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ORIGINAL ARTICLE

Effect of atomic decay on purity, total correlations and entanglement of pure and mixed states

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Abstract The atomic decay for a two level atom interacting with a single mode of electromagnetic field is considered. In particular for a coherent state or statistical mixture (SM) of two opposite coherent states as initial field states, the exact solution of the master equation is found. Effect of the atomic damping on the partial entropies of the atom or the field and the total entropy as a measures of the purity loss is investigated. The degree of entanglement by the negativity and the mutual information and the atomic coherence through the master equation is studied.

1. Introduction

The study of quantum correlations is a key issue in quantum information theory [1] and quantum entanglement represents the indispensable physical resource for the description and performance of quantum information processing tasks, like quantum teleportation, cryptography, superdense coding and quantum computation [2]. Thus a great deal of efforts have been devoted to study and characterizing entanglement in the recent years. The central task of quantum information theory is to characterize and quantify entanglement of a given system. On the other hand, for the realistic quantum systems, they will interact with the environments inevitably. The interaction between the system and the environment usually leads to a decoherence process [3,4]. This is one fundamental obstacle to perform quantum computation. Unfortunately, decoherence destroys the quantumness of the system and hence will decreases the useful entanglement between parts of the system [5–7]. There are several approaches to consider decoherence in a quantum system which is responsible for quantum–classical transition. One of these approaches is based on modifying master equation in such a way that the quantum coherence be automatically destroyed as the system evolves.

The interaction between radiation and matter is a central problem in quantum optics. The simplest physical situation can be successfully described by a rather simplified but nontrivial model is popularly known as the Jaynes–Cummings (JC) model [8], which describing the interaction between a single two-level atomic system and a quantized radiation mode. Stimulated by the success of the JC model, more and more people have paid special attention to extending and generalizing the model in order to explore a new quantum effects [9]. Such a generalization is of considerable interest because of its relevance to the study of the nonlinear coupling between a two-level atom and the radiation field [10–13].

In the last two decades, there are more studies has been focused on a dissipative effects of JC model [14–23]. Also there

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are exists a theoretical motivation to include relevant damping mechanism to JC model because its dynamics becomes more interesting. Because of the damping of the field is larger than the damping of the atom many authors [18–21] neglect the damping of the atom. Also there are many studies that focused on the properties of the entanglement [22–28] neglect the damping of the atom. But the entanglement induced by the atomic damping of a dispersive reservoir γ_P dissected in [29]. Therefore, in our work, one shall show that the atomic damping γ leads to losing the purity, decreasing both of the entanglement and total correlations, and disappearance of the interference between atomic states for a nonlinear qubit-field system when the field initially is a coherent state or statistical mixture (*SM*) of two coherent states.

To measure the purity, total correlations, and entanglement, one uses the von Neumann reduced entropy, the mutual information [30], the sum of negative eigenvalues of the partially transposed density matrix [31]. The main objective of this paper is to present the exact solution of the master equation in the case of a high-Q cavity with atomic damping through the dressed-state representation. Therefore, the effects of atomic damping on the purity, total correlation, entanglement and the atomic coherence are studied.

2. Solution of the master equation for the atomic damping

Here, one considers a qubit interacting nonlinearly with a quantum harmonic oscillator. The nonlinear qubit-field system [32] described in the interaction picture by

$$\widehat{H} = \omega\left(\frac{\hat{\sigma}_z}{2} + \hat{\psi}^\dagger\hat{\psi}\right) + \lambda\left[\sqrt{\hat{\psi}^\dagger\hat{\psi}}\hat{\psi}^\dagger\sigma_- + \hat{\psi}\sqrt{\hat{\psi}^\dagger\hat{\psi}}\sigma_+\right], \tag{1}$$

where $\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_- = |g\rangle\langle e|$ with $|e\rangle$ and $|g\rangle$ being the excited and ground states of the atom of transition frequency ω. This the atom interacting resonantly with the field which is described by the bosonic operators $\hat{\psi}^\dagger$ and $\hat{\psi}$, the creation and annihilation operator, respectively.

