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ORIGINAL ARTICLE

Traveling waves for a dissipative modified KdV equation

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Abstract In this paper we consider a dispersive–dissipative nonlinear equation which can be regarded as a dissipation perturbed modified KdV equation, governing the evolution of long waves in an elastic rod immersed inside a viscoelastic medium. Using geometric singular perturbation theory, a construction of traveling waves for the equation is shown. This also is illustrated by presenting some numerical calculations.

1. Introduction

When attempting to describing the propagation of small-amplitude long waves in nonlinear dispersive media, it is frequently necessary to take account of dissipative mechanisms to accurately reflect real situations. In such cases one may consider the following equation

$$\frac{\partial u}{\partial t} + \alpha u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial^3 u}{\partial x^3} + \tau \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right) + \delta \frac{\partial^4 u}{\partial x^4} = 0, \quad (1.1)$$

as model for long wave propagation in nonlinear media with dispersion, dissipation and backward quadratic diffusion. Here, α, β, γ, τ and δ are constants. This model equation arises in many physical systems and describes weakly waves with certain dissipative effects. The coefficients in (1.1) depend upon parameters of the system. Similar equations [1–6] for (1.1) ware studied by using the asymptotic expansion methods and exact solutions, describing both the evolution of a solitary wave and kink-shaped waves, were obtained. In this paper we consider (1.1) as dissipation perturbed modified KdV equation. We assume that all coefficients of the dissipative terms in (1.1) are small relative to the other coefficients, i.e., $\beta = \epsilon b,\ \tau = \epsilon r,\ \delta = \epsilon s, \epsilon \ll 1$. Then (1.1) may be written as

$$\frac{\partial u}{\partial t} + \alpha u^2 \frac{\partial u}{\partial x} + \gamma \frac{\partial^3 u}{\partial x^3} + \epsilon\left(b \frac{\partial^2 u}{\partial x^2} + r \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right) + s \frac{\partial^4 u}{\partial x^4}\right) = 0. \quad (1.2)$$

With the methods of geometric singular perturbation theory as attractive methods, developed in [7,8] and used in [9–12], we provide a geometric construction of the traveling solitary wave solutions for (1.2).

The paper is organized as follows. In Section 2, we present preliminaries. In Section 3, we describe how geometric singular perturbation theory is used to construct a locally invariant manifold for the traveling wave equation when $\epsilon > 0$. In Section 4 we use this manifold to obtain a traveling solitary wave solution. In Section 5 we present some numerical calculations. Section 6 contains a brief conclusion.

2. Preliminaries

In traveling wave form, with $u(x, t) = u(z)$, $z = x - ct$, and $c \geqslant 0$ without loss of generality, Eq. (1.2), after one integration and setting the integration constant equal to zero, reads

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$$-cu + \frac{\alpha}{3}u^3 + \gamma\frac{d^2u}{dz^2} + \epsilon\left(b\frac{du}{dz} + ru\frac{du}{dz} + s\frac{d^3u}{dz^3}\right) = 0. \tag{2.1}$$

Here, the imposed boundary conditions u, du/dz, d^2u/dz^2, $d^3u/dz^3 \to 0$ as $z \to \infty$ which describe the traveling solitary wave imply the integration constants are zero. Clearly, for practical applications, we are interested only in real bounded solution $u(z)$ of (2.1) which can be written as a system of first order equations

$$\begin{aligned}\frac{du}{dz} &= v,\\ \frac{dv}{dz} &= w,\\ s\epsilon\frac{dw}{dz} &= cu - \frac{\alpha}{3}u^3 - b\epsilon v - r\epsilon uv - \gamma w.\end{aligned} \tag{2.2}$$

Note that if $\epsilon = 0$, then (2.2) reduces to

$$\begin{aligned}\frac{du}{dz} &= v,\\ \frac{dv}{dz} &= \frac{1}{\gamma}\left(cu - \frac{\alpha}{3}u^3\right),\end{aligned}$$

which is the dynamical system of ODEs for solitary wave solutions of the modified KdV equation.

