Journal of the Egyptian Mathematical Society (2016) 24, 8–14

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On Jordan left-*I*-centralizers of prime and semiprime gamma rings with involution

Kalyan Kumar Dey [a,*], Akhil Chandra Paul [a], Bijan Davvaz [b]

[a] *Department of Mathematics, Rajshahi University, Rajshahi 6205, Bangladesh*
[b] *Department of Mathematics, Yazd University, Yazd, Iran*

Received 17 January 2014; revised 19 June 2014; accepted 2 August 2014
Available online 11 February 2015

KEYWORDS
Prime Γ-ring;
Semiprime Γ-ring;
Involution

Abstract Let M be a 2-torsion free Γ-ring with involution I satisfying the condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. The object of our paper is to show that every Jordan left-I-centralizer on a semiprime Γ-ring with involution I, is a reverse left-I-centralizer. We solve some functional equations in prime and semiprime Γ-rings with involution I by means of the above results. Moreover, we discuss some more related results.

2010 AMS MATHEMATICS SUBJECT CLASSIFICATION: 16N60; 16W10

1. Introduction

The notion of a Γ-ring was first introduced by Nobusawa [6] as a generalization of a ring and then Barnes [2] generalized the definition of Nabusawa's Γ-ring in a more general nature. For the Γ-rings we refer to Barnes [2]. Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \to M$ (sending $(a, \alpha, b) \mapsto a\alpha b$) which satisfies the conditions (1) $(a+b)\alpha c = a\alpha c + b\alpha c, a(\alpha+\beta)b = a\alpha b + a\beta b, a\alpha(b+c) = a\alpha b + a\alpha c$; (2) $(a\alpha b)\beta c = a\alpha(b\beta c)$; for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Then, M is a Γ-ring in the sense of Barnes [2]. Let M be a Γ-ring. A mapping $I : M \to M$ is called an *involution* if (1) $I(a+b) = I(a) + I(b)$; (2) $I(a\alpha b) = I(b)\alpha I(a)$; (3) $I^2(a) = a$; for all $a, b \in M$ and $\alpha \in \Gamma$. In the ring theory, Ali, Dar and Vukman [1] proved that every Jordan left ★-centralizer on a semiprime ring with involution, of char R different from 2 is a reverse left ★-centralizer. They used this result to make it possible to solve some fundamental equations in prime and semiprime rings. Also, see [3,4,7]. This paper deals with the study of Jordan left-I-centralizers of prime and semiprime Γ-rings with involution I, and was motivated by the work of [1]. Throughout, M will represent a Γ-ring with center $Z(M)$. We shall denote by $C(M)$ the extended centroid of a prime Γ-ring M. For the explanation of $C(M)$ we refer to the reader in the paper of Soyturk [8]. Given an integer $n \geqslant 2$, a Γ-ring M is said to be *n-torsion free*, if for $x \in M, nx = 0$ implies $x = 0$. As usual $[x, y]_\alpha$ and $\langle x, y\rangle_\alpha$ will denote the commutator $x\alpha y - y\alpha x$ and anti-commutator $x\alpha y + y\alpha x$, respectively for all $x, y \in M$ and $\alpha \in \Gamma$. An additive mapping $T : M \to M$ is called a *left centralizer* in case $T(x\alpha y) = T(x)\alpha y$ holds for all

* Corresponding author.
E-mail addresses: kkdmath@yahoo.com (K.K. Dey), acpaulrubd_math@yahoo.com (A.C. Paul), davvaz@yazd.ac.ir (B. Davvaz).
Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2014.08.002
1110-256X