In order to describe dissipation, one uses the master equation, which in the interaction picture governs the dynamics of a two-level atom coupled with an electromagnetic field as [16,17]

$$\begin{aligned}\frac{\partial\hat{\rho}(t)}{\partial t} = & -i[\widehat{H},\hat{\rho}(t)] + \gamma([\sigma_-,\rho\sigma_+] + [\sigma_-\rho,\sigma_+]) \\ & + \gamma_F([\hat{\psi}\hat{\rho},\hat{\psi}^\dagger] + [\hat{\psi},\hat{\rho}\hat{\psi}^\dagger]) + \frac{1}{2}\gamma_P([\sigma_z,\rho\sigma_z] + [\sigma_z\rho,\sigma_z]).\end{aligned} \tag{2}$$

where the damping of the atom and the field are treated separately, with γ and γ_F being their respective decay constants. A dephasing of the atomic coherence is the last term which is usually written as $\gamma_P(\sigma_z\rho\sigma_z - \rho)$. The environment is assumed to be at zero temperature, hence no thermic excitation is included. The master Eq. (2) conserves $tr\rho = 1$ and guarantees that the density operator remains a Hermitian and positive operator as shown by Lindblad [15]. If γ_F and γ_P are neglected, Eq. (2) can be solved by analytic method in the case of a high-Q cavity ($\gamma \langle\langle \lambda$), in which the so-called dressed-states representation [18], i.e. representation consisting of the complete set of eigenstates of the Hamiltonian is used. These eigenstates of the Hamiltonian is given by

$$|\beta_n^\pm\rangle = \frac{1}{\sqrt{2}}(|e,n\rangle \pm |g,n+1\rangle) \quad (n = 0,1,2,\ldots), \tag{3}$$

with

$$\begin{aligned}&\widehat{H}|g,0\rangle = -\frac{\omega}{2}, \quad \widehat{H}|\beta_n^\pm\rangle = E_n^\pm|\beta_n^\pm\rangle, \\ &E_n^\pm = \omega\left(n+\frac{1}{2}\right) \pm \lambda(n+1) = \omega\left(n+\frac{1}{2}\right) \pm \mu_n.\end{aligned} \tag{4}$$

To derive the equation in a dressed-states representation, one first writes the atomic operators appearing in Eq. (2) in terms of the dressed-states and using the following representation

$$\dot{x}(t) = e^{i\widehat{H}t}\frac{\partial\hat{\rho}(t)}{\partial t}e^{-i\widehat{H}t} + i[\hat{H},x(t)], \tag{5}$$

one can rewrite the master equation Eq. (2) in the form

$$\begin{aligned}\dot{x}(t) = & \gamma(|g,0\rangle\langle\beta_0^+|x|\beta_0^+\rangle\langle g,0| + |g,0\rangle\langle\beta_0^-|x|\beta_0^-\rangle\langle g,0|) \\ & + \frac{\gamma}{2}\sum_{n=1}^{\infty}[|\beta_{n-1}^+\rangle\langle\beta_n^+|x|\beta_n^+\rangle\langle\beta_{n-1}^+| + |\beta_{n-1}^-\rangle\langle\beta_n^-|x|\beta_n^-\rangle \\ & \langle\beta_{n-1}^-| + |\beta_{n-1}^+\rangle\langle\beta_n^-|x|\beta_n^-\rangle\langle\beta_{n-1}^+| + |\beta_{n-1}^-\rangle\langle\beta_n^+|x|\beta_n^+\rangle\langle\beta_{n-1}^-|] \\ & - \frac{\gamma}{2}\sum_{n=0}^{\infty}\left(|\beta_n^+\rangle\langle\beta_n^+| + |\beta_n^-\rangle\langle\beta_n^-|\right)x + x\left(|\beta_n^+\rangle\langle\beta_n^+| + |\beta_n^-\rangle\langle\beta_n^-|\right).\end{aligned} \tag{6}$$

