Journal of the Egyptian Mathematical Society (2012) **20**, 98–107

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Characterization of *TL*-uniform spaces by coverings

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Received 30 April 2009; revised 28 December 2010
Available online 29 November 2012

KEYWORDS

Triangular norm;
Covering *TL*-uniform spaces;
Fuzzy *TL*-uniform spaces;
I-topological spaces

Abstract In this paper we introduce the concepts of fuzzy *TL*-uniform spaces by means of coverings, where *T* stands for any continuous triangular norm. We show that the structure of covering *TL*-uniform spaces are isomorphic to fuzzy *TL*-uniform spaces as defined by Hashem and Morsi (2006) [5]. In particular, we study the continuity of functions between covering *TL*-uniform spaces, the *I*-topological space associated with a covering *TL*-uniform space. Also, we define the notions of level covering uniformities for a covering *TL*-uniformity. Moreover, we deduce a number of functors between categories of covering *TL*-uniform spaces, fuzzy *TL*-uniform spaces, covering uniform spaces and uniform spaces.

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1. Introduction

In [5], Hashem and Morsi deduced the fuzzy *TL*-uniform spaces, for each continuous triangular norm *T*. In this manuscript, we introduce a new structure of covering *TL*-uniform spaces that conforms well with fuzzy *TL*-uniform spaces and with the *I*-topological spaces [7]. We deduce the notion of *C*-uniformly maps and here we show that the class of all covering *TL*-uniform spaces together with *C*-uniformly maps as arrows forms a concrete category. We study the level covering uniformities for a covering *TL*-uniformity and conversely, we show that every covering uniformity generates a covering *TL*-uniformity. Also, we will make clear there are correlation and compatibility between the following structures: covering *TL*-uniform spaces, fuzzy *TL*-uniform spaces, covering uniform spaces and uniform spaces.

We proceed as follows:

In Section 2, we present some basic definitions and ideas on fuzzy sets, *I*-topological spaces, residuated implication and fuzzy *TL*-uniform spaces.

In Section 3, we deduce some important definitions and results for the classes of fuzzy sets which will be used in the sequel.

In Section 4, we introduce the concepts of covering *TL*-uniform spaces and the *I*-topology associated with a covering *TL*-uniformity, together with illustrative examples. We define and study the *C*-uniformly continuous functions between covering *TL*-uniform spaces. Also, we show that there is an isomorphism between category of covering *TL*-uniform spaces and category of fuzzy *TL*-uniform spaces.

In Section 5, we introduce the notion of the α-levels for a covering *TL*-uniformity and we study the relationships between them. Also, we define the functors between category of covering *TL*-uniform spaces and category of covering uniform spaces.

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2012.08.003

2. Prerequisites

In this section, we will recall some definitions related to fuzzy sets, residuated implication, fuzzy TL-uniform spaces and I-topological spaces.

A triangular norm (cf. [12]) is a binary operation on the unit interval $I = [0, 1]$ that is associative, symmetric, monotone in each argument and has the neutral element 1.

For a continuous triangular norm T, the following binary operation on I,

$$\hat{J}(\alpha, \gamma) = \sup\{\theta \in I : \alpha T\theta \leqslant \gamma\}, \quad \alpha, \gamma \in I,$$

is called the residual implication of T[9].

Proposition 2.1. [11].*For the residual implication $\hat{\mathbf{J}}$, we have*

(i) $\hat{\mathbf{J}}(1, \gamma) = \gamma$, *for every* $\gamma \in I$;
(ii) $\hat{\mathbf{J}}$ *is antimonotone in the left argument and monotone in the right argument.*

A fuzzy set λ in a universe set X, introduced by Zadeh in [14], is a function $\lambda{:}X \to I$. The collection of all fuzzy sets of X is denoted by I^X. The height of a fuzzy set λ is the following real number: $hgt\lambda = \sup\{\lambda(x){:}x \in X\}$.

If H is a subset of X, then we shall denote to its characteristic function by the symbol $\mathbf{1}_H$, said to be a crisp fuzzy subset of X. We also denote the constant fuzzy set of X with value $\alpha \in I$ by $\underline{\alpha}$.

Given a fuzzy set $\lambda \in I^X$ and a real number $\alpha \in I_1 = [0, 1[$, the strong α-cut of λ is the following subset of X: $\lambda^\alpha = \{x \in X{:}\lambda(x) > \alpha\}$.

For a given two fuzzy sets μ, $\lambda \in I^X$ we denote by $\mu T\lambda$ the following fuzzy set of X:

$(\mu T\lambda)(x) = \mu(x)T\lambda(x)$, $x \in X$. The degree of containment of μ in λ according to $\hat{\mathbf{J}}$ defined as the real number in I [3], given by:

$$\hat{\mathbf{J}}\langle\mu, \lambda\rangle = \inf_{x\in X}(\mu(x), \quad \lambda(x)) \tag{1}$$

We follow Lowen's definition of a fuzzy interior operator on a set X [7]. This is an operator ${}^o{:}I^X \to I^X$ that satisfies $\mu^o \leqslant \mu$, $(\mu \bigwedge \lambda)^o = \mu^o \bigwedge \lambda^o$ for all μ, $\lambda \in I^X$ and $\underline{\alpha}^o = \underline{\alpha}$ for all $\alpha \in I$. We may define an I-topology in the usual way, namely assuming a fuzzy set μ to be open if and only if $\mu^o = \mu$. We denote this I-topology by τ. The pair (X, τ) is called an I-topological space (cf.[2]).

A function $f{:}(X, {}^o) = (X, \tau) \to (Y, {}^{o\prime}) = (Y, \tau')$, between two I-topological spaces, is said to be continuous, if $f^{\leftarrow}(\mu) \in \tau$, for all $\mu \in \tau'$, where $(f^{\leftarrow}(\mu))(x) = \mu(f(x))$, for every $x \in X$.

I-filters and I-filterbases were introduced by Lowen in [8]. An I-filter in a universe X is a nonempty collection $\Im \subset I^X$ which satisfies: $\underline{0} \notin \Im$, $\Im$ is closed under finite meets and contains all the fuzzy supersets of its individual members. An I-filterbase in X is a nonempty collection $ß \subset I^X$ which satisfies: $\underline{0} \notin ß$ and the meet of two members of ß contain a member of ß.

Definition 2.1. [10]. The T-saturation operator is the operator $\sim T$ which sends an I-filterbase ß in X to the following subset of I^X

$$ß^{\sim T} = \left\{\mu \in I^X : \bigvee_{\gamma\in I}(\underline{\gamma}T\mu_\gamma) \leqslant \mu, \text{ where } \mu_\gamma \in ß \quad \forall\, \gamma \in I\right\},$$

said to be the T-saturation of ß. An I-filterbase ß is called T-saturated when $ß^{\sim T} = ß$.

In [4], Höhle defines for every $\psi, \varphi \in I^{X\times X}$ and $\lambda \in I^X$:

The T-section of ψ over λ by $(\psi\langle\lambda\rangle_T)(x) = \sup_{z\in X}[\lambda(z)T\psi(z, x)]$, $x \in X$.

The T-composition of ψ, φ by $(\psi o_T\varphi)(x, y) = \sup_{z\in X}[\varphi(x, z) T\psi(z, y)]$, $x, y \in X$.

The symmetric of ψ by ${}_s\psi(x, y) = \psi(y, x)$, x, $y \in$ X.

Notice, it is easy to see that, for every $\psi \in I^{X\times X}$ and x, $y \in$ X,

$$(\psi\langle\mathbf{1}_y\rangle_T)(x) = \psi(y, x). \tag{2}$$

The fuzzy TL-uniform spaces (TL-uniform spaces, for short) were introduced by Hashem and Morsi, for more definitions and properties, we can refer to [5].

Definition 2.2. [5].

(i) A TL-uniform base on a set X is a subset $\upsilon \subset I^{X\times X}$ which fulfills the following properties:

(TLUB1) υ is an I-filterbase;
(TLUB2) For all $\psi \in \upsilon$ and $x \in X$, we have $\psi(x, x) = 1$;
(TLUB3) For all $\psi \in \upsilon$ and $\gamma \in I_1$, there is $\psi_\gamma \in \upsilon$ such that $\underline{\gamma}T\psi_\gamma \leqslant {}_s\psi$;
(TLUB4) For all $\psi \in \upsilon$ and $\gamma \in I_1$, there is $\psi_\gamma \in \upsilon$ such that $\underline{\gamma}\ T(\psi_\gamma\ o_T\psi_\gamma) \leqslant \psi$.

(ii) A TL-uniformity on X is a T-saturated TL-uniform base on X.
(iii) If Ω is a TL-uniformity on X, then we shall say that υ is a basis for Ω if υ is an I-filterbase and $\upsilon^{\sim T} = \Omega$.It follows that for a TL-uniformity Ω on a set X and all $\psi \in \Omega$, we find that ${}_s\psi \in \Omega$. The pair (X, Ω) consisting of a set X and a TL-uniformity Ω on X is called TL-uniform space.

Definition 2.3. [5]. Let (X, Ω) and (Y, ϖ) be TL-uniform spaces and f: $X \to Y$ be a function. We say that f is uniformly continuous if for every $\varphi \in \varpi$ there is $\psi \in \Omega$ such that $\psi \leqslant (f\times f)^{\leftarrow}(\varphi)$. Equivalent for every $\varphi \in \varpi$, $(f\times f)^{\leftarrow}(\varphi) \in \Omega$.

We denote by TL-US the category of TL-uniform spaces and as morphisms are uniformly continuous functions between these spaces.

Proposition 2.2. [5].*If (X, Ω) is a TL-uniform space, then the fuzzy interior operator ^{o} of an I-topological space (X, $\tau(\Omega)$) is given by:*

$$\lambda^o(x) = \sup_{\varphi\in\Omega}\hat{\mathbf{J}}\langle\varphi\langle\mathbf{1}_x\rangle_T, \lambda\rangle, \quad \lambda \in I^X, x \in X.$$

Now, we give the following lemma, which is needed in the sequel.

Lemma 2.1. *If υ is a TL-uniform base on a set X and $\varphi \in \upsilon$, then for every $\gamma \in I_1$, there is $\psi \in \upsilon$ such that $\underline{\gamma}T\psi \leqslant \underline{\gamma}\ T(\psi \bigvee {}_s\psi) \leqslant \varphi$.*

Proof. Let $\varphi \in \upsilon$ and $\gamma \in I_1$. Then there is $\varphi_\gamma \in \upsilon$ such that

$$\underline{\gamma} T\varphi_\gamma \leqslant s\varphi. \tag{3}$$

Since υ is an *I*-filterbase, then there is $\psi \in \upsilon$ such that $\psi \leqslant \varphi \bigwedge \varphi_\gamma$, that is $\psi \leqslant \varphi$ and ${}_s\psi \leqslant {}_s\varphi_\gamma$.