When ϵ is non-zero, Eq. (2.2) defines a dynamical system of ODEs whose solutions evolve in the three-dimensional (u, v, w) phase space. In this phase space, there are critical points at

$$(u, v, w) = (0, 0, 0) \quad \text{and} \quad (u_e, 0, 0),$$

where $u_e = \pm\sqrt{3c/\alpha}$ and these equilibria are independent of ϵ.

A traveling wave solution of the original equation will exist if among the solutions of (2.2), there exists a homoclinic (heteroclinic) orbit. The plausibility of this can be seen by a preliminary calculation of the dimension of the stable and unstable manifolds of the critical point $(0,0,0)$. The linearized matrix J of system (2.2) is

$$J(u, v, p, q) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{s\epsilon}(c - \alpha u^2 - r\epsilon v) & -\frac{1}{s\epsilon}(b\epsilon + r\epsilon u) & -\frac{\gamma}{s\epsilon} \end{pmatrix}.$$

At the steady state $(0,0,0)$, the eigenvalues λ of this matrix satisfy

$$\epsilon\lambda^3 + \frac{r}{s}\lambda^2 + \frac{\epsilon b}{s}\lambda - \frac{c}{s} = 0.$$

For ϵ sufficiently small it is easily seen that this equation has two real negative roots, and one real positive root. Hence the dimension of the stable manifold of the steady state $(0,0,0)$ is two and the dimension of the unstable manifold is one. However, this does not rigorously establish the existence of a homoclinic (heteroclinic) orbit, but it does lend plausibility to the idea that two manifolds might intersect along a one-dimensional curve in R^3. The existence of a homoclinic (heteroclinic) orbit will now be confirmed by showing the existence of a two-dimensional invariant manifolds for (2.2) and analyzing the system reduced to this manifold.

3. Existence of an invariant manifold

We note that when $\epsilon = 0$, system (2.2) does not define a dynamical system in R^3. This problem may be overcome by the transformation $z = \epsilon\zeta$, under which the system becomes

$$\begin{aligned}\frac{du}{d\zeta} &= \epsilon v,\\ \frac{dv}{d\zeta} &= \epsilon w,\\ s\frac{dw}{d\zeta} &= cu - \frac{\alpha}{3}u^3 - b\epsilon v - r\epsilon uv - \gamma w.\end{aligned} \tag{3.1}$$

While the two systems are equivalent for $\epsilon > 0$, the different time-scales give rise to different limiting systems. Letting $\epsilon \to 0$ in (2.2), we obtain

$$\begin{aligned}\frac{du}{dz} &= v,\\ \frac{dv}{dz} &= w,\\ 0 &= cu - \frac{\alpha}{3}u^3 - \gamma w.\end{aligned} \tag{3.2}$$

Thus the flow of system (3.2) is confined to the set

$$M_0 = \left\{(u, v, w) \in R^3 : cu - \frac{\alpha}{3}u^3 - \gamma w = 0\right\} \tag{3.3}$$

and its dynamics are determined by the first two equations only. On the other hand, setting $\epsilon \to 0$ in (3.1) yields the system

$$\begin{aligned}\frac{du}{d\zeta} &= 0,\\ \frac{dv}{d\zeta} &= 0,\\ \frac{dw}{d\zeta} &= cu - \frac{\alpha}{3}u^3 - \gamma w.\end{aligned} \tag{3.4}$$

Any points in M_0 are the equilibria of system (3.4). Generally, system (2.2) is referred to as the slow system, since the time-scale z is slow, and (3.1) is referred to as the fast system, since the time-scale ζ is fast. M_0 is the slow manifold.

If M_0 is normally hyperbolic, then the geometric singular perturbation theory of Fenichel [7] applies and provides us with a two-dimensional invariant manifold M_ϵ for the flow when $\epsilon > 0$. The idea is then to study the flow of (2.2) restricted to this manifold, and the resulting system will be two-dimensional. This does not in itself establish the existence of a traveling wave, we still have to study the system reduced to M_ϵ and show it possesses a homoclinic (heteroclinic) orbit.