In this work, one investigates the effect of atomic damping on the temporal evolution of various measures of entanglement, and suppose that the atom is initially in the excited state, i.e., $\hat{\rho}_a(0) = |e\rangle\langle e|$, and the field density operator is assumed to be initially in a coherent state, i.e.,

$$\hat{\rho}^f(0) = |\alpha\rangle\langle\alpha|, \quad |\alpha\rangle = \sum_{n=0}^{\infty}q_n|n\rangle = \sum_{n=0}^{\infty}e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{7}$$

As well know that when dissipation is included, the interference terms in the density matrix of state (7) rapidly decay to a statistical mixture *SM* of the coherent states, so one wants to see the behavior of the system if the input states are statistical mixture of the states $|\alpha\rangle$ and $|-\alpha\rangle$, i.e.,

$$\hat{\rho}^f_{SM}(0) = \frac{1}{2}(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|), \tag{8}$$

Therefore, the initial density matrix $x(0)$ for the system in the dressed-states representation is given by

$$x(0) = \frac{1}{2}\sum_{l,j=0}^{\infty}\pi_{l,j}\left(|\beta_l^+\rangle\langle\beta_j^+| + |\beta_l^-\rangle\langle\beta_j^-| + |\beta_l^+\rangle\langle\beta_j^-| + |\beta_l^-\rangle\langle\beta_j^+|\right), \tag{9}$$

where $\pi_{l,j} = q_lq_j^*$ for coherent state case and $\pi_{l,j} = q_lq_j^*[1+(-1)^{(l+j)}]$ for *SM* of the coherent states case. The solution of master equation in the high-Q limit with pervious initial states is given by:

$$\hat{\rho}(t) = \sum_{m,n=e,g}\sum_{l,j=0}\rho_{l,j}^{mn}(t)|\varphi_l^m\rangle\langle\varphi_j^n|, \tag{10}$$

where $|\varphi_n^g\rangle = |n+1,g\rangle$, $|\varphi_n^e\rangle = |n,e\rangle$, $\rho_{l,j}^{mn}(t) = \left\langle\varphi_l^m|e^{-i\widehat{H}t}x(t)e^{i\widehat{H}t}|\varphi_j^n\right\rangle$ and $\bar{n} = |\alpha|^2$. The coherence properties and entanglement will be study by use the eigenvalues of the a pervious density matrix $\rho(t)$ and its marginal density matrices of the field and atom $\rho_{F(A)}(t)$ and its partially transposed density matrix. In the following section one uses this the eigenvalues to study the dynamical properties of entropies and entanglement for atomic damping JC model.

3. Measures of the entanglement and the interference

To study the purity, entanglement, total correlations and the interference between states $|e\rangle$ and $|g\rangle$, we offer the following measures;

One firstly investigates the purity loss of the system states caused by the atomic-damping reservoir, by using the von Neumann entropy $\mathcal{E}_T = -\sum_{i=1}^{\infty}\lambda_i^S\, ln(\lambda_i^S)$, where λ_i^S are the eigenvalues of the density matrix of the system $\hat{\rho}(t)$. Also another measure to quantify the entanglement in states (10), it is based on the trace norm of the partial transpose $\hat{\rho}^T$ of the bipartite mixed state $\hat{\rho}$, which define as [33]

$$\mathcal{E}_N = \frac{\|\hat{\rho}^T\| - 1}{2}, \tag{11}$$

which corresponds to the absolute value of the sum of negative eigenvalues of $\hat{\rho}^T$, and which vanishes for unentangled states. As one shall prove here, it is an entanglement monotone, and as such it can be used to quantify the degree of the entanglement in composite systems.