Consequently, by (3), we have $\underline{\gamma} T_s\psi \leqslant \underline{\gamma} T_s\varphi_\gamma = {}_s(\underline{\gamma} T\varphi_\gamma) \leqslant \varphi$.

This clearly, implies that

$$\underline{\gamma} T\psi \leqslant \underline{\gamma} T(\psi \bigvee {}_s\psi) = (\underline{\gamma} T\psi) \bigvee (\underline{\gamma} T_s\psi) \leqslant (\underline{\gamma} T\varphi) \bigvee \varphi = \varphi.$$

This winds up the proof. □

3. Definitions and general properties

In this section, we give some additional properties for the classes of fuzzy sets, which needed in the following sections. In order to avoid complicated notations, we will write $F(X)$ for the family of all subsets of I^X.

Definition 3.1. For the subsets $\Re, \Im \subset I^X$, we say that:

(i) $\Re$ is a fuzzy covering of X, if $(\bigvee_{\mu \in \Re} \mu) = \underline{1}$.The set of all fuzzy covering of X will be denoted $F_c(X)$.
(ii) $\Re$ is coarser than $\Im$ and write $\Re \ll \Im$, if for every $\lambda \in \Re$ there is $v \in \Im$ such that $\lambda \leqslant v$.

Definition 3.2. If $\Sigma \subset F(X)$, we define

$[\Sigma] = \{\Re \subset I^X$: there is $\Im \in \Sigma$ with $\Im \ll \Re\}$;

$\Sigma^\wedge = \{\bigvee_{\gamma \in I} (\underline{\gamma} T\Re_\gamma) : \Re_\gamma \in \Sigma\}$;

where, $\underline{\gamma} T\Re = \{(\underline{\gamma} T\lambda) \in I^X : \lambda \in \Re\}$.

Also, the T-saturation $\Sigma^{\sim T}$ of Σ by: $\Sigma^{\sim T} = [\Sigma^\wedge]$.

A collection Σ is called T-saturated when $\Sigma^{\sim T} = \Sigma$.

Lemma 3.1. *For every Σ, $£ \subset F(X)$, we have*

(i) *If $\Sigma \subseteq £$, then $[\Sigma] \subseteq [£]$, $\Sigma^\wedge \subseteq £^\wedge$ and $\Sigma^{\sim T} \subseteq £^{\sim T}$;*
(ii) *$\Sigma \subseteq [\Sigma] \subseteq \Sigma^{\sim T}$;*
(iii) *$[[\Sigma]] = [\Sigma]$;*
(iv) *$[\Sigma]^\wedge \subseteq [\Sigma^\wedge]$;*
(v) *$(\Sigma^\wedge)^\wedge \subseteq [\Sigma^\wedge]$;*
(vi) *$(\Sigma^{\sim T})^{\sim T} = \Sigma^{\sim T}$;*
(vii) *If $\Sigma \subseteq [£]$, then $\Sigma^{\sim T} \subseteq £^{\sim T}$.*

Proof. The assertions (i), (ii) and (iii) are immediately from definitions.

(iv) Let $\Re \in [\Sigma]^\wedge$. Then there are two families

$\{\Im_\gamma : \gamma \in I_1\} \subset [\Sigma]$ and $\{\Im'_\gamma : \gamma \in I_1\} \subset \Sigma$, for which $\Re = \bigvee_{\gamma \in I_1} (\underline{\gamma} T\Im_\gamma)$ and $\Im'_\gamma \ll \Im_\gamma$.

Hence, the element $\Im = \bigvee_{\gamma \in I_1} (\underline{\gamma} T\Im'_\gamma)$ in $\Sigma^\wedge$ satisfies $\Im \ll \bigvee_{\gamma \in I_1} (\underline{\gamma} T\Im_\gamma) = \Re$, which implies that $\Re \in [\Sigma^\wedge]$, that is $[\Sigma]^\wedge \subseteq [\Sigma^\wedge]$.

(v) If $\Re \in (\Sigma^\wedge)^\wedge$, then $\Re = \bigvee_{\gamma \in I_1} (\underline{\gamma} T\Re_\gamma)$, where $\Re_\gamma \in \Sigma^\wedge$, for all $\gamma \in I_1$.

In turn, there is a family $\{\Re_\gamma^\theta : \theta \in I_1\} \subset \Sigma$ such that $\Re_\gamma = \bigvee_{\theta \in I_1} (\underline{\theta} T\Re_\gamma^\theta)$. So, $\Re = \bigvee_{\gamma \in I_1} (\underline{\gamma} T\Re_\gamma) = \bigvee_{\gamma \in I_1} [\bigvee_{\theta \in I_1} (\underline{\gamma} T\underline{\theta} T\Re_\gamma^\theta)] = \bigvee_{\alpha = \gamma T\theta} (\underline{\alpha} T\Re_\gamma^\theta) = \bigvee_{\alpha \in I_1} (\underline{\alpha} T\Im_\alpha)$, where $\Im_\alpha = \bigvee_{\alpha = \gamma T\theta \in I_1} \Re_\gamma^\theta$ and clearly $\Im_\alpha \in [\Sigma]$.

Consequently, $\Re \in [\Sigma]^\wedge$, thus by (iv), we get $\Re \in [\Sigma^\wedge]$, which proves our assertion.

(vi)
$$\begin{aligned}
(\Sigma^{\sim T})^{\sim T} &= [[\Sigma^\wedge]^\wedge] \\
&\subseteq [[(\Sigma^\wedge)^\wedge]], \quad \text{by (iv)} \\
&= [(\Sigma^\wedge)^\wedge], \quad \text{by (iii)} \\
&\subseteq [[\Sigma^\wedge]], \quad \text{by (v)} \\
&= [\Sigma^\wedge], \quad \text{by (iii)} \\
&= \Sigma^{\sim T} \\
&\subseteq (\Sigma^{\sim T})^{\sim T}, \quad \text{by (ii)}.
\end{aligned}$$

Thus the equality holds.

(vii) Suppose that $\Sigma \subseteq [£]$, hence by (iv) then (iii), we have

$$\Sigma^{\sim T} = [\Sigma^\wedge] \subseteq [[£]^\wedge] \subseteq [[£^\wedge]] = [£^\wedge] = £^{\sim T}.$$

Which rendering the proof. □

Proposition 3.1

(i) *If $\Re \in F_c(X)$, $\Im \in F(X)$ and $\Re \ll \Im$, then $\Im \in F_c(X)$.*
(ii) *If $\Re_\gamma \in F_c(X)$, for every $\gamma \in I_1$, then so $\bigvee_{\gamma \in I} (\underline{\gamma} T\Re_\gamma) \in F_c(X)$.*

The proof follows immediately from definitions.

Definition 3.3

(i) Let $\lambda \in I^X, \Re \subset I^X$, we define the star $\lambda^*(\Re)$ of λ with respect to $\Re$ as $\lambda^*(\Re) \in I^X$, by:

$$\lambda^*(\Re) = \sup_{v \in \Re} [hgt(\lambda T v) T v], \text{ that is } (\lambda^*(\Re))(x) = \sup_{z \in X, v \in \Re} [\lambda(z) T v(z) T v(x)], \forall\, x \in X.$$

(ii) The star $\Re^*$ of $\Re \subset I^X$ is defined by: $\Re^* = \{\lambda^*(\Re) : \lambda \in \Re\}$.

Lemma 3.2. *For every $\Re, \Im \subset I^X$, we have the following:*

(i) *If $\Re \ll \Im$, then $\lambda^*(\Re) \leqslant \lambda^*(\Im)$, for all $\lambda \in I^X$;*
(ii) *If $\Re \ll \Im$, then $\Re^* \ll \Im^*$;*
(iii) *If λ, $v \in I^X$ with $\lambda \leqslant v$, then $\lambda^*(\Re) \leqslant v^*(\Re)$;*
(iv) *$(\underline{\gamma} T\Re)^* = (\underline{\gamma} T\underline{\gamma} T\underline{\gamma}) T\Re^*, \forall \gamma \in I$.*

Proof.

(i) Let $\lambda \in I^X, \Re \ll \Im$ and x $\in X$. Then

$$\begin{aligned}
(\lambda^*(\Re))(x) &= \sup_{z \in X, v \in \Re} [\lambda(z) T v(z) T v(x)] \\
&\leqslant \sup_{z \in X, \mu \in \Im} [\lambda(z) T \mu(z) T \mu(x)], \quad \text{by hypothesis} \\
&= (\lambda^*(\Im))(x).
\end{aligned}$$

(ii) Follows from (i).
(iii) Obviously holds.
(iv) Can be proved as follows

$$\begin{aligned}(\underline{\gamma}T\mathfrak{R})^* &= \{v^*(\underline{\gamma}T\mathfrak{R}) : v \in \underline{\gamma}T\mathfrak{R}\}\\ &= \{(\underline{\gamma}T\lambda)^*(\underline{\gamma}T\mathfrak{R}) : \lambda \in \mathfrak{R}\}\\ &= \{\sup_{\vartheta\in\underline{\gamma}T\mathfrak{R}}[hgt\ (\underline{\gamma}T\lambda T\vartheta)T\vartheta]\}\\ &= \{\sup_{\mu\in\mathfrak{R}}[hgt\ ((\underline{\gamma}T\lambda)T(\underline{\gamma}T\mu))T(\underline{\gamma}T\mu)]\}\\ &= \{\sup_{\mu\in\mathfrak{R}}[(\underline{\gamma}T\underline{\gamma}T\underline{\gamma})T\, hgt\ (\lambda T\mu)T\mu]\}, \quad \text{by continuity of } T\\ &= (\underline{\gamma}T\underline{\gamma}T\underline{\gamma})\{\sup_{\mu\in\mathfrak{R}}[hgt\ (\lambda T\mu)T\mu]\}\\ &= (\underline{\gamma}T\underline{\gamma}T\underline{\gamma})T\mathfrak{R}^*.\end{aligned}$$

Which completes the proof. □

Lemma 3.3. *For every $\mathfrak{R} \subset I^X$, the following are equivalent statements:*

(i) $\lambda \leqslant \lambda^*(\mathfrak{R})$, *for all* $\lambda \in I^X$;
(ii) $hgt\ \lambda = hgt\ \lambda^*(\mathfrak{R})$, *for all* $\lambda \in I^X$;
(iii) $\mathfrak{R} \in F_c(X)$;
(iv) $\mathfrak{R}^* \in F_c(X)$.

