(\cdot) \in L^1(I,E)$. Then we can write $q_{n_k} = \mathcal{C}$ on $[-d,0]$ and

$$q_{n_k}(t) = \begin{cases} \mathcal{N}(t,0)\mathcal{C}(0) & \text{if } 0 \leqslant t \leqslant \frac{T}{n_k} \\ \mathcal{N}(t,0)\mathcal{C}(0) + \int_0^{t-\frac{T}{n_k}} \mathcal{N}(t,s)q_{n_k}(s)ds & \text{if } \frac{T}{n_k} \leqslant t \leqslant T. \end{cases}$$

As in the proof of Theorem 4.1 we obtain $\gamma(\{q_{n_k}(t) : k \in \mathbb{N}\}) = 0$ for $t \in I$, so $q_{n_k} \to q_0 \in L^1(I,E)$ and from Lemma 2.6 $q_0(t) \in F^d(t,\theta_t v_0)$. Thus

$$v_0(t) = \mathcal{N}(t,0)\mathcal{C}(0) + \int_0^t \mathcal{N}(t,s)q_0(s)ds$$

and consequently $\mathcal{A} = \{\mathcal{C} \in C_E([-d,0]) : S_{\mathcal{C}} \cap Z \neq \emptyset\}$ is closed in $C_E([-d,0])$ thus S is upper semicontinuous. Further, by the same arguments we can show that $\mathcal{P} = \{\mathcal{C} \in C_E([-d,0]) : S_t(\mathcal{C}) \cap Z \neq \emptyset\}$ is closed so, $S_t(\mathcal{C})$ is upper semicontinuous. Since $S(\mathcal{C})$ is compact, then both $S_{\mathcal{C}}$ and S_t has compact values. Lastly the set $\mathcal{Q} = \{\text{t} \in I : S_{\mathcal{C}}(t) \cap Z\}$ is closed, then from Berge's Theorem [16] $\cup_{t\in I} S_{\mathcal{C}}(t)$ is compact in E. □

Let Z be a compact subset of $C_E([-d,0])$ and $\gamma : E \to \mathbb{R}$ be lower semicontinuous. Now we consider the following control problem

$$(Q^c)\begin{cases} \dot{x}(t) \in A(t)x(t) + F^d(t,\theta_t x) \\ x = \mathcal{C} \in Z \end{cases}$$
$$\textit{minimise } \gamma(x(T)).$$

The problem (Q^c) has an optimal solution if there exist $\mathcal{C} \in Z$ and $v \in S(\mathcal{C})$ such that $\gamma(v(T)) = \inf\{\gamma(x(T)) : x \in S(\mathcal{C})\}$.

Theorem 4.4. *Under the assumptions of* Theorem 4.1, *the problem (Q^c) has an optimal solution.*

Proof. If $\mathcal{C}_0 \in Z \subseteq C_E([-d,0])$, then there exists a continuous function $v \in S(\mathcal{C}_0)$ and so, $v(T) \in S_T(\mathcal{C}_0)$. But S_T is upper semicontinuous and has compact values, from Berge's Theorem [16] we have $S_T(Z)$ is compact and so γ has its minimum T_0 on $S_T(Z)$. Now there exists $\mathcal{C} \in Z$ such that $v_0 \in S_T(\mathcal{C})$, where $\gamma(v_0) = T_0$ and $v_0 \in S_T(Z)$, thus $v_0 \in S_{\mathcal{C}}(T)$ which means that $v_0 = v(T)$ for some $v \in S(\mathcal{C})$. Therefore $\gamma(v(T)) = \inf\{\gamma(x(T)) : x \in S(\mathcal{C})\}$. □

5. Conclusion

The problem (P) was investigated by many authors without delay (θ is the zero mapping), for instance, in [1] the author deals with the existence of weak and strong solutions while in [21,22] the authors deal with the existence of strong solutions. Cichón [23] deals with some existence theorems using different types of integrals and its properties, Szep [24] considered a Peano type theorem of ordinary differential equations in reflexive Banach spaces and the result of Cramer–Lakshmikantham–Mitchell [25] is stronger than that of Szep [24]. We concern with the problem (Q) on account of its great practical interest since this problem investigated, without delay, by many authors see [3,10,6,9,8,26] and the references therein.

When $A(t) = 0$, $\mathcal{N}(t,s) = id$ and a mild solution is a Carathéodory one, we have a generalization to the existence theorems of Deimling [3] Ibrahim and Gomaa [4], Kisielewicz [5], Papageorgiou [6,7]. As $A(t) \neq 0$ our results extend that of [10,6,8,9]. Moreover much work has been done to study the topological properties of the solution set for the differential inclusions (see, for instance, [27–36]). Recent results with a finite delay $d > 0$ in Banach spaces are obtained by Syam [37], Castaing and Ibrahim [6] and Gomaa [38,39].

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