The reduced entropy theory about interaction of the field with the atom has been introduced by [34,35]. They have shown that the reduced entropy is very useful and is a sensitive operational measure of the purity of the quantum state, which automatically includes all the moments of the density operator. The purity loss of the field states investigate by use the field entropy $\mathcal{E}_F$, which defined as: $\mathcal{E}_F = -\sum_{i=1}^{\infty}\lambda_i^F\, ln(\lambda_i^F)$. Here the eigenvalues λ_i^F of the reduced field density matrix $\hat{\rho}_F(t)$ are computed by using a numerical calculations. The atomic reduced density operator $\hat{\rho}_A(t)$ is given by $\hat{\rho}_A(t) = Tr_F\{\hat{\rho}(t)\}$, and then, the eigenvalues $\lambda_{1,2}^A$ for $\hat{\rho}_A(t)$ are

$$\lambda_{1,2}^A = \frac{1}{2}\left\{1 \pm \sqrt{\left(\sum_{i=1}\rho_i^{ee}(t) - \sum_{i=1}\rho_i^{gg}(t)\right)^2 + 4\left|\sum_{i=1}\rho_i^{eg}(t)\right|^2}\right\}. \tag{12}$$

So to see the temporal evolution of the purity loss of atomic states, one uses the atomic entropy $\mathcal{E}_A$: $\mathcal{E}_A = -\lambda_1^A\, ln(\lambda_1^A) - \lambda_2^A\, ln(\lambda_2^A)$. If $\mathcal{E}_A(\mathcal{E}_F) = 0$, then the states are separable states. When the initial state is factored with both the atom and radiation taken to be pure states as is often done in the literature [34,35], these two entropies must be the same, as guaranteed by the Araki-Lieb theorem [36]. In the present case where the initial state is also taken to be pure states, but the system

