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SHORT COMMUNICATION

Units in finite dihedral and quaternion group algebras

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Abstract Let $\mathbb{F}_q G$ be the group algebra of a finite group G over $\mathbb{F}_q = GF(q)$. Using the Wedderburn decomposition of $\mathbb{F}_{2^k} D_{2n}/J(\mathbb{F}_{2^k} D_{2n})$, we establish the structure of the unit group of $\mathbb{F}_{2^k} G$ when G is either D_{4n}, the dihedral group of order $4n$ or Q_{4n}, the generalized quaternion group of order $4n, n$ odd.

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1. Introduction

Let FG be the group algebra of a finite group G over a field F and $\mathcal{U}(FG)$ be its unit group. The study of the group of units is one of the classical topics in group ring theory. Results obtained in this direction are useful for the investigation of Lie properties of group rings, isomorphism problem and other open questions in this area [1]. In [2], Bovdi gave a comprehensive survey of results concerning the group of units of a modular group algebra of characteristic p. There is a long tradition on the study of the unit group of finite group algebras [3–12]. In general, the structure of $\mathcal{U}(FG)$ is elusive if $|G| = 0$ in F.

Let us introduce the background of our investigation. The structure of $\mathcal{U}(\mathbb{F}_2 D_{2p})$ was determined by Kaur and Khan in [13] for an odd prime p. Recently, the authors generalized this result and computed the structure of the unit group of $\mathbb{F}_{2^k} D_{2n}$ when n is odd. In this note, we use the Wedderburn decomposition of $\mathbb{F}_{2^k} D_{2n}/J(\mathbb{F}_{2^k} D_{2n})$ obtained in [14] to study the unit group of $\mathbb{F}_{2^k} D_{4n}$ and $\mathbb{F}_{2^k} Q_{4n}$ when n is odd.

In what follows, $q = 2^k, ord_l(m)$ denotes the multiplicative order of m modulo l when $(l, m) = 1$ and $\varphi(n)$ denotes the Euler's phi function on a positive integer n.

2. Main results

In this section, we begin by considering the lemmas that are essential for developing the proof of main results.

Lemma 2.1. *Let F be a perfect field, G be a finite group and $J(FG)$ be the Jacobson radical of FG. Then*

$$\mathcal{U}(FG) \cong (1 + J(FG)) \times \mathcal{U}(FG/J(FG))$$

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Proof. Observe that

$$1 \longrightarrow 1 + J(FG) \xrightarrow{inc} \mathcal{U}(FG) \xrightarrow{\psi} \mathcal{U}(FG/J(FG)) \longrightarrow 1$$

is a short exact sequence of groups, where $\psi(x) = x + J(FG) \quad \forall\, x \in \mathcal{U}(FG)$.

By *Wedderburn–Malcev* theorem [15, Thm. 6.2.1], it follows that there exists a semisimple subalgebra B of FG such that

$$FG = B \oplus J(FG)$$

and thus for each $x + J(FG) \in FG/J(FG)$, there exists a unique $x_B \in B$ such that

$$x + J(FG) = x_B + J(FG)$$

Define $\theta : \mathcal{U}(FG/J(FG)) \to \mathcal{U}(FG)$ as

$$\theta(x + J(FG)) = x_B \quad \forall\, x + J(FG) \in \mathcal{U}(FG/J(FG))$$

Then θ is a group homomorphism such that $\psi o \theta = id\,|_{\mathcal{U}(FG/J(FG))}$ and hence

$\mathcal{U}(FG) \cong (1 + J(FG)) \times \mathcal{U}(FG/J(FG))$ □

For a normal subgroup H of G, the natural homomorphism $\varepsilon_H : G \to G/H$ can be extended to an F-algebra epimorphism $\varepsilon_H^* : FG \to F(G/H)$. The kernel of ε_H^* is denoted by $\Delta(G, H)$ and $\Delta(G) = \Delta(G, G)$.