Proof. (i) ⇒ (ii): Let $\mathfrak{R} \subset I^X$ and $\lambda \in I^X$, with $\lambda \leqslant \lambda^*(\mathfrak{R})$. Then

$$\begin{aligned}hgt\ \lambda \leqslant hgt\ \lambda^*(\mathfrak{R}) &= \sup_{x\in X}(\lambda^*(\mathfrak{R}))(x)\\ &= \sup_{x\in X}\{\sup_{z\in X, v\in\mathfrak{R}}[\lambda(z)Tv(z)Tv(x)]\}\\ &\leqslant \sup_{z\in X, v\in\mathfrak{R}}[\lambda(z)Tv(z)], \quad \text{clear}\\ &\leqslant \sup_{z\in X}\lambda(z)\\ &= hgt\ \lambda.\end{aligned}$$

Which holds the equality.

(ii) ⇒ (iii): If $\mathfrak{R} \notin F_c(X)$, then there are $x_o \in X$ and $\gamma \in I_1$ such that $(\bigvee_{v\in\mathfrak{R}}v)(x_o) < \gamma < 1$, and therefore

$$\begin{aligned}hgt[(\mathbf{1}_{x_o})^*(\mathfrak{R})] &= \sup_{x\in X}[(\mathbf{1}_{x_o})^*(\mathfrak{R})](x)\\ &= \sup_{x\in X}\{\sup_{z\in X, v\in\mathfrak{R}}[(\mathbf{1}_{x_o})(z)Tv(z)Tv(x)]\}\\ &= \sup_{x\in X}\{\sup_{v\in\mathfrak{R}}[(\mathbf{1}_{x_o})(x_o)Tv(x_o)Tv(x)]\}\\ &= \sup_{x\in X}\{\sup_{v\in\mathfrak{R}}[v(x_o)Tv(x)]\}\\ &\leqslant \sup_{v\in\mathfrak{R}}[v(x_o)]\\ &< \gamma < 1\\ &= hgt\ (\mathbf{1}_{x_o}).\end{aligned}$$

Which contradiction with (ii).

(iii) ⇒ (iv): Obviously hold from (i).

(iv) ⇒ (i): Let $\mathfrak{R}^* \in F_c(X)$, $\lambda \in I^X$ and $x \in X$. Then there is $v \in \mathfrak{R}$ such that $(v^*(\mathfrak{R}))(x) > \gamma,\ \forall\, \gamma \in I_1$. Therefore,

$$\sup_{\mu\in\mathfrak{R}}(\mu(x)) \geqslant \sup_{z\in X, \mu\in\mathfrak{R}}[v(z)T\mu(z)T\mu(x)] = (v^*(\mathfrak{R}))(x) > \gamma.$$

By choosing $\mu' \in \mathfrak{R}$ for which $\mu'(x) > \gamma$, we obtain

$$\begin{aligned}(\lambda^*(\mathfrak{R}))(x) &= \sup_{z\in X, v\in\mathfrak{R}}[\lambda(z)Tv(z)Tv(x)]\\ &\geqslant \lambda(x)T\mu'(x)T\mu'(x)\\ &\geqslant \lambda(x)T\gamma T\gamma.\end{aligned}$$

By the arbitrariness of γ and x, we get $\lambda^*(\mathfrak{R}) \geqslant \lambda$, which rendering (i).

This completes the proof. □

In order to obtain simple expressions for some of the functors we encounter, it will be convenient to reformulate some of the foregoing definitions and properties.

Definition 3.4

(i) Let $\psi \in I^{X\times X}$ and define $\sigma(\psi) \subset I^X$ by: $\sigma(\psi) = (\psi\langle\mathbf{1}_x\rangle_T)_{x\in X}$, where $\psi\langle\mathbf{1}_x\rangle_T$ is the T-section of ψ over $\mathbf{1}_x$.If $\upsilon \subset I^{X\times X}$, we define $\sigma(\upsilon) \subset I^X$ by: $\sigma(\upsilon) = \{\sigma(\psi){:}\psi \in \upsilon\} \subset F(X)$.
(ii) If $\mathfrak{R} \subset I^X$, we define $\Gamma(\mathfrak{R}) \in I^{X\times X}$ by: $(\Gamma(\mathfrak{R}))(x,y) = \sup_{v\in\mathfrak{R}}[v(x)Tv(y)]$, $x,\ y \in X$; and if $\Sigma \subset F(X)$, we define $\Gamma(\Sigma)$ by: $\Gamma(\Sigma) = \{\Gamma(\mathfrak{R}) : \mathfrak{R} \in \Sigma\} \subset I^{X\times X}$.

Lemma 3.4. *For every $\mathfrak{R}, \mathfrak{I} \in F_c(X)$, we have the following:*

(i) $\underline{\gamma}T\sigma(\Gamma(\mathfrak{R})) \ll \mathfrak{R}^*, \forall\gamma \in I_1$;
(ii) $\underline{\gamma}T\mathfrak{R} \ll \sigma(\Gamma(\mathfrak{R})), \forall\gamma \in I_1$;
(iii) *If* $\psi \in I^{X\times X}$, *then* $\Gamma(\sigma(\psi)) = \psi o_{Ts}\psi$;
(iv) $(\Gamma(\mathfrak{R}))(x,x) = 1,\ \forall\ x \in X$;
(v) $\Gamma(\underline{\gamma}T\mathfrak{R}) = \underline{\gamma}T\underline{\gamma}T\Gamma(\mathfrak{R}),\ \forall\ \gamma \in I_1$;
(vi) *If* $\mathfrak{R} \ll \mathfrak{I}$, *then* $\Gamma(\mathfrak{R}) \leqslant \Gamma(\mathfrak{I})$;
(vii) *If* $\psi,\varphi \in I^{X\times X}$, *for which* $\psi \leqslant \varphi$, *then* $\sigma(\psi) \ll \sigma(\varphi)$.

Proof

(i) Let $\lambda \in [\underline{\gamma}T\sigma(\Gamma(\mathfrak{R}))]$, where $\gamma \in I_1$. Then there is $x_o \in X$ such that $\lambda = [\underline{\gamma}T\Gamma(\mathfrak{R})\langle\mathbf{1}_{xo}\rangle_T]$. Also by hypothesis, there is $\mu_o \in \mathfrak{R}$ for which $\mu_o(x_o) = 1$.Hence, for every $y \in X$, we have

$$\begin{aligned}\lambda(y) &= [\underline{\gamma}T\Gamma(\mathfrak{R})\langle\mathbf{1}_{x_o}\rangle_T](y)\\ &= \gamma T\sup_{z\in X}[(\mathbf{1}_{x_o})(z)T(\Gamma(\mathfrak{R}))(z,y)]\\ &= \gamma T(\Gamma(\mathfrak{R}))(x_o,y)\\ &= \gamma T\sup_{v\in\mathfrak{R}}[v(x_o)Tv(y)]\\ &\leqslant \mu_o(x_o)T\sup_{v\in\mathfrak{R}}[v(x_o)Tv(y)]\\ &= \sup_{v\in\mathfrak{R}}[\mu_o(x_o)Tv(x_o)Tv(y)]\\ &\leqslant \sup_{x\in X, v\in\mathfrak{R}}[\mu_o(x)Tv(x)Tv(y)]\\ &= (\mu_o^*(\mathfrak{R}))(y).\end{aligned}$$

This shows the existence of an element $\mu_o^*(\mathfrak{R})$ of $\mathfrak{R}^*$ which greater or equal to λ.
(ii) As the same manner of (i).
(iii) Let $\psi \in I^{X\times X}$. Then, for every $(x,y) \in X\times X$, we have

$$\begin{aligned}(\Gamma(\sigma(\psi)))(x,y) &= \sup_{v\in\sigma(\psi)}[v(x)Tv(y)] \\ &= \sup_{z\in X}[\psi\langle \mathbf{1}_z\rangle_T(x)T(\psi\langle \mathbf{1}_z\rangle_T)(y)], \quad \text{by definition} \\ &= \sup_{z\in X}[\psi(z,x)T\psi(z,y)], \quad \text{by (2)} \\ &= \sup_{z\in X}[{}_s\psi(x,z)T\psi(z,y)] \\ &= (\psi o_{Ts}\psi)(x,y).\end{aligned}$$

which proves the required equality.The proofs of other parts are trivially hold. □

4. Covering TL-uniform spaces

In this section, the covering TL-uniform spaces are introduce and some of their properties are given, together with illustrative examples. Also, the C-uniformly continuous functions are define. Moreover, an isomorphism between category of TL-uniform spaces and category of TL-uniform spaces is holds.

Definition 4.1

(i) A covering TL-uniform base on a set X is a subset $\mathcal{H} \subset F_c(X)$ which fulfills the following properties:

(CTLUB1) For all $\mathfrak{R}, \mathfrak{I} \in \mathcal{H}$, there is $£ \in \mathcal{H}$ such that $£ \ll \mathfrak{R}$ and $£ \ll \mathfrak{I}$;
(CTLUB2) For all $\mathfrak{R} \in \mathcal{H}$ and $\gamma \in I_1$, there is $£ \in \mathcal{H}$ such that $(\underline{\gamma}T£^*) \ll \mathfrak{R}$.

(ii) A covering TL-uniformity on a set X is a T-saturated covering TL-uniform base on X.
(iii) If $\mathcal{K}$ is a covering TL-uniformity on X, then we say that $\mathcal{H} \subset F_c(X)$ is a basis for $\mathcal{K}$ if $\mathcal{H}^{\sim T} = \mathcal{K}$.
(iv) A covering TL-uniform space is a couple $(X, \mathcal{K})$, where X is a set and $\mathcal{K}$ is a covering TL-uniformity on X.

Definition 4.2. Let $(X,\mathcal{K})$ and $(Y,\mathcal{K}')$ be covering TL-uniform spaces and f: $X \to Y$ be a function. We say that f is C-uniformly continuous (C-uniformly maps) if for each $\mathfrak{R}' \in \mathcal{K}'$ there is $\mathfrak{R} \in \mathcal{K}$ such that $\mathfrak{R} \subseteq f^{\leftarrow}(\mathfrak{R}')$. Where

$$f^{\leftarrow}(\mathfrak{R}') = \{f^{\leftarrow}(\lambda) : \lambda \in \mathfrak{R}'\}.$$

The composite of two C-uniformly continuous functions f: $(X,\mathcal{K}) \to (Y,\mathcal{K}')$ and g: $(Y,\mathcal{K}') \to (Y,\mathcal{K}'')$ is again C-uniformly continuous, since for every $\mathfrak{R}'' \in \mathcal{K}''$, we have

$$\begin{aligned}(f\ o\ g)^{\leftarrow}(\mathfrak{R}'') &= f^{\leftarrow}(g^{\leftarrow}(\mathfrak{R}'')) \\ &\supseteq f^{\leftarrow}(\mathfrak{R}'), \quad \text{for some} \quad \mathfrak{R}' \in \mathcal{K}' \\ &\supseteq \mathfrak{R}, \quad \text{for some } \mathfrak{R} \in \mathcal{K}.\end{aligned}$$

Also, for a covering TL-uniform space $(X,\mathcal{K})$, it is easy to see that the identity map
Id_X: $(X,\mathcal{K}) \to (X,\mathcal{K})$ is C-uniformly continuous function.