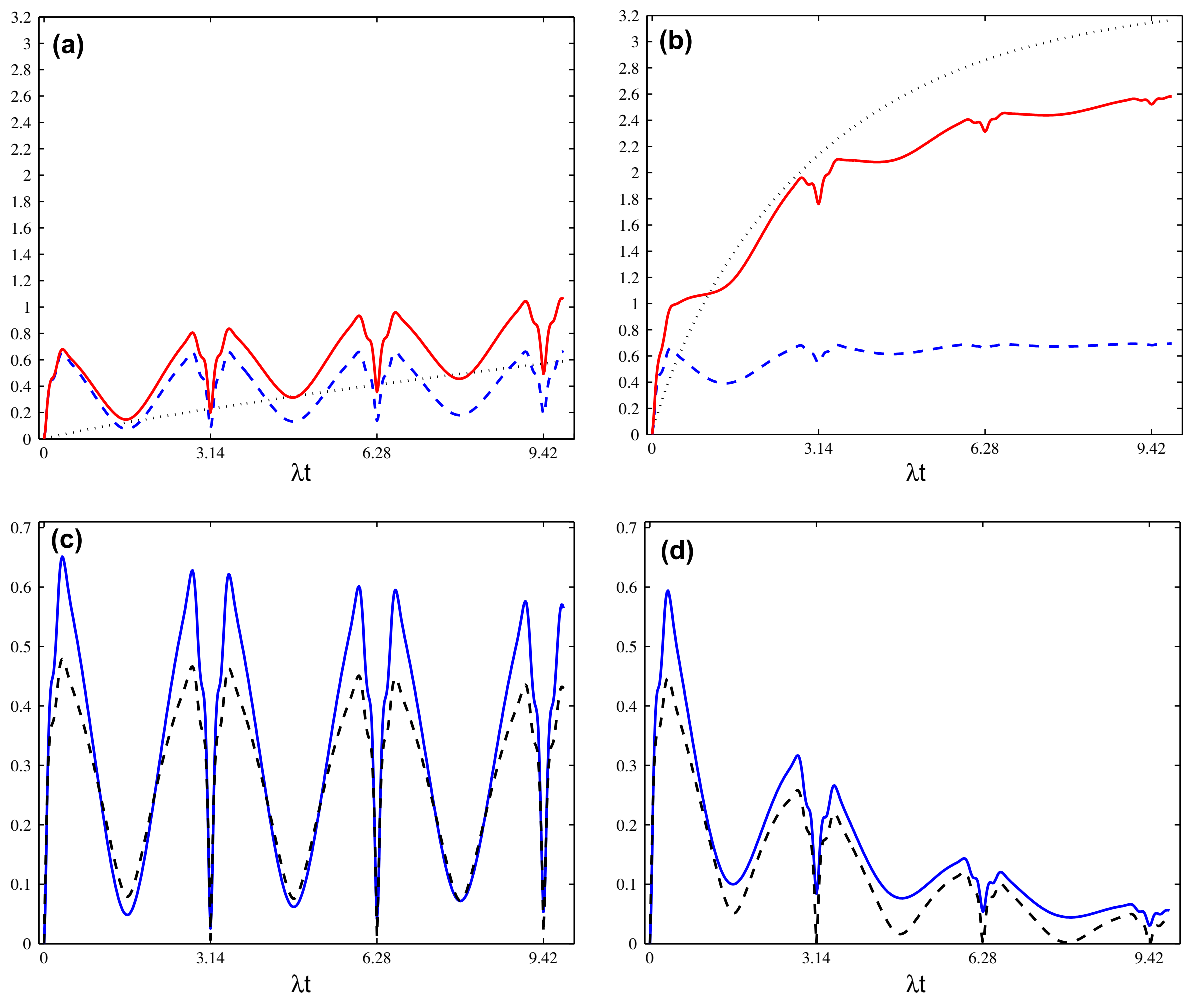

Figure 1 The time evolutions of $\mathcal{E}_T$ (dot curve), $\mathcal{E}_A$ (dash curve) and $\mathcal{E}_F$ (solid curve) as shown in (a) and (b). The time evolutions of $\mathcal{E}_N$ (dashed curve) in comparison to $\mathcal{E}_M$ (solid curve) as shown in (c) and (d). For the coherent state with $|\alpha|^2 = 10$ with different values of atomic damping parameter; for (a and c) $\frac{\gamma}{\lambda} = 0.01$, and (b and d) $\frac{\gamma}{\lambda} = 0.2$.

coupling with atomic decay. One probes the total correlation through the mutual information is also called index of classical correlation $\mathcal{E}_M$ [30],

$$\mathcal{E}_M = \frac{1}{4}(\mathcal{E}_F + \mathcal{E}_A - \mathcal{E}_T). \tag{13}$$

One shall compare this mutual information $\mathcal{E}_M$ with the negativity $\mathcal{E}_N$.

The off-diagonal elements describe atomic quantum coherences. In particular, to see what happens to the atomic interference of the states $|e\rangle$ and $|g\rangle$ with the atomic damping, one finds from the general solution of the absolute value of these non-diagonal elements $\xi(t) = |\langle e|\rho_A|g\rangle|$, then atomic quantum coherences is given by:

$$\xi(t) = \left|\sum_{i=0}^{\infty} \rho_i^{eg}(t)\right| = \left|\sum_{i=0}^{\infty} \pi_{i,i+1}\, e^{-\gamma t} \sin \mu_{i+1} t \cos \mu_i t\right|, \quad \xi(t \to \infty) = 0 \tag{14}$$