Lemma 2.2 [16, Lemma 1.17]. *Let G be a locally finite p-group and F be a field of characteristic p. Then $J(FG) = \Delta(G)$.*

Lemma 2.3 [17, Ch. 1, Prop. 6.16]. *Let $f : R_1 \to R_2$ be a surjective homomorphism of rings. Then*

$$f(J(R_1)) \subseteq J(R_2)$$

with equality if $ker\, f \subseteq J(R_1)$.

Lemma 2.4 [18, Theorem 7.2.7]. *Let H be a normal subgroup of G with $[G : H] = n < \infty$. Then $J(FG)^n \subseteq J(FH)FG \subseteq J(FG)$. If in addition $n \neq 0$ in F, then $J(FG) = J(FH)FG$.*

Lemma 2.5. *Let $N = 2^t n$ such that $2 \nmid n$. Then*

$$\begin{aligned}\mathbb{F}_q Q_{4N}/J(\mathbb{F}_q Q_{4N}) &\cong \mathbb{F}_q D_{2N}/J(\mathbb{F}_q D_{2N}) \\ &\cong \mathbb{F}_q \oplus \bigoplus_{m|n,\ m>1} M(2, \mathbb{F}_{q^{e_m}})^{\frac{\varphi(m)}{2e_m}}\end{aligned}$$

where

$$e_m = \begin{cases} d_m/2 & \text{if } d_m \text{ is even and } q^{d_m/2} \equiv -1 \bmod m \\ d_m & \text{otherwise} \end{cases}$$

and $d_m = ord_m(q)$.

Proof. To distinguish the elements of D_{2N} from those of D_{2n}, let D_{2N} be presented by

$$\langle A, B \mid A^N, B^2, B^{-1}AB = A^{-1} \rangle$$

and D_{2n} by

$$\langle a, b \mid a^n, b^2, b^{-1}ab = a^{-1} \rangle$$

From [14], it is known that

$$\mathbb{F}_q D_{2n}/J(\mathbb{F}_q D_{2n}) \cong \mathbb{F}_q \oplus \bigoplus_{m|n,\ m>1} M(2, \mathbb{F}_{q^{e_m}})^{\frac{\varphi(m)}{2e_m}}$$

Now

$$\Delta(D_{2N}, \langle A^n \rangle) = \Delta(\langle A^n \rangle)\mathbb{F}_q D_{2N} = J(\mathbb{F}_q \langle A^n \rangle)\mathbb{F}_q D_{2N} \subseteq J(\mathbb{F}_q D_{2N})$$

showing that $dim_{\mathbb{F}_q} J(\mathbb{F}_q D_{2N}) \geqslant 2N - 2n$.

Since $D_{2N}/\langle A^n \rangle \cong D_{2n}$, there exists an onto $\mathbb{F}_q$-algebra homomorphism

$$\phi : \mathbb{F}_q D_{2N} \to \mathbb{F}_q D_{2n}/J(\mathbb{F}_q D_{2n})$$

given by the assignment $A \mapsto a + J(\mathbb{F}_q D_{2N}), B \mapsto b + J(\mathbb{F}_q D_{2N})$ whence $J(\mathbb{F}_q D_{2N}) \subseteq ker\ \phi$ and

$dim_{\mathbb{F}_q} J(\mathbb{F}_q D_{2N}) \leqslant 2N - (2n - 1) = 2N - 2n + 1$

But there is only one 1-dimensional representation of D_{2N} over $\mathbb{F}_q$. This proves that $dim_{\mathbb{F}_q} J(\mathbb{F}_q D_{2N}) = 2N - 2n + 1$ and $J(\mathbb{F}_q D_{2N}) = ker\ \phi$ giving

$\mathbb{F}_q D_{2N}/J(\mathbb{F}_q D_{2N}) \cong \mathbb{F}_q D_{2n}/J(\mathbb{F}_q D_{2n})$

The decomposition of $\mathbb{F}_q Q_{4N}/J(\mathbb{F}_q Q_{4N})$ can be obtained by working on parallel lines. □