$x, y \in M$ and $\alpha \in \Gamma$. The definition of a *right centralizer* should be self-explanatory. An additive mapping T is called a *two-sided centralizer* in case $T: M \to M$ is a left and a right centralizer. An additive mapping $T: M \to M$ is called a *Jordan left centralizer* if $T(x\alpha x) = T(x)\alpha x$ holds for all $x \in M$ and $\alpha \in \Gamma$. Let M be a Γ-ring with involution I. An additive mapping $T: M \to M$ is said to be a *left-I-centralizer* (resp. *reverse left-I-centralizer*) if $T(x\alpha y) = T(x)\alpha I(y)$ (resp. $T(x\alpha y) = T(y)\alpha I(x)$) holds for all $x, y \in M$ and $\alpha \in \Gamma$. An additive mapping $T: M \to M$ is called a *Jordan left-I-centralizer* in case $T(x\alpha x) = T(x)\alpha I(x)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. The definition of a right-I-centralizer and a Jordan right-I-centralizer should be self-explanatory. For some fixed element $a \in M$, the map $M \to M$ defined by $x \mapsto a\alpha I(x)$ is a Jordan left-I-centralizer and the map $x \mapsto I(x)\alpha a$ is a Jordan right-I-centralizer on M. Clearly, every left-I-centralizer on a Γ-ring M is a Jordan left-I-centralizer. Further, we establish a result concerning additive mapping $T: M \to M$ satisfying the relation $T(x\alpha x\alpha x) = I(x)\alpha T(x)\alpha I(x)$ for all $x \in M$ and $\alpha \in \Gamma$. We showed that if M is a 2-torsion free semiprime Γ-ring and $S, T: M \to M$ are left centralizers such that $[S(x), T(x)]_\alpha \beta S(x) + S(x)\beta[S(x), T(x)]_\alpha = 0$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then $[S(x), T(x)]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$. In case M is a prime Γ-ring and $S \neq 0$ ($T \neq 0$), then there exists $p, q \in C(M)$ such that $T = pS$ ($S = qT$). We shall restrict our attention on Jordan left-I-centralizers, since all results presented in this article are also true for Jordan right-I-centralizers because of left–right symmetry. Throughout this paper, M will represent a Γ-ring and $Z(M)$ will be its center. A Γ-ring M is prime if $x\Gamma M\Gamma y = 0$ implies that $x = 0$ or $y = 0$, and is semiprime if $x\Gamma M\Gamma x = 0$ implies $x = 0$. Let $x, y \in M$ and $\alpha \in \Gamma$, the commutator $x\alpha y - y\alpha x$ will be denoted by $[x, y]_\alpha$. We know that $[x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha \beta y + x[\beta, \alpha]_z y$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha \beta z + y[\beta, \alpha]_x z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. We shall take the assumption $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Using this assumption, the above identities reduce to $[x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha \beta y$, $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha \beta z$, for all $x, y, z \in M$ and $\alpha, \beta \in \gamma$. Also, $\langle x, y\beta z\rangle_\alpha = \langle x, y\rangle_\alpha \beta z - y\beta[x, z]_\alpha = y\beta\langle x, z\rangle_\alpha + [x, y]_\alpha \beta z$, $\langle x\beta y, z\rangle_\alpha = x\beta\langle y, z\rangle_\alpha - [x, z]_\alpha \beta y = \langle x, z\rangle_\alpha \beta y + x\beta[y, z]_\alpha$, which are used extensively in our results.

2. Basic results

Lemma 2.1. *Let M be a prime Γ-ring with the central closure $C(M)$. Suppose that the elements $a_i, b_i \in C(M)$ satisfying the condition $\sum a_i \alpha x \beta b_i = 0$ for all $x \in M$ and $\alpha \in \Gamma$. If $b_i = 0$ for some i, then a_i's are $C(M)$-independent.*

Proof. We show that a_i's are linearly independent over $C(M)$. If not, there are a minimal n elements $a_1, a_2, \ldots, a_n \in M$ linearly independent over $C(M)$ such that $\sum_{i=1}^n a_i \alpha x \beta b_i = 0$ for all $x \in M$ and $\alpha, \beta \in \Gamma$, where b_i are non-zero elements of M. Since M is prime, $n > 1$. Suppose that $x_i, y_i \in M$ are such that $\sum_{i=1}^n x_i \gamma x \delta y_i = 0$. If $r \in M$, then $\sum_{i=1}^n a_i \alpha r \beta x_j \gamma b_i \delta y_j = \sum_{i=1}^n a_i \alpha r \beta b_i \delta y_j = \sum_{i=1}^n a_i \alpha r \beta \left(\sum x_j \gamma b_i \delta y_j\right) = 0$. Since $\sum_{i=1}^n a_i \alpha r \beta x_j \gamma b_i = 0$, we obtain a shorter relation than n, we have that $\sum_{i=1}^n x_i \gamma b_i \delta y_j = 0$ for all i. Hence, the map $\psi_i : M\Gamma b_i \Gamma M \to M$ defined by $\psi_i\left(\sum_{i=1}^n u_j \gamma b_i \delta v_j\right) = \sum_{i=1}^n u_j \gamma b_i \delta v_j = 0$ is well defined. It is trivial that ψ_i is an additive map of the ideal $M\Gamma b_i \Gamma M$ into M. Hence, ψ_i gives an element b_i such that $\psi_i(b_i) \in C(M)$. Moreover, by definition $\psi_i(b_i) = b_i$. Thus, $\sum_{i=1}^n a_i \alpha x \beta b_i = \sum_{i=1}^n a_i \alpha x \beta \psi_i(b_i) = \left(\sum_{i=1}^n \psi_i(a_i)\right)\alpha x \beta b_i = 0$. By the primeness of M, we get that $\sum_{i=1}^n a_i \alpha x \beta b_i = 0$, since a_i are linearly independent over $C(M)$, we must have $\psi_i = 0$. But then by definition of ψ_i, $M\Gamma b_i \Gamma M = 0$, gives a contradiction $b_i = 0$. $\square$