Corollary 4.1. *It is clear by above result that the class of all covering TL-uniform spaces together with C-uniformly maps as arrows forms a concrete category.*

We denote by CTL-US the category of covering TL-uniform spaces and as morphisms are C-uniformly continuous functions between these spaces.

First, we show that there is an isomorphism between category of TL-uniform spaces and category of covering TL-uniform spaces. We make use of the constructions and notations introduced above.

Theorem 4.1. *If υ is a TL-uniform base on a set X, then $\sigma(\upsilon)$ is a covering TL-uniform base on X.*

Proof. To prove (CTLUB1), let $\varphi,\varphi' \in \upsilon$. Then by (TLUB1), we can find $\psi \in \upsilon$ such that $\psi \leqslant \varphi \bigwedge \varphi'$, it follows for every $x \in X$ that,

$\psi\langle \mathbf{1}_x\rangle_T \leqslant \varphi\langle \mathbf{1}_x\rangle_T \bigwedge \varphi'\langle \mathbf{1}_x\rangle_T$. So $\sigma(\psi) \ll \sigma(\varphi)$ and $\sigma(\psi) \ll \sigma(\varphi')$.

(CTLUB2) Let $\varphi \in \upsilon$ and $\gamma \in I_1$. By continuity of T, we can get $\theta \in I_1$ for which $\gamma = \theta T\theta T\theta T\theta T\theta T\theta$. Then by applying (TLUB4) twice, we can find $\phi \in \upsilon$ such that

$$\begin{aligned}&(\underline{\theta}T\underline{\theta}T\underline{\theta})T(\phi o_T\ \phi\ o_T\ \phi\ o_T\ \phi) \leqslant \varphi, \text{ thus clearly} \\ &\quad (\underline{\theta}T\underline{\theta}T\underline{\theta})T(\phi\ o_T\ \phi\ o_T\ \phi) \leqslant \varphi,\end{aligned} \tag{4}$$

Also, by Lemma 2.1, there is $\psi \in \upsilon$ such that $(\underline{\theta}T\ \psi) \leqslant [\underline{\theta}T(\psi \bigvee {}_s\psi)] \leqslant \phi$.

Now, by putting $\psi' = (\psi \bigvee {}_s\psi)$ and consider an arbitrary element

$\lambda = \{\underline{\gamma}T\,[(\psi\langle \mathbf{1}_x\rangle_T)^*(\sigma(\psi))]\}$ of $\underline{\gamma}\,(T(\sigma(\psi))^*$, we have for every $y \in X$,

$$\begin{aligned}\lambda(y) &= \{\underline{\gamma}T\ [(\psi\langle \mathbf{1}_x\rangle_T)^*(\sigma(\psi))]\}(y) \\ &= \gamma\ T\sup_{z,r\in X}[(\psi\langle \mathbf{1}_x\rangle_T)(r)T\ (\psi\langle \mathbf{1}_z\rangle_T)(r)\ \ (T(\psi\langle \mathbf{1}_z\rangle_T)(y)] \\ &= \gamma T\sup_{z,r\in X}[\psi(x,r)T\psi(z,r)T\ \psi(z,y)], \quad \text{by (2)} \\ &= \gamma\ T\ \sup_{z,r\in X}[\psi(x,r)T_s\psi(r,z)T\psi(z,y)] \\ &= \gamma\ T\ [\psi\ o_T\ ({}_s\psi\ o_T\ \psi)](x,y) \\ &\leqslant \gamma\ T\ (\psi'\ o_T\ \psi'\ o_T\psi')(x,y) \\ &= (\theta T\theta T\theta T)[(\underline{\theta}T\ \psi')\ o_T(\underline{\theta}T\ \psi')\ o_T(\underline{\theta}T\ \psi')](x,y) \\ &\leqslant (\theta T\theta T\theta T)\ (\phi\ o_T\ \phi\ o_T\ \phi)(x,y) \\ &\leqslant \varphi(x,y), \quad \text{by (4)} \\ &= (\varphi\langle \mathbf{1}_x\rangle_T)(y), \quad \text{by (2) again.}\end{aligned}$$

Which shows the existence of a member $\varphi\langle \mathbf{1}_x\rangle_T$ of $\sigma(\varphi)$, which greater or equal to λ, therefore $\underline{\gamma}(T\ (\sigma\ (\psi))^* \ll \sigma(\varphi)$.

Which proves (CTLUB2) and completes the proof. □

Theorem 4.2. *If $\mathcal{H}$ is a covering TL-uniform base on a set X, then $\Gamma(\mathcal{H})$ is a TL-uniform base on X.*

Proof. To prove (TLUB1), let $\mathfrak{R}, \mathfrak{I} \in \mathcal{H}$. Then by (CTLUB1), there is $£ \in \mathcal{H}$ such that $£ \ll \mathfrak{R}$ and $£ \ll \mathfrak{I}$.

Hence, from Lemma 3.4 (vi), it follows that $\Gamma(£) \ll \Gamma(\mathfrak{R}) \bigwedge \Gamma(\mathfrak{I})$.

(TLUB2) Follows immediately from the fact that every $\mathfrak{R} \in \mathcal{H}$ is a fuzzy covering of X.

(TLUB3) Obviously holds, because every $\Gamma(\mathfrak{R})$ is symmetric.

(TLUB4) Let $\mathfrak{R} \in \mathcal{H}$ and $\gamma \in I_1$. Then by continuity of T, we can get $\theta \in I_1$ for which $\gamma = (\theta\ T\theta)$. Thus by (CTLUB2), we can find $\mathfrak{I} \in \mathcal{H}$ such that $(\underline{\theta} T \mathfrak{I}^*) \ll \mathfrak{R}$, this meaning that, for every $\lambda \in \mathfrak{I}$ there is $\vartheta \in \mathfrak{R}$ such that

$$[\underline{\theta} T\ \lambda^*(\mathfrak{I})] \leqslant \vartheta. \tag{5}$$

Now, for every x, $y \in X$, we have

$$\begin{aligned}
[\underline{\gamma} T\,(\Gamma(\mathfrak{I})\ o_T\ \Gamma(\mathfrak{I}))](x,y) &= \gamma\ T \sup_{z\in X}[(\Gamma(\mathfrak{I})(x,z)\, T\, \Gamma(\mathfrak{I}))(z,y)]\} \\
&= \gamma\ T \sup_{z\in X}\{[\sup_{\lambda\in\mathfrak{I}}(\lambda(x)\ T\lambda(z))]\, T\, [\sup_{v\in\mathfrak{I}}(v(z)\ T\, v(y))]\} \\
&= \gamma\ T \sup_{z\in X}\{\sup_{\lambda,v\in\mathfrak{I}}[\lambda(x)\ T\,\lambda(z)\, T\, v(z)\ T\, v(y)]\} \\
&= \gamma\ T \sup_{\lambda\in\mathfrak{I}} \lambda(x) T \sup_{z\in X, v\in\mathfrak{I}}[\lambda(z)\ T\, v(z)\ T\, v(y)]\} \\
&= (\theta\ T\theta)\ T \sup_{\lambda\in\mathfrak{I}}\{[\lambda(x) T\,(\lambda^*(\mathfrak{I}))(y)]\} \\
&\leqslant (\theta\ T\theta)\ T \sup_{\lambda\in\mathfrak{I}}[(\lambda^*(\mathfrak{I}))(x) T\,(\lambda^*(\mathfrak{I}))(y)], \text{ by Lemma 3.3 (i)} \\
&= \sup_{\lambda\in\mathfrak{I}}\{[(\underline{\theta} T\,(\lambda^*(\mathfrak{I}))(x)]\, T[(\underline{\theta}(T\,(\lambda^*(\mathfrak{I}))(y)]\} \\
&\leqslant \sup_{\vartheta\in\mathfrak{R}}[\vartheta(x) T\, \vartheta(y)], \quad \text{by (5)} \\
&= (\Gamma(\mathfrak{R}))(x,y),
\end{aligned}$$

that is, $\underline{\gamma} T[\Gamma(\mathfrak{I})\ o_T \Gamma(\mathfrak{I})] \leqslant \Gamma(\mathfrak{R})$.

Which completes the proof that $\Gamma(\mathcal{H})$ is a TL-uniform base on X.

Proposition 4.1

(i) *If $\mathcal{H} \subset F_c(X)$ satisfies (CTLUB1) and $\mathcal{H}^{\sim T}$ is a covering TL-uniformity on X, then $\mathcal{H}$ is a covering TL-uniform base (and a basis for $\mathcal{H}^{\sim T}$).*

(ii) *If $\mathcal{H} \subset F_c\ (X)$ and $[\mathcal{H}]$ is a covering TL-uniformity on X, then $\mathcal{H}$ is a covering TL-uniform base (and a basis for $[\mathcal{H}]$).*

Proof

(i) For the condition (CTLUB2), let $\mathfrak{R} \in \mathcal{H} \subseteq \mathcal{H}^{\sim T}$ and $\gamma \in I_1$. Then by continuity of T, we can find $\theta \in I_1$ for which $\gamma = (\theta T\theta T\theta T\theta)$ and by hypothesis, there is $\mathfrak{I} \in \mathcal{H}^{\sim T}$ such that $(\underline{\theta} T\ \mathfrak{I}^*) \ll \mathfrak{R}$.Also, there is a family $\{\mathfrak{I}_\epsilon : \epsilon \in I_1\}$ such that $[\bigvee_{\epsilon\in I}(\underline{\epsilon} T\mathfrak{I}_\epsilon)] \ll \mathfrak{I}$.In particular, it follows that $(\underline{\theta} T\mathfrak{I}_\theta) \ll \mathfrak{I}$, so $(\underline{\theta} T\mathfrak{I}_\theta)^* \ll \mathfrak{I}^*$. Hence

$$\begin{aligned}
(\underline{\gamma} T\mathfrak{I}^*_\theta) &= [(\underline{\theta} T\underline{\theta} T\underline{\theta} T\underline{\theta}) T\mathfrak{I}^*_\theta] \\
&= [\underline{\theta} T(\underline{\theta} T\mathfrak{I}^*_\theta)], \text{ by Lemma 3.2 (iv)} \\
&\ll (\underline{\theta} T\ \mathfrak{I}^*) \\
&\ll \mathfrak{R}.
\end{aligned}$$

(ii) Let $\mathfrak{R}, \mathfrak{I} \in \mathcal{H} \subseteq [\mathcal{H}]$. Then there is $£ \in [\mathcal{H}]$ satisfies $£ \ll \mathfrak{R}$ and $£ \ll \mathfrak{I}$.Consequently there is $£' \in \mathcal{H}$ for which $£' \ll £$.Thus $\mathcal{H}$ satisfies (CTLUB1).Now, by hypothesis and Lemma 3.1, we have

$$[\mathcal{H}] = [\mathcal{H}]^{\sim T} = [[\mathcal{H}]^\wedge] \subseteq [[\mathcal{H}^\wedge]] = [\mathcal{H}^\wedge] = \mathcal{H}^{\sim T} \subseteq [\mathcal{H}]^{\sim T} = [\mathcal{H}].$$

It follows that $\mathcal{H}^{\sim T} = [\mathcal{H}]$ is a covering TL-uniformity and then we can apply (i) to reach our assertion.