Theorem 2.6. *If n is odd, then*

$$\mathcal{U}(\mathbb{F}_q D_{4n}) \cong C_2^{(2n+1)k} \times \left(C_{q-1} \times \prod_{m|n,\ m>1} GL(2, \mathbb{F}_{q^{e_m}})^{\frac{\varphi(m)}{2e_m}} \right)$$

where

$$e_m = \begin{cases} d_m/2 & \text{if } d_m \text{ is even and } q^{d_m/2} \equiv -1 \bmod m \\ d_m & \text{otherwise} \end{cases}$$

and $d_m = ord_m(q)$.

Proof. Let

$$D_{4n} = \langle\, \alpha,\ \beta \mid \alpha^{2n},\ \beta^2,\ \beta^{-1}\alpha\beta = \alpha^{-1} \rangle$$

and $X = 1 + \alpha^n$ and then $\{X, \alpha X, \cdots, \alpha^{n-1}X, \beta X, \beta\alpha X, \cdots, \beta\alpha^{n-1}X\}$ is a basis of $\Delta(D_{4n}, \langle \alpha^n \rangle)$.

Observe that any $W \in \Delta(D_{4n}, \langle \alpha^n \rangle)$ is expressible as

$$W = \left(A_1 + A_2\alpha + \cdots + A_n\alpha^{n-1} + A_{n+1}\beta + A_{n+2}\beta\alpha + \cdots + A_{2n}\beta\alpha^{n-1}\right)X$$

for some $A_i \in \mathbb{F}_q$ so that

$$\begin{aligned} W^2 = \big(&A_1 + A_2\alpha + \cdots + A_n\alpha^{n-1} + A_{n+1}\beta \\ &+ A_{n+2}\beta\alpha + \cdots + A_{2n}\beta\alpha^{n-1}\big)^2 (1 + \alpha^n)^2 = 0 \end{aligned}$$

That is, $1 + \Delta(D_{4n}, \langle \alpha^n \rangle) \cong C_2^{2nk}$.

The $\mathbb{F}_q$ algebra homomorphism

$$\theta : \mathbb{F}_q D_{4n} \to \mathbb{F}_q D_{2n}/J(\mathbb{F}_q D_{2n})$$

given by the assignment

$$\alpha \mapsto a + J(\mathbb{F}_q D_{2n}), \beta \mapsto b + J(\mathbb{F}_q D_{2n})$$

is onto.

It is known that $\widehat{D_{2n}} \in J(\mathbb{F}_q D_{2n})$. Thus if $B = (1 + \alpha + \cdots + \alpha^{n-1})(1 + \beta)$, then $\theta(B) = 0 + J(\mathbb{F}_q D_{2n})$ showing that $B \in ker\theta = J(\mathbb{F}_q D_{4n})$. In fact, $J(\mathbb{F}_q D_{4n}) = \Delta(D_{4n}, \langle\alpha^n\rangle) \oplus \mathbb{F}_q B$ as a vector space over $\mathbb{F}_q$.

Since

$$\begin{aligned} B^2 &= ((1+\beta)(1+\alpha+\cdots+\alpha^{n-1}))^2 \\ &= (1+\beta)(1+\beta\alpha^{n+1})(1+\alpha+\cdots+\alpha^{n-1})^2 \\ &= (1+\beta+\alpha^{n+1}+\beta\alpha^{n+1})(1+\alpha^2+\cdots+\alpha^{2n-2}) \\ &= 1+\alpha^2+\cdots+\alpha^{2n-2}+\beta+\beta\alpha^2+\cdots+\beta\alpha^{2n-2}+\alpha^{n+1} \\ &\quad +\alpha^{n+3}+\cdots+\alpha^{3n-1}+\beta\alpha^{n+1}+\beta\alpha^{n+3}+\cdots+\beta\alpha^{3n-1} \\ &= 0 \text{ because } n \text{ is odd and } \alpha^{2n}=1. \end{aligned}$$

and

$$XB = (1+\alpha^n)(1+\alpha+\cdots+\alpha^{n-1})(1+\beta) = \widehat{\alpha}(1+\beta),$$

we find that $WB = BW$ so that

$$\begin{aligned} V &= 1+J(\mathbb{F}_q D_{4n}) = 1+\Delta(D_{4n}, \langle\alpha^n\rangle) \times \{1+\eta B \mid \eta \in \mathbb{F}_q\} \\ &\cong C_2^{(2n+1)k} \end{aligned}$$

This completes the proof. □

A group G is said to be a *general product* of its subgroups L and M if

$$G = LM, \ \ L \cap M = \{1\}$$

In this case, we write $G = LoM$.

In the subsequent theorem, it is established that $1+J(\mathbb{F}_q Q_{4n})$ is a general product of two of its proper subgroups. As a consequence, the structure of $\mathcal{U}(\mathbb{F}_q Q_{4n})$ is obtained.

Lemma 2.7 [3, Lemma 1.1]. *Let G be a finite abelian p-group,* $G^{p^i} = \{x^{p^i} \mid x \in G\}$ *and* $p^{m_i} = |G^{p^i}|$. *If* $G \cong \prod_{i=1}^{k} C_{p^i}^{n_i}$, *then*

$$n_i = m_{i-1} - 2m_i + m_{i+1} \quad \forall\ i,\ 1 \le i \le k$$

Theorem 2.8. *If n is odd, then*

$$\mathcal{U}(\mathbb{F}_q Q_{4n}) \cong \left(C_2^{(2n-2)k} o\left(C_2^k \times C_4^k\right)\right) \rtimes \left(C_{q-1} \times \prod_{m|n,\, m>1} GL(2, \mathbb{F}_{q^{e_m}})^{\frac{\varphi(m)}{2e_m}}\right)$$

where e_m as in Theorem 2.6.

Proof. Let Q_{4n} be presented by

$$\langle C, D \mid C^{2n}, D^2 = C^n, D^{-1}CD = C^{-1}\rangle$$

and $U = (1+D)(1+C+\ldots+C^{n-1})$. Then via similar arguments as in the previous theorem, $U \in J(\mathbb{F}_q Q_{4n})$ and

$$\begin{aligned} U^2 &= \left((1+D)(1+C^n)+D\widehat{C}\right)(1+C+\ldots+C^{n-1}) \\ &= (1+D)\widehat{C}+D\widehat{C} = \widehat{C} \end{aligned}$$

Notice that if $Y = 1+C^n$, then $\{Y, CY, \cdots, C^{n-2}Y, DY, DCY, \cdots, DC^{n-2}Y, \widehat{C}, (1+D)\widehat{C}, U\}$ is a basis of $J(\mathbb{F}_q Q_{4n})$ over $\mathbb{F}_q$.

Since $Y \in \mathcal{Z}(\mathbb{F}_q Q_{4n})$ and $Y^2 = 0$, it is therefore evident that

$$\begin{aligned} H &= \left\{1+\left(\sum_{i=0}^{n-2} A_i C^i + \sum_{i=0}^{n-2} B_i D C^i\right) Y \;\middle|\; A_i, B_i \in \mathbb{F}_q\right\} \\ &\leqslant 1+J(\mathbb{F}_q Q_{4n}) \end{aligned}$$

and $H \cong C_2^{(2n-2)k}$.

Also $K = \left\{1+A_1 C + A_2\widehat{C} + A_3(1+D)\widehat{C} \mid A_i \in \mathbb{F}_q\right\} \leqslant 1+J(\mathbb{F}_q Q_{4n})$ and by Lemma 2.7, we find $K \cong C_2^k \times C_4^k$.

Since $H \cap K = \{1\}$ and $|1+J(\mathbb{F}_q Q_{4n})| = |HK|$, it follows that $1+J(\mathbb{F}_q Q_{4n}) = HoK$. This completes the proof. □

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