where these measure of atomic interference is zero for statistical mixture of coherent states. In Figs. 1 and 2, the functions $\mathcal{E}_T$, $\mathcal{E}_A$, $\mathcal{E}_F, \mathcal{E}_N$ and $\mathcal{E}_M$ are ploted against time λt for the coherent state and statistical mixture of coherent states respectively, with ($|\alpha|^2 = 10$), with different values of the atomic damping parameter $\frac{\gamma}{\lambda}$. From Fig. 1a, one finds that, at weak damping $\frac{\gamma}{\lambda} = 0.01$, the field entropy splits up from the atomic entropy, the amplitudes of the fast oscillations diminish. The minima (maxima) at half-revival time are noticed for the mentioned measures of degree of entanglement are still apparent. While at the large value of $\frac{\gamma}{\lambda} = 0.2$, (see Fig. 1b and d) the more rapid in this phenomenon and the more rapid the field will reach a statistically mixed state. In Fig. 1c and d, one compares the mutual information (solid curve) $\mathcal{E}_M$ with the negativity $\mathcal{E}_N$ (dashed curve). It is shown that, both $\mathcal{E}_M$ and $\mathcal{E}_N$ have almost the same behavior for the time interval considered. They arrive to their maximum values at half the revival time. This means that $\mathcal{E}_M$ can represent the entanglement of the system in almost the same way as the negativity measure $\mathcal{E}_N$. And from this figures one notes that, the total correlations larger than a mount of the entanglement. One notes that, at strong damping the maximum values of $\mathcal{E}_M$ and $\mathcal{E}_N$ decrease, and they approximate together to same maximum value.

From the above, the mixed state induced by strong damping, in which the interference terms have a rapid decay, and

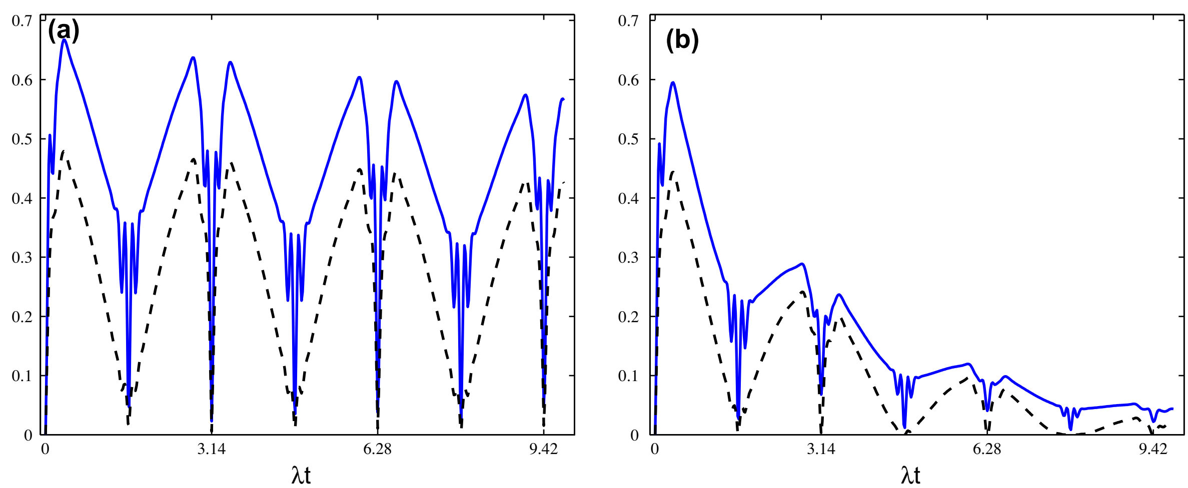

Figure 2 The same as in Fig. 1c and d but the field initially prepared in *SM*.