Proposition 4.2. *If $\mathcal{H}$ is a covering TL-uniform base on a set X and $\Upsilon \subset F_c(X)$ satisfies $\Upsilon \subset [\mathcal{H}]$ and $\mathcal{H} \subset [\Upsilon]$, then Υ is also a covering TL-uniform base on X.*

Proof. Let $\mathfrak{R}, \mathfrak{I} \in \Upsilon$, since $\Upsilon \subset [\mathcal{H}]$. Then there exist $\mathfrak{R}', \mathfrak{I}' \in \mathcal{H}$ such that $\mathfrak{R}' \ll \mathfrak{R}$ and $\mathfrak{I}' \ll \mathfrak{I}$.

Since $\mathcal{H}$ is a covering TL-uniform base, then there is $\varrho' \in \mathcal{H}$ such that

$\varrho' \ll \mathfrak{R}'$ and $\varrho' \ll \mathfrak{I}'$.

We can in turn, since $\mathcal{H} \subset [\Upsilon]$, then there is $\varrho \in \Upsilon$ such that $\varrho\ \ \varrho'$.

Hence $\varrho \ll \mathfrak{R}$ and $\varrho \ll \mathfrak{I}$,

which proves that Υ satisfies (CTLUB1).

(CTLUB2) Follows from Proposition 4.1 (i), since by Lemma 3.1 (vii), $\Upsilon^{\sim T} = \mathcal{H}^{\sim T}$ and by hypothesis we have, $\mathcal{H}^{\sim T}$ is a covering TL-uniformity on X. □

Proposition 4.3. *If $\mathcal{H}$ is a covering TL-uniform base on a set X, then $\sigma(\Gamma(\mathcal{H}))$ is a basis for $\mathcal{H}^{\sim T}$, that is $(\sigma(\Gamma(\mathcal{H})))^{\sim T} = \mathcal{H}^{\sim T}$.*

Proof. That $\sigma(\Gamma(\mathcal{H})))$ is a covering TL-uniform base, follows from Theorems 4.1 and 4.2.

Now, from Lemma 3.4 (ii), it follows that $(\underline{\gamma} T\mathfrak{R}) \ll \sigma(\Gamma(\mathfrak{R}))$, for every $\mathfrak{R} \in \mathcal{H}, \gamma \in I_1$, so that $\sigma(\Gamma(\mathfrak{R})) \in \mathcal{H}^{\sim T}$, hence $\sigma(\Gamma(\mathcal{H})) \subseteq \mathcal{H}^{\sim T}$ and thus, by Lemma 3.1 (i), (vi), we get $(\sigma(\Gamma(\mathcal{H})))^{\sim T} \subseteq \mathcal{H}^{\sim T}$.

On the other hand, for every $\mathfrak{R} \in \mathcal{H}$ and $\gamma = (\theta\ T\theta) \in I_1$, we get an element $\mathfrak{R}_\gamma \in \mathcal{H}$ such that $[\underline{\theta}\ T(\mathfrak{R}_\gamma)^*] \ll \mathfrak{R}$.

By Lemma 2.4 (i), it follows that, $[\underline{\theta} T\sigma(\Gamma(\mathfrak{R}_\gamma))] \ll \mathfrak{R}^*_\gamma$.

Consequently,

$$[\underline{\gamma} T\sigma(\Gamma(\mathfrak{R}_\gamma))] = \{\underline{\theta} T[\underline{\theta} T\sigma(\Gamma(\mathfrak{R}_\gamma))]\} \ll [\underline{\theta} T(\mathfrak{R}_\gamma)^*] \ll \mathfrak{R},$$

this implies that $\{\bigvee_{\gamma\in I_1}[\underline{\gamma} T\sigma(\Gamma(\mathfrak{R}_\gamma))]\} \ll \mathfrak{R}$, thus $\mathfrak{R} \in (\sigma(\Gamma(\mathcal{H})))^{\sim T}$.

Hence $\mathcal{H} \subseteq (\sigma(\Gamma(\mathcal{H})))^{\sim T}$. Which proves our assertion.

Proposition 4.4. *If $\upsilon$is a TL-uniform base on a set X, then $\Gamma(\sigma(\upsilon))$ is a basis for $\upsilon^{\sim T}$, that is $(\Gamma(\sigma(\upsilon)))^{\sim T} = \upsilon^{\sim T}$.*

Proof. We already know from Theorems 4.1 and 4.2, that $\Gamma(\sigma(\upsilon))$ is a TL-uniform base.

Let $\varphi \in \upsilon$ and $\gamma \in I_1$, by continuity of T, we can get $\theta \in I_1$ for which $\gamma = (\theta T\theta T\theta)$. Then there is $\varphi_\theta \in \upsilon$ such that

$$[\underline{\theta} T(\varphi_\theta\ o_T\ \varphi_\theta)] \leqslant \varphi. \tag{6}$$

By Lemma 2.1, there is $\psi_\gamma \in \upsilon$ with $(\underline{\theta} T\psi_\gamma) \leqslant [\underline{\theta} T(\psi_\gamma \bigvee_s \psi_\gamma)] \leqslant \varphi_\theta$.

It follows that

$$\underline{\gamma}T\Gamma(\sigma(\psi_\gamma)) = \underline{\gamma}T(\psi_\gamma\ o_{Ts}\psi_\gamma),\ \text{by Lemma 3.4 (iii)}$$
$$= (\underline{\theta}T\underline{\theta}T\underline{\theta})T(\psi_\gamma\ o_{Ts}\psi_\gamma)$$
$$\leqslant (\underline{\theta}T\underline{\theta}T\underline{\theta})T[(\psi_\gamma \bigvee {}_s\psi_\gamma)\ o_T\ (\psi_\gamma \bigvee {}_s\psi_\gamma)]$$
$$= \underline{\theta}T[\underline{\theta}T(\psi_\gamma \bigvee {}_s\psi_\gamma)]\ o_T[\underline{\theta}T(\psi_\gamma \bigvee {}_s\psi_\gamma)]\}$$
$$= \underline{\theta}T(\varphi_\theta\ o_T\ \varphi_\theta)$$
$$\leqslant \varphi, \quad \text{by (6)}$$

Consequently, $\bigvee_{\gamma\in I_1}[\underline{\gamma}T\Gamma(\sigma(\psi_\gamma))] \leqslant \varphi$, so $\upsilon \subset (\Gamma(\sigma(\upsilon)))^{\sim T}$.
Hence, by Lemma 3.1, $\upsilon^{\sim T} \subset (\Gamma(\sigma(\upsilon)))^{\sim T}$.

On the other hand, for every $\varphi \in \upsilon$, we have from Lemma 3.4 (iii), that

$$\Gamma(\sigma(\varphi)) = (\varphi\ o_{Ts}\varphi) \geqslant \varphi.$$

Therefore, $\Gamma(\sigma(\varphi)) \in [\upsilon]$,. hence by Lemma 3.1 (vii), we have $(\Gamma(\sigma(\upsilon)))^{\sim T} \subset \upsilon^{\sim T}$.
This winds up the proof.

Lemma 4.1. *If $\upsilon \subset I^{X\times X}$ and $\mathcal{H} \subset F_c(X)$, then $\sigma(\upsilon^{\sim T}) = (\sigma(\upsilon))^{\sim T}$ and $(\Gamma(\mathcal{H}))^{\sim T} \subseteq \Gamma(\mathcal{H}^{\sim T})$. Moreover, if $\mathcal{H}$ is a T-saturated, then $\Gamma(\mathcal{H}^{\sim T}) = (\Gamma(\mathcal{H}))^{\sim T}$.*

Proof. First, we show that $\sigma(\upsilon^{\sim T}) = (\sigma(\upsilon))^{\sim T}$, as follows. Let $\mu \in I^X$, we have

$\mu \in \sigma(\upsilon^{\sim T})$

iff $\exists\ \psi \in \upsilon^{\sim T}$ such that $\mu = \sigma(\psi)$

iff $\exists\ \psi_\gamma \in \upsilon$ such that $\psi \geqslant \bigvee_{\gamma\in I_1}(\underline{\gamma}T\ \psi_\gamma), \mu = \sigma(\psi)$

iff $\exists\ \psi_\gamma \in \upsilon, x_o \in X$ such that $\psi \geqslant \bigvee_{\gamma\in I_1}(\underline{\gamma}T\psi_\gamma), \mu = \psi\langle \mathbf{1}_{x_o}\rangle_T$

iff $\exists\ \psi_\gamma \in \upsilon, x_o \in X$ such that $\mu \geqslant [\bigvee_{\gamma\in I_1}(\underline{\gamma}T\psi_\gamma)]\langle \mathbf{1}_{x_o}\rangle_T$

iff $\exists\ \psi_\gamma \in \upsilon, x_o \in X$ such that $\mu \geqslant \bigvee_{\gamma\in I_1}[\underline{\gamma}T\psi_\gamma\langle \mathbf{1}_{x_o}\rangle_T]$

iff $\exists\ \psi_\gamma \in \upsilon, x_o \in X$ and $\lambda_\gamma = \psi_\gamma\langle \mathbf{1}_{x_o}\rangle_T$ such that $\mu \geqslant \bigvee_{\gamma\in I_1}(\underline{\gamma}T\lambda_\gamma)$

iff $\exists\ \psi_\gamma \in \upsilon$ and $\lambda_\gamma = \sigma(\psi_\gamma)$ such that $\mu \geqslant \bigvee_{\gamma\in I_1}(\underline{\gamma}T\ \lambda_\gamma)$

iff $\exists\ \lambda_\gamma \in \sigma(\upsilon)$ such that $\mu \geqslant \bigvee_{\gamma\in I_1}(\underline{\gamma}T\ \lambda_\gamma)$

iff $\mu \in \sigma(\upsilon))^{\sim T}$.