From Fenichel [7], M_0 is a normally hyperbolic manifold if the linearization of the fast system (3.1), restricted to M_0, has exactly dim M_0 eigenvalues with zero real part. The linearization of the fast system, restricted to M_0, has the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c - \alpha u & 0 & -\gamma \end{pmatrix}$$

and the eigenvalues of which are $0, 0, -\gamma$. Thus, M_0 is normally hyperbolic and the geometric singular perturbation theory implies that there exists a two-dimensional manifold M_ϵ for $\epsilon > 0$. To determine M_ϵ explicitly, we have

$$M_\epsilon = \left\{(u, v, w) \in R^3 : \ w = h(u, v, \epsilon) + \frac{1}{\gamma}\left(cu - \frac{\alpha}{3}u^3\right)\right\}, \tag{3.5}$$

where the function h is to be determined and satisfies

$$h(u, v, 0) = 0. \tag{3.6}$$

By substituting into the slow system (2.2), we see that h must satisfy

$$s\epsilon\left[v\frac{\partial h}{\partial u}+\frac{\partial h}{\partial v}\left(h+\frac{1}{\gamma}\left(cu-\frac{\alpha}{3}u^3\right)\right)+\frac{1}{\gamma}(c-\alpha u^2)v\right]=-b\epsilon v-r\epsilon uv-\gamma h.$$

Since ϵ is small, we attempt solutions of this partial differential equation in the form of regular perturbation expansion in ϵ. Since h is zero when $\epsilon = 0$, we set

$$h(u,v,\epsilon) = \epsilon h_1(u,v) + O(\epsilon^2). \tag{3.7}$$

Substituting $h(u, v, \epsilon)$ into the above equation and setting the coefficient of ϵ to zero, we obtain

$$h_1(u,v) = \frac{1}{\gamma^2}((\alpha su - \gamma r)u - (b\gamma + sc))v. \tag{3.8}$$

Therefore, the slow system (2.2) restricted to M_ϵ is given by

$$\frac{du}{dz} = v,$$
$$\frac{dv}{dz} = \frac{1}{\gamma}\left(cu - \frac{\alpha}{3}u^3\right) + \epsilon\left(\frac{1}{\gamma^2}((\alpha su - \gamma r)u - (b\gamma + sc))v\right) + O(\epsilon^2). \tag{3.9}$$

4. The flow on the manifold M_ϵ

Note that when $\epsilon = 0$, this system reduce to the corresponding system for the mKdV equation. We now show, for $\epsilon > 0$ sufficiently small, that a homoclinic orbit for (3.9) exists. For this we use the argument of Melnikov function [13,14]. For $\epsilon = 0$, the homoclinic orbit $q_h(z)$ is given by

$$u(z) = \mu\ sech(kz), \quad \mu = \sqrt{\frac{6c}{\alpha}}, \quad k = \sqrt{\frac{c}{\gamma}}$$

and

$$v(z) = \frac{du}{dz} = -\mu k\ sech(kz)\ tanh(kz),$$

Thus, the Melnikov function is described as

$$\begin{aligned} M(c,z_0) &= \int_{-\infty}^{\infty} f(q_h(z)) \wedge g(q_h(z), z+z_0))dz \\ &= \int_{-\infty}^{\infty} \begin{pmatrix} v \\ \frac{1}{\gamma}\left(cu - \frac{\alpha}{3}u^3\right) \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \frac{1}{\gamma^2}((\alpha su - \gamma r)u - (b\gamma + sc))v \end{pmatrix} dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\gamma^2}\left(\alpha su^2v^2 - \gamma ruv^2 - (b\gamma + sc)v^2\right)dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\gamma^2}\left(\alpha s\mu^4k^2\ sech^4(kz)tanh^2(kz)\right. \\ &\quad \left. -\gamma r\mu^3k^2\ sech^3(kz)tanh^2(kz)\right) \\ &\quad - \int_{-\infty}^{\infty} \frac{1}{\gamma^2}\left((b\gamma + sc)\mu^2k^2\ sech^4(kz)tanh^2(kz)\right)dz \\ &= \frac{1}{120\gamma}\left(32\alpha s\mu^4k - 15\pi\gamma r\mu^3k - 80(b\gamma + sc)\mu^2k\right) \end{aligned} \tag{4.1}$$

where the wedge operator $\wedge$ is defined as $f \wedge g = f_1\, g_2 - f_2 g_1$. Eq. (4.1) shows that the melnikov function $M(c)$ has a unique zero and hence the solvability condition for the existence of a homoclinic orbit for ϵ sufficiently small. This homoclinic orbit corresponds to solitary wave solution for (1.2). Therefore, we arrive at the following theorem.