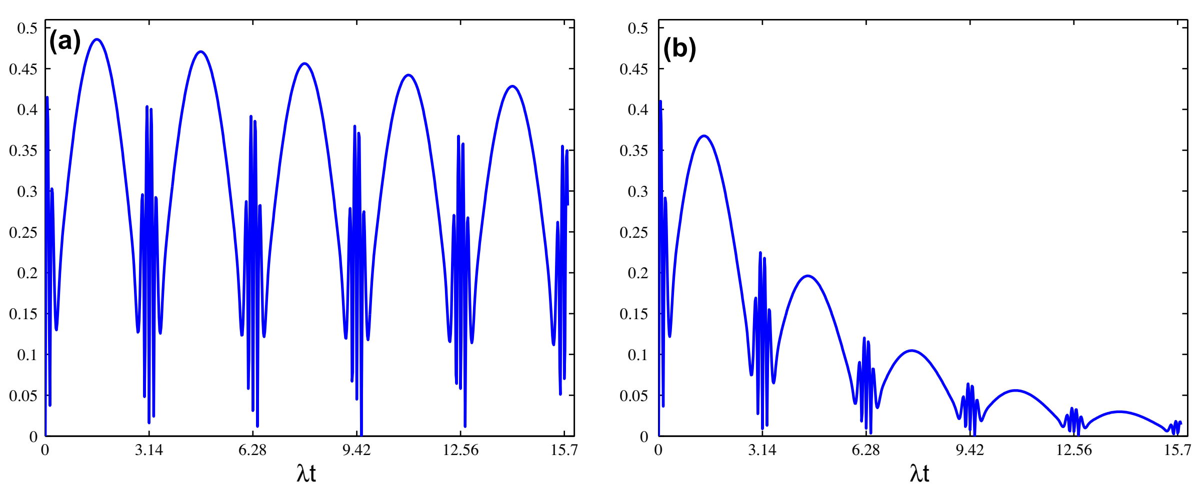

Figure 3 The time evolution of the atomic coherence ξ when the field initially prepared in coherent state for $\frac{\gamma}{\lambda} = 0.01$ in (a) and $\frac{\gamma}{\lambda} = 0.2$ in (b).

the figures have the following properties; (i) The field entropy splits up from the atomic entropy, i.e., ($\mathcal{E}_F > \mathcal{E}_A$). (ii) The maximum values of $\mathcal{E}_M$ run low to its counterparts of $\mathcal{E}_N$. (iii) The total entropy $\mathcal{E}_T$ increases monotonically from 0 (in which $\frac{\gamma}{\lambda} = 0$) to its asymptotic limit, i.e., $\mathcal{E}_T \neq 0$. Now one can ask, *what happens when one start with mixed state?*. To look for the answer of this question see Fig. 2. One sees the influence of the initial field SM with atomic damping on the $\mathcal{E}_M$ and $\mathcal{E}_N$ for this case. One notes that, the behaviors of the curve of $\mathcal{E}_M$ and $\mathcal{E}_N$ are different from its counterparts of case the coherent state. With the atomic damping (see Fig. 2), the minimum values of both $\mathcal{E}_M$ and $\mathcal{E}_N$ decrease with strong damping and in the last the fluctuations diminish. One notes, degradation of the entanglement of states (10) is appeared with strong damping.

To see the influence of the atomic damping on the measure of atomic interference, the function ξ is plotted with different values for the damping parameter, $\frac{\gamma}{\lambda} = 0.01, 0.2$ in Fig. 3. From Fig. 1a–c, the atomic interference with strong atomic damping decreases with time, and it disappears at particular time, this means that the atomic state is statistical mixture state, which have $\xi = 0$.

4. Conclusions

One has studied how atomic damping leads to growing entropy and a strong degradation of the entanglement. The atomic entropy affected by atomic damping less than the field entropy is observed. Its found that the degradation of the entanglement and the total correlations is generated by atomic damping. One concludes that the mutual information and the negativity shows a good measure to the degree of the entanglement. The interference between the atomic states is introduced by special measure, therefore, the effect of the atomic damping on the atomic coherence is studied. It is worth noting that, one of the pioneering experimental works on the JC-like dynamics in the context of trapped ions was reported in Ref.[37]. They observed the Rabi oscillations among two hyperfine levels of a $^9Be^+$ ion by measuring the probability of finding the ion in its lower electronic level. Results of this article may be relevant to such experiments.

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