Second, as the same steps of the first part and using Lemma 3.4 (v), we can show that $(\Gamma(\mathcal{H}))^{\sim T} \subseteq \Gamma(\mathcal{H}^{\sim T})$.
Moreover, if $\mathcal{H}$ is a T-saturated, then by Lemma 3.1 (ii), we have

$$\Gamma(\mathcal{H}^{\sim T}) \subseteq (\Gamma(\mathcal{H}^{\sim T}))^{\sim T} = (\Gamma(\mathcal{H}))^{\sim T}.$$

Hence, we get the required equality and completes the proof. □

The preceding results entails the following proposition

Proposition 4.5

(i) *If Ω is a TL-uniformity on a set X, then $\Gamma(\sigma(\Omega)) = \Omega$.*
(ii) *If $\mathcal{H}$ is a covering TL-uniformity on X, then $\sigma(\Gamma(\mathcal{H})) = \mathcal{H}$.*

Theorem 4.3

(i) *If the function f: $(X,\Omega) \to (Y,\Omega')$ between TL-uniform spaces, is uniformly continuous, then f: $(X,\sigma(\Omega)) \to (Y,\sigma(\Omega'))$ is C-uniformly continuous.*
(ii) *If the function f: $(X,\mathcal{K}) \to (Y,\mathcal{K}')$ between covering TL-uniform spaces, is C-uniformly continuous, then f: $(X,\Gamma(\mathcal{K})) \to (Y,\Gamma(\mathcal{K}'))$ is uniformly continuous.*

Proof

(i) Let f: $(X,\Omega) \to (Y,\Omega')$ be uniformly continuous and consider an arbitrary element $\mathfrak{R}' \in \sigma(\Omega')$, then there is $\varphi' \in \Omega'$ such that $\mathfrak{R}' = (\varphi'\langle \mathbf{1}_y\rangle_T)_{y\in Y}$.Now, for every x, $z \in X$, we have

$$(f^{\leftarrow}(\varphi'\langle \mathbf{1}_{f(x)}\rangle_T))(z) = (\varphi'\langle \mathbf{1}_{f(x)}\rangle_T)(f(z))$$
$$= \varphi'(f(x),f(z)),\ \text{by (2)}$$
$$= [(f\times f)^{\leftarrow}(\varphi')](x,z)$$
$$= [((f\times f)^{\leftarrow}(\varphi'))\langle \mathbf{1}_x\rangle_T](z),\ \text{by(2)again}$$

If we take $\mathfrak{R} = (((f\times f)^{\leftarrow}(\varphi'))\langle \mathbf{1}_x\rangle_T)_{x\in X}$, we get by hypothesis that $\mathfrak{R} \in \sigma(\Omega)$ and$f^{\leftarrow}(\mathfrak{R}') = f^{\leftarrow}((\varphi'\langle \mathbf{1}_y\rangle_T)_{y\in Y})$

$$\supseteq (f^{\leftarrow}(\varphi'\langle \mathbf{1}_{f(x)}\rangle_T)_{x\in X},\ \text{for range}\ f \subseteq\ Y$$
$$= (((f\times f)^{\leftarrow}(\varphi'))\langle \mathbf{1}_x\rangle_T)_{x\in X}$$
$$= \mathfrak{R}.$$

This proves that f: $(X,\sigma(\Omega)) \to (Y,\sigma(\Omega'))$ is C-uniformly continuous.
(ii) Let f: $(X,\mathcal{K}) \to (Y,\mathcal{K}')$ be C-uniformly continuous and $\psi' \in \Gamma(\mathcal{K}')$. Then there is $\mathfrak{R}' \in \mathcal{K}'$ for which $\psi' = \Gamma(\mathfrak{R}')$.Hence, for every x, $y \in X$, we have

$$[(f\times f)^{\leftarrow}(\psi')](x,y) = [(f\times f)^{\leftarrow}(\Gamma(\mathfrak{R}'))](x,y)$$
$$= (\Gamma(\mathfrak{R}'))(f(x),f(y))$$
$$= \sup_{\lambda\in\mathfrak{R}'}[\lambda(f(x))T\lambda(f(y))]$$
$$= \sup_{\lambda\in\mathfrak{R}'}[(f^{\leftarrow}(\lambda))(x)T(f^{\leftarrow}(\lambda))(y)]$$
$$= \sup_{v\in f^{\leftarrow}(\mathfrak{R}')}[v(x)Tv(y)]$$
$$\geqslant \sup_{\mu\in\mathfrak{R}}[\mu(x)T\mu(y)],\ \text{by hypothesis}$$
$$= (\Gamma(\mathfrak{R}))(x,y)$$
$$= \psi(x,y),\ \text{for some}\ \psi \in \Gamma(\mathcal{K}).$$

This shows the existences of an element ψ in $\Gamma(\mathcal{K})$ satisfies $\psi \leqslant (f\times f)^{\leftarrow}(\psi')$.Which proves the uniformly continuous of f: $(X,\Gamma(\mathcal{K})) \to (Y,\Gamma(\mathcal{K}'))$.Rendering (ii) and completes the proof. □

Now, we see how a covering TL-uniformity can generate an I-topology. The formula of fuzzy interior operator, consequently I-topology associated with a covering TL-uniformity, in particularly simple, through the TL-uniformity which defined by covering TL-uniformity, as is shown in the next result.

Theorem 4.4. *If $(X,\mathcal{K})$ is a covering TL-uniform space, then the fuzzy interior operator which defines the I-topology $\tau(\mathcal{K})$ is given by:*

$$\lambda^o(x) = \sup_{\mathfrak{R}\in\mathcal{K}} \hat{\mathbf{J}}\langle(\Gamma(\mathfrak{R}))\langle \mathbf{1}_x\rangle_T), \lambda\rangle, \lambda \in I^X, x \in X.$$

Proof. By Theorem 4.2, we have $\Gamma(\mathcal{K})$ is a TL-uniformity, it follows from Proposition 2.2, that for every $\lambda \in I^X$ and x ∈ X,

$$\lambda^o(x) = \sup_{\varphi\in\Gamma(\mathcal{K})} \hat{\mathbf{J}}\langle\varphi\langle \mathbf{1}_x\rangle_T, \lambda\rangle = \sup_{\mathfrak{R}\in\mathcal{K}} \hat{\mathbf{J}}\langle(\Gamma(\mathfrak{R}))\langle \mathbf{1}_x\rangle_T, \lambda\rangle.$$

which renders the proof. □

By conjunction of Theorem 4.3 (ii) and [5. Theorem 3.10], we arrive to

Theorem 4.5. *Let $(X,\mathcal{K})$ and $(Y,\mathcal{K}')$ be covering TL-uniform spaces and f: $X \to Y$ is C-uniformly continuous, then f is continuous with respect to the I-topologies associated with $\mathcal{K}$ and $\mathcal{K}'$, respectively.*

Example 1. Let $(X, \leqslant)$ be a directed set, define $H_r = \{x \in X{:}x > r\}$ for every $r \in X$, and $\mathfrak{I}_r = \mathbf{1}_{Hr} \bigvee \{\{\mathbf{1}_x\}{:} x \leqslant r\}$. It is easy to verify that

$\mathcal{H} = \{\mathfrak{I}_r : r \in X\}$ is a covering TL-uniform base on X.

Obviously, $\mathcal{H} \subset F_c(X)$, to verify that (CTLUB1) holds, let $\mathfrak{I}_{r_1}, \mathfrak{I}_{r_2} \in \mathcal{H}$ and take $r > r_1 > r_2$. Therefore $\mathfrak{I}_r \in \mathcal{H}$, which satisfies $\mathfrak{I}_r \ll \mathfrak{I}_{r_1}$ and $\mathfrak{I}_r \ll \mathfrak{I}_{r_2}$.

(CTLUB2) Let $\mathfrak{I}_r \in \mathcal{H}$ and $\gamma \in I_1$, we can choose $t > r$, for which $\mathfrak{I}_t \in \mathcal{H}$ and $\mathfrak{I}_t \ll \mathfrak{I}_r$. Then, for every $\lambda \in \mathfrak{I}_t$, $x \in X$, we have

$$\begin{aligned}
[\underline{\gamma} T\lambda^*(\mathfrak{I}_t)](x) &\leqslant (\lambda^*(\mathfrak{I}_t))(x) \\
&= \sup_{z\in X, \nu\in\mathfrak{I}_t} [\lambda(z)T\nu(z)T\nu(x)] \\
&\leqslant \sup_{z\in X, \mu\in\mathfrak{I}_r} [\lambda(z)T\mu(z)T\mu(x)], \quad \text{clear} \\
&= \sup_{\mu\in\mathfrak{I}_r} [\lambda(x)T\mu(x)T\mu(x)] \\
&= \lambda(x) \\
&\leqslant \lambda'(x), \quad \text{for some } \lambda' \in \mathfrak{I}_r, \text{ since } \mathfrak{I}_t \ll \mathfrak{I}_r.
\end{aligned}$$

This shows that, $[\underline{\gamma}\mathrm{T}(\mathfrak{I}_t)^*] \ll \mathfrak{I}_r$.

In particular, if X is any set, then $\mathcal{H} = \{\{\mathbf{1}_x\} : x \in X\}$ is a covering TL-uniform base on X, which generated the discrete I-topology on X, since we can show that each fuzzy set is open, as follows:

For every $\lambda \in I^X$ and $x \in X$, we have

$$\begin{aligned}
\lambda^o(x) &= \sup_{\mathfrak{R}\in\mathcal{H}} \hat{\mathbf{J}}\langle(\Gamma(\mathfrak{R}))\langle \mathbf{1}_x\rangle_T, \lambda\rangle \\
&= \hat{\mathbf{J}}\langle(\Gamma(\mathcal{H}))\langle \mathbf{1}_x\rangle_T, \lambda\rangle \\
&= \inf_{y\in X} \hat{\mathbf{J}}(((\Gamma(\mathcal{H}))\langle \mathbf{1}_x\rangle_T))(y), \lambda(y)), \quad \text{by (1)} \\
&= \inf_{y\in X} \hat{\mathbf{J}}(((\Gamma(\mathcal{H}))(x,y), \lambda(y)), \quad \text{by (2)} \\
&= \inf_{y\in X} \hat{\mathbf{J}}(\sup_{z\in X}((\mathbf{1}_z)(x)T(\mathbf{1}_z)(y)), \lambda(y)) \\
&= \hat{\mathbf{J}}((\mathbf{1}_x)(x)T(\mathbf{1}_x)(x), \lambda(x)), \quad \text{by Proposition 2.1 (ii)} \\
&= \hat{\mathbf{J}}(1, \lambda(x)), \\
&= \lambda(x), \quad \text{by Proposition 2.1 (i).}
\end{aligned}$$

That is, $\lambda^o = \lambda$.

Example 2. If X is a nonempty set, then the singleton $\mathcal{H} = \{\underline{1}\}$ is a covering TL-uniform base on X, which induces the indiscrete I-topology, because the open sets are exactly the constant fuzzy sets. Since as the same steps in Example 1, it is easy to see that for every $\lambda \in I^X$, we have $\lambda^o = \underline{\alpha}$, for some $\alpha \in I$.

Now, if we define the map $\sigma^{\sim}{:}TL\text{-US} \to CTL\text{-US}$ by setting $\sigma^{\sim}(X,\Omega) = (X, \sigma(\Omega))$ and $\sigma^{\sim}(f) = f$, we get $\sigma^{\sim}$ is a well defined functor.

Also, if we define the map $\Gamma^{\sim}$: CTL-US $\to$ TL-US by setting $\Gamma^{\sim}(X,\mathcal{K}) = (X, \Gamma(\mathcal{K}))$ and $\Gamma^{\sim}(f) = f$, we get $\Gamma^{\sim}$ is well defined functor.

5. The α-levels of a covering TL-uniformity

In this section, we introduce the concepts of α-level covering uniformities for a covering TL-uniformity and we study the relationships between them. Also, we study the corresponding functors between category of covering TL-uniform spaces and category of covering uniform spaces. We denote by US (CUS) the category of uniform spaces (covering uniform spaces).

The functors $\mathcal{P}^{\sim}$: US → CUS and $\mathcal{Q}^{\sim}$: CUS → US are defined in [13] as follows:

(i) if $(X,\mathcal{U}) \in |$ US $|$, the image object is $(X, \mathcal{P}(\mathcal{U}))$, where $\mathcal{P}(\mathcal{U})$ is the unique covering uniformity on X, having a basis

$$\mathcal{P}(\mathcal{B}) = \{(V\langle x\rangle)_{x\in X} : \mathrm{V} \in \mathcal{B}\}, \quad \text{whenever } \mathcal{B} \text{ is any basis for } \mathcal{U}.$$

(ii) if $(X,\mathcal{C}) \in |$ CUS $|$, the image object is $(X, \mathcal{Q}(\mathcal{C}))$, where $\mathcal{Q}(\mathcal{C})$ is the unique uniformity on X, having a basis

$$\mathcal{Q}(\mathcal{E}) = \{\bigcup_{H\in\Lambda}(H \times H) : \Lambda \in \mathcal{E}\}, \quad \text{whenever } \mathcal{E} \text{ is any basis for } \mathcal{C};$$

both functors $\mathcal{P}^{\sim}$ and $\mathcal{Q}^{\sim}$ leaving morphisms unchanged.

The functors $\iota^{\sim}_{u,\alpha}{:}TL\text{-US} \to \text{US}$ and $\omega^{\sim}_u$: US → TL-US are defined in [5] as follows:

(i) if $(X,\Omega) \in |\,TL\text{-US}|$ and $\alpha \in I_1$, the image object is $(X, \iota_{u,\alpha}(\Omega))$, where

$$\iota_{u,\alpha}(\Omega) = \{\psi^{J(\beta,\alpha)} \subseteq X \times X : \psi \in \Omega \text{ and } \beta \in]\alpha, 1]\},$$

is a uniformity on X, called the α-level uniformity of Ω;

(ii) if $(X,\mathcal{U}) \in |\text{US}|$, the image object is $(X, \omega_u(\mathcal{U}))$, where

$$\omega_u(\mathcal{U}) = \{\psi \in I^{X\times X} : \psi^{\gamma} \in \mathcal{U}, \forall \quad \gamma \in I_1\},$$

is a TL-uniformity on X.In the following, we introduce the notions of α-level covering uniformities for a covering TL-uniformity and we study the relationships between these structures.

Definition 5.1

(i) For a covering TL-uniform base $\mathcal{H}$ on a set X and $\alpha \in I_1$, we define

$\iota_{c,\alpha}(\mathcal{H}) = \{\mathfrak{R}^{J(\beta,\alpha)} : \mathfrak{R} \in \mathcal{H} \text{ and } \beta \in]\alpha, 1]\}$,

called the α-level covering uniformity of $\mathcal{H}$, where

$\mathfrak{R}^{J(\beta,\alpha)} = \{\lambda^{J(\beta,\alpha)} \subseteq X : \lambda \in \mathfrak{R}\}$.

(ii) For a covering uniform base $\mathcal{E}$ on a set X and $\alpha \in I$, we define

$\omega_{c,\alpha}(\mathcal{E}) = \{\omega_{c,\alpha}(\Lambda) : \Lambda \in \mathcal{E}\}$,

where $\omega_{c,\alpha}(\Lambda) = \{(\underline{1-\alpha} \bigvee \mathbf{1}_H) \in I^X : H \in \Lambda\}$.

Next, we show that $\iota_{c,\alpha}(\mathcal{H})$ is a covering uniform base and $\omega_{c,\alpha}(\mathcal{E})$ is a covering TL-uniform base on X.

It is not difficult to prove directly that we can obtain the desired functors with the help of the above, quite natural, definitions. However, given a covering TL-uniform base $\mathcal{H}$, we already know an associated TL-uniform base $\Gamma(\mathcal{H})$, so a uniform base $\iota_{u,\alpha}(\Gamma(\mathcal{H}))$ and in turn a basis $\mathcal{P}(\iota_{u,\alpha}(\Gamma(\mathcal{H})))$ for a covering uniformity and it is to be expected that this last basis should be equivalent to $\iota_{c,\alpha}(\mathcal{H})$. Analogous considerations can be made about the transition in the other direction. In order to prove all the desired results, it appears to be necessary to study the functors can be derived from Definition 5.1.

Theorem 5.1. *If $\mathcal{H}$ is a covering TL-uniform base on a set X and $\alpha \in I_1$, then both $\iota_{c,\alpha}(\mathcal{H})$ and $\mathcal{P}(\iota_{u,\alpha}(\Gamma(\mathcal{H})))$ are basis for the same covering uniformity $\mathcal{C}$ on X.*

Proof. We already know that $\mathcal{P}(\iota_{u,\alpha}(\Gamma(\mathcal{H})))$ is a basis for a covering uniformity $\mathcal{C}$ on X, moreover, as $\mathcal{H} \subset F_c(X)$, that every element of $\iota_{c,\alpha}(\mathcal{H})$ is a covering of the set X. Thus, it is sufficient to prove that each element of $\iota_{c,\alpha}(\mathcal{H})$ (resp. $\mathcal{P}(\iota_{u,\alpha}(\Gamma(\mathcal{H})))$) is refined by an element of $\mathcal{P}(\iota_{u,\alpha}(\Gamma(\mathcal{H})))$ (resp. $\iota_{c,\alpha}(\mathcal{H})$).

In order to do, we first take an arbitrary element $\mathfrak{R}^{J(\beta,\alpha)}$ in $\iota_{c,\alpha}(\mathcal{H})$, where $\beta \in]\alpha, 1]$. Now, by continuity of T and Proposition 2.1 (ii), we can find γ, $\theta \in I_1$, such that

$\gamma T\hat{\mathbf{J}}(\theta T\beta, \alpha) T\hat{\mathbf{J}}(\theta T\beta, \alpha) > \hat{\mathbf{J}}(\beta, \alpha)$.

Consider the element $(\Gamma(\mathfrak{R}))^{J(\theta T\beta,\alpha)}\langle x_o\rangle$ in $((\Gamma(\mathfrak{R}))^{J(\theta T\beta,\alpha)}\langle x\rangle)_{x\in X}$.

Choosing $v_o \in \mathfrak{R}$ such that

$$v_o(x_o) \geqslant [v_o(x_o)Tv_o(y)] > \hat{\mathbf{J}}(\theta T\beta, \alpha). \tag{7}$$

For every $y \in (\Gamma(\mathfrak{R}))^{J(\theta T\beta,\alpha)}\langle x_o\rangle$, we can get $v \in \mathfrak{R}$ such that

$$v(x_o)\ Tv(y) \geqslant \hat{\mathbf{J}}(\theta T\beta, \alpha). \tag{8}$$

Moreover, by (CTLUB2), we can find $\lambda \in \mathfrak{R}$ for which $[\underline{\gamma}Tv_o^*(\mathfrak{R})] \leqslant \lambda$. Hence

$$\begin{aligned}\lambda(y) &\geqslant \underline{\gamma}(y)T\{\sup_{z\in X,\mu\in\mathfrak{R}}[v_o(z)T\mu(z)T\mu(y)]\},\\ &\geqslant \gamma\ T[v_o(x_o)Tv(x_o)Tv(y)]\\ &\geqslant \gamma\ T\hat{\mathbf{J}}(\theta T\beta,\alpha)\ T\hat{\mathbf{J}}(\theta T\beta,\alpha), \quad \text{by (7), (8)}\\ &> \hat{\mathbf{J}}(\beta,\alpha),\end{aligned}$$

that is, $y \in \lambda^{J(\beta,\alpha)}$.

Which implies that $(\Gamma(\mathfrak{R}))^{J(\theta T\beta,\alpha)}\langle x_o\rangle \subseteq \lambda^{J(\beta,\alpha)}$, and therefore $\mathfrak{R}^{J(\beta,\alpha)}$ is refined by $((\Gamma(\mathfrak{R}))^{J(\theta T\beta,\alpha)}\langle x\rangle)_{x\in X}$ in $\mathcal{P}(\iota_{u,\alpha}(\Gamma(\mathcal{H})))$.

Conversely, let $(((\Gamma(\mathfrak{R}))^{J(\beta,\alpha)}\langle x\rangle)_{x\in X} \in \mathcal{P}(\iota_{u,\alpha}(\Gamma(\mathcal{H})))$ and $\mu^{J(\beta,\alpha)} \in \iota_{c,\alpha}(\mathfrak{R})$ for some $\mu \in \mathfrak{R}, \beta \in]\alpha, 1]$.

Then choosing $x_o \in \mu^{J(\beta,\alpha)}$, since $\mathfrak{R}$ is a fuzzy covering of X, we have

$$\begin{aligned}(\Gamma(\mathfrak{R}))(x_o, x_o) &= 1, \quad \text{by Lemma 3.4 } (iv)\\ &> \hat{\mathbf{J}}(\beta,\alpha),\ \forall\ \beta \in]\alpha,\ 1].\end{aligned}$$

Hence, $(x_o, x_o) \in (\Gamma(\mathfrak{R}))^{J(\beta,\alpha)}$, that is $x_o \in (\Gamma(\mathfrak{R}))^{J(\beta,\alpha)}\langle x_o\rangle$, therefore

$\mu^{J(\beta,\alpha)} \subseteq (\Gamma(\mathfrak{R}))^{J(\beta,\alpha)}\langle x_o\rangle$.

Which proves that $(((\Gamma(\mathfrak{R}))^{J(\beta,\alpha)}\langle x\rangle)_{x\in X}$ is refined by the element $\mathfrak{R}^{J(\beta,\alpha)}$ in $\iota_{c,\alpha}(\mathcal{H})$.

Proposition 5.1. *If $\mathcal{B}$ is a uniform base on a set X and $\alpha \in I$, then$\omega_{u,\alpha}(\mathcal{B}) = \{(\underline{1-\alpha} \bigvee \mathbf{1}_U) \in I^{X\times X} : U \in \mathcal{B}\}$, is a TL – uniform base on$X$.*

Proof. (TLUB1) Obviously $\omega_{u,\alpha}(\mathcal{B})$ is an I-filterbase, because $\underline{0} \notin \omega_{u,\alpha}(\mathcal{B})$, also $[\underline{1-\alpha} \bigvee \mathbf{1}_U)] \bigwedge [\underline{1-\alpha} \bigvee \mathbf{1}_V] = [\underline{1-\alpha} \bigvee \mathbf{1}_{(U\cap V)}], \forall\ U, V \in \mathcal{B} \geqslant \underline{1-\alpha} \bigvee \mathbf{1}_W$, for some$W \in \mathcal{B}$ with$W \subseteq U \cap V$.(TLUB2) For every $(\underline{1-\alpha} \bigvee \mathbf{1}_U) \in \omega_{u,\alpha}(\mathcal{B})$ and $x \in X$, we have $(x, x) \in U$ and hence $[\underline{1-\alpha} \bigvee \mathbf{1}_U](x, x) = 1$.Now, for every $(\underline{1-\alpha} \bigvee \mathbf{1}_U) \in \omega_{u,\alpha}(\mathcal{B})$ and $\gamma \in I_1$, we can get $W \in \mathcal{B}$ with

(W o W) $\subseteq_s$U which implies, for every $x, y \in X$, that

$$\begin{aligned}&\{\underline{\gamma}T[(\underline{1-\alpha} \bigvee \mathbf{1}_W) o_T (\underline{1-\alpha} \bigvee \mathbf{1}_W)]\}(x, y)\\ &= \gamma T\sup_{z\in X}[(\underline{1-\alpha} \bigvee \mathbf{1}_W)(x, z)T(\underline{1-\alpha} \bigvee \mathbf{1}_W)(z, y)]\\ &= \gamma T\sup_{z\in X}\{\underline{1-\alpha} \bigvee [\mathbf{1}_W(x, z)T\mathbf{1}_W(z, y)]\}\\ &\leqslant \sup_{z\in X}\{\underline{1-\alpha} \bigvee [\mathbf{1}_W(x, z) \bigwedge \mathbf{1}_W(z, y)]\}\\ &\leqslant [\underline{1-\alpha} \bigvee_s \mathbf{1}_U](x, y)\\ &= {}_s[\underline{1-\alpha} \bigvee \mathbf{1}_U](x, y)\end{aligned}$$

This obviously, rendering both (TLUB3) and (TLUB4).

Which completes the proof. □

Theorem 5.2. *If $\mathcal{E}$ is a covering uniform base on a set X and $\alpha \in I$, then both $\omega_{c,\alpha}(\mathcal{E})$ and $\sigma(\omega_{u,\alpha}(\mathcal{Q}(\mathcal{E})))$ are basis for the same covering TL-uniformity $\mathcal{K}$ on X.*

Proof. We already know that $\sigma(\omega_{u,\alpha}(\mathcal{Q}(\mathcal{E})))$ is a basis for a covering TL-uniformity $\mathcal{K}$, moreover $\omega_{c,\alpha}(\mathcal{E}) \subset F_c$(X) is trivial.

From above definitions, we can write for simply

$\sigma(\omega_{u,\alpha}(\mathcal{Q}(\mathcal{E}))) = \{((\eta(\Lambda))\langle \mathbf{1}_x\rangle_T)_{x\in X} : \Lambda \in \mathcal{E}\}$,

where $\eta(\Lambda) = \bigvee_{H\in\Lambda}(\underline{1-\alpha} \bigvee \mathbf{1}_{H\times H})$.

Now, suppose that $\Lambda \in \mathcal{E}$ and $M \in \Lambda$, by choosing $x_o \in M$, we have, for every $y \in X$,

$$\begin{aligned}(\underline{1-\alpha} \bigvee \mathbf{1}_M)(y) &= (\underline{1-\alpha} \bigvee \mathbf{1}_{M\times M})(x_o, y), \quad \text{because } \mathbf{1}_M(x_o) = 1\\ &= ((\underline{1-\alpha} \bigvee \mathbf{1}_{M\times M})\langle \mathbf{1}_{x_o}\rangle_T)(y), \text{ by (2)}\\ &\leqslant ([\bigvee_{H\in\Lambda} (\underline{1-\alpha} \bigvee \mathbf{1}_{H\times H})]\langle \mathbf{1}_{x_o}\rangle_T)(y).\end{aligned}$$

This shows the existence of an element $[\bigvee_{H\in\Lambda} (\underline{1-\alpha} \bigvee \mathbf{1}_{H\times H})]\langle \mathbf{1}_{x_o}\rangle_T$ of $((\eta(\Lambda))\langle \mathbf{1}_x\rangle_T)_{x\in X}$ which greater or equal to $(\underline{1-\alpha} \bigvee \mathbf{1}_M)$.

Therefore, $\omega_{c,\alpha}(\Lambda) \ll ((\eta(\Lambda))\langle \mathbf{1}_x\rangle_T)_{x\in X}$, from which it follows that $\sigma(\omega_{u,\alpha}(\mathcal{Q}(\mathcal{E}))) \subseteq \omega_{c,\alpha}(\mathcal{E})$.

On the other hand, let $\Lambda \in \mathcal{E}$ and $x_o \in X$. Now if any $y \in X$, we get $([\bigvee_{H\in\Lambda}(\underline{1-\alpha} \bigvee \mathbf{1}_{H\times H})]\langle \mathbf{1}_{x_o}\rangle_T)(y) = 1$, i.e., (by (2)), $[\bigvee_{H\in\Lambda}(\underline{1-\alpha} \bigvee \mathbf{1}_{H\times H})](x_o, y) = 1$, then, there is $M \in \Lambda$ in such a way that $\{x_o, y\} \subseteq M$.

Consequently $(\underline{1-\alpha} \bigvee \mathbf{1}_M)(y) \geqslant (\mathbf{1}_M)(y) = 1$, which shows the existence of an element $(\underline{1-\alpha} \bigvee \mathbf{1}_M)$ of $\omega_{c,\alpha}(\Lambda)$ which greater or equal to $[\bigvee_{H\in\Lambda}(\underline{1-\alpha} \bigvee \mathbf{1}_{H\times H})]\langle \mathbf{1}_{x_o}\rangle_T$.

Therefore $((\eta(\Lambda))\langle \mathbf{1}_x\rangle_T)_{x\in X} \ll \omega_{c,\alpha}(\Lambda)$, from which it follows that $\omega_{c,\alpha}(\Lambda) \subseteq \sigma(\omega_{u,\alpha}(\mathcal{Q}(\mathcal{E})))$.

Hence, the theorem is follows from Proposition 4.2. □

Now, if $\alpha \in I_1$, and we define the map $\iota^{\sim}_{c,\alpha}: CTL\text{-US} \to \text{CUS}$ by setting $\iota^{\sim}_{c,\alpha}(X, \mathcal{K}) = (X, \iota_{c,\alpha}(\mathcal{K}))$ and $\iota^{\sim}_{c,\alpha}(f) = f$, we get $\iota^{\sim}_{c,\alpha}$ is well defined functor.

Also, if $\alpha \in I$, and we define the map $\omega^{\sim}_{c,\alpha}$: $\text{CUS} \to CTL\text{-US}$ by setting $\omega^{\sim}_{c,\alpha}(X, \mathcal{C}) = (X, \omega_{c,\alpha}(\mathcal{C}))$ and $\omega^{\sim}_{c,\alpha}(f) = f$, we get $\omega^{\sim}_{c,\alpha}$ is well defined functor.

Proposition 5.2. *For the functors defined above, we get the following relations:*

(i) *For all* $\alpha \in I_1$, *we have* $\iota^{\sim}_{c,\alpha} = \mathcal{P}^{\sim} o \iota^{\sim}_{u,\alpha} o \Gamma^{\sim}$ *and* $\iota^{\sim}_{u,\alpha} = \mathcal{Q}^{\sim} o \iota^{\sim}_{c,\alpha} o \sigma^{\sim}$;consequently, $\mathcal{Q}^{\sim}$ o $\iota^{\sim}_{c,\alpha} = \iota^{\sim}_{u,\alpha}$ o $\Gamma^{\sim}$ and $\iota^{\sim}_{c,\alpha}$ o $\sigma^{\sim} = \mathcal{P}^{\sim}$ o $\iota^{\sim}_{u,\alpha}$.

(ii) *For all* $\alpha \in I$, *we have* $\omega^{\sim}_{c,\alpha} = \sigma^{\sim}$ *o* $\omega^{\sim}_{u,\alpha}$ *o* $\mathcal{Q}^{\sim}$ *and* $\omega^{\sim}_{u,\alpha} = \Gamma^{\sim}$ *o* $\omega^{\sim}_{c,\alpha}$ *o* $\mathcal{P}^{\sim}$;therefore $\Gamma^{\sim}$ o $\omega^{\sim}_{c,\alpha} = \omega^{\sim}_{u,\alpha}$ o $\mathcal{Q}^{\sim}$ and $\sigma^{\sim}$ o $\omega^{\sim}_{u,\alpha} = \omega^{\sim}_{c,\alpha}$ o $\mathcal{P}^{\sim}$.The proof follows immediately.

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