Theorem 4.1. *For* $\epsilon > 0$ *sufficiently small*, Eq. (1.2) *admits a traveling solitary wave solution* $u(x, t) = u(z)$, $z = x - ct$, $c > 0$.

5. Numerical results

In this section we illustrate the above analysis by presenting some numerical calculations. We solve (2.2) as an initial-value problem. The initial condition is approximated by a point on the unstable manifold of the steady state $(0,0,0)$ of (2.2). The results of numerical solution using an adaptive step Runge–Kutta scheme of order fourth are shown in Figs. 1–4. Fig. 1 shows the graph of the projection of a homoclinic orbit corresponding to solitary wave solutions for $\alpha = \gamma = 2.0$, $b = -1.0$, $r = s = 1.0$, $c = 1.0$ and $\epsilon = 0.001$ while Fig. 2 shows the graph of the projection of a homoclinic orbit for large $\epsilon = 0.1$ with the same values as in Fig. 1. Clearly, both

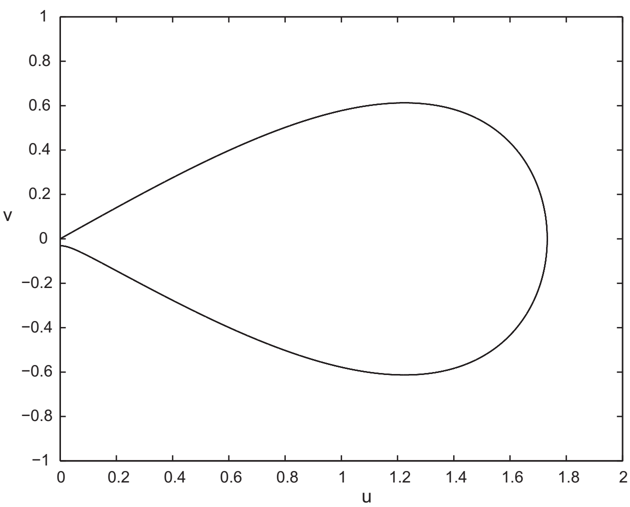

Figure 1 The dynamics of solitary waves. Sketch of the graph of the projection of the homoclinic orbit in the three-dimensional phase space onto the u–v-plane for small ϵ, corresponding to solitary wave.

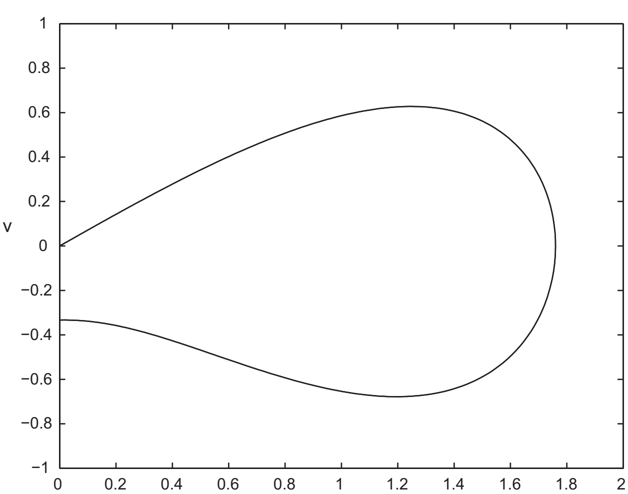

Figure 2 Sketch of the graph of the projection of the homoclinic orbit in the three-dimensional phase space onto the u–v-plane for large ϵ. Clearly the homoclinic orbit breaks when ϵ becomes larger.