Lemma 2.2. *Let M be a prime Γ-ring with involution I satisfying the condition $x\alpha y\beta z = x\beta y\alpha z$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$ and let $T: M \to M$ be a Jordan left-I-centralizer on M. If $T(x) \in Z(M)$ for all $x \in M$, then $T = 0$.*

Proof. By the assumption we have $[T(x), y]_\alpha = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$. Substituting $x\beta x$ for x in the above relation, then we obtain $0 = [T(x\beta x), y]_\alpha = [T(x)\beta I(x), y]_\alpha = [T(x), y]_\alpha \beta I(x) + T(x)\beta[I(x), y]_\alpha$, for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. In view of our hypothesis, the last expression yields that $T(x)\beta[I(x), y]_\alpha = 0$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Since the center of a prime Γ-ring is free from zero divisors, either $T(x) = 0$ or $[I(x), y]_\alpha = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$. Let $A = \{x \in M \mid T(x) = 0\}$ and $B = \{x \in M \mid [I(x), y]_\alpha = 0$ for all $y \in M$ and $\alpha \in \Gamma\}$. It can be easily seen that A and B are two additive subgroups of M whose union is M and hence by Brauer's trick, we get $A = M$ or $B = M$. If $B = M$, then M is commutative, which gives a contradiction. Thus, the only possibility remains that $A = M$. That is, $T(x) = 0$ for all $x \in M$. This completes the proof. $\square$

Theorem 2.3. *Let M be a semiprime Γ-ring with involution I satisfying the condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, of characteristic different from two and $T: M \to M$ an additive mapping which satisfies $T(x\alpha x) = T(x)\alpha I(x)$ for all $x \in M$ and $\alpha \in \Gamma$. Then, T is a reverse left-I-centralizer, that is, $T(x\alpha y) = T(y)\alpha I(x)$ for all $x, y \in M$ and $\alpha \in \Gamma$.*

Proof. We have $T(x\alpha x) = T(x)\alpha I(x)$ for all $x \in M$ and $\alpha \in \Gamma$. Applying involution I both sides to the above expression, we obtain $I(T(x\alpha x)) = x\alpha I(T(x))$ for all $x \in M$ and $\alpha \in \Gamma$. Define a new map $S: M \to M$ such that $S(x) = I(T(x))$ for all $x \in M$ and $\alpha \in \Gamma$. Then, we see that $S(x\alpha x) = I(T(x\alpha x)) = I(T(x)\alpha I(x)) = x\alpha I(T(x)) = x\alpha S(x)$ for all $x \in M$ and $\alpha \in \Gamma$. Hence, we obtain $S(x\alpha x) = x\alpha S(x)$ for all $x \in M$ and $\alpha \in \Gamma$. Thus, S is a Jordan right-centralizer on M. In view of [5], S is a right-centralizer that is, $S(x\alpha y) = x\alpha S(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. This implies that $I(T(x\alpha y)) = x\alpha I(T(y))$ for all $x, y \in M$ and $\alpha \in \Gamma$. By applying involution to the both sides of the last relation, we find that $T(x\alpha y) = T(y)\alpha I(x)$ for all $x, y \in M$ and $\alpha \in \Gamma$. This completes the proof. $\square$

Lemma 2.4. *Let M be a prime Γ-ring with involution I satisfying the condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, of characteristic