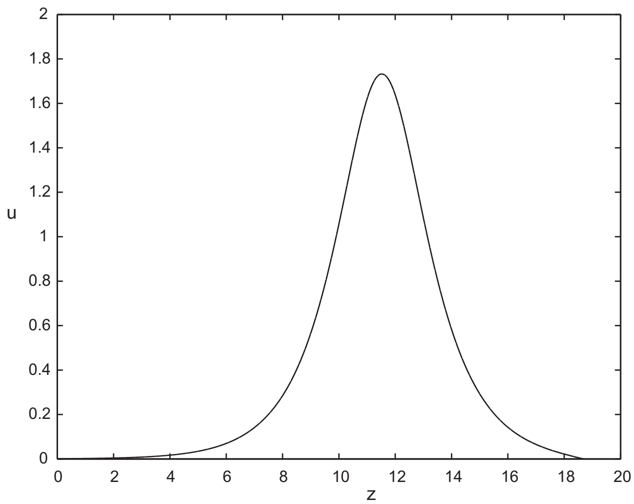

Figure 3 Sketch of the graph of traveling solitary wave solution, corresponding to the homoclinic orbit shown in Fig. 1.

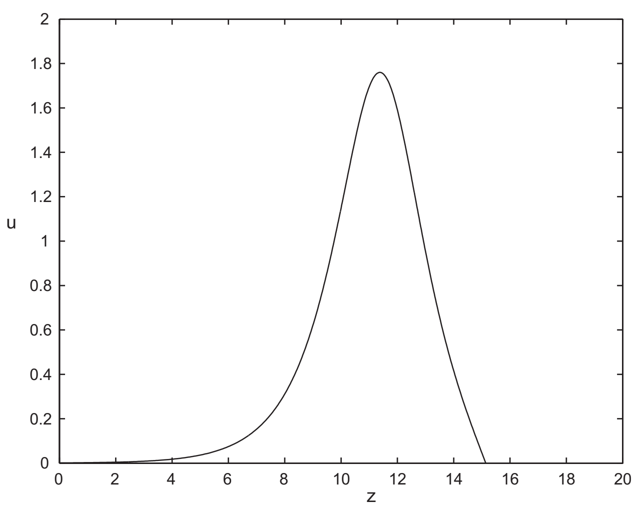

Figure 4 This figure shows the graph of traveling solitary wave solution corresponding to the homoclinic orbit shown in Fig. 2, when ϵ becomes larger.

Figs. 1 and 2 show the dynamics of solitary waves. Figs. 3 and 4 show the graph of the corresponding traveling solitary wave solutions.

Moreover, we note that for b positive, kink (oscillatory kink) type waves also exist on the two-dimensional manifold within the three-dimensional phase space. These waves correspond to heteroclinic orbits connecting the critical points $(0,0,0)$ and $(u_e,0,0)$, see Fig. 5, and no other waves exist.

6. Conclusion

In this paper we have considered a dispersive–dissipative nonlinear model equation which can be regarded as a dissipation perturbed modified KdV equation. Using dynamical systems theory, specifically geometric singular perturbation theory as attractive methods, we have constructed traveling wave solutions for the equation. For this, we have shown that the traveling waves exist on a two-dimensional slow manifold within the resulting higher-dimensional system. We have proved persistence of the slow manifold under perturbation, and then we have constructed the wave as homoclinic (or heteroclinic) orbit in the transverse intersection of appropriate stable and unstable manifolds in this slow manifold. Further, we have presented some numerical calculations by solving an initial-value problem, showing approximations for such solitary waves as well as oscillatory kink waves.

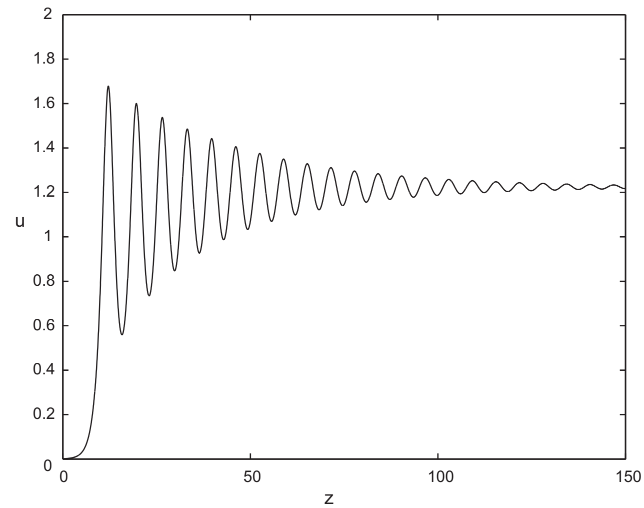

Figure 5 Sketch of the graph of an oscillatory kink type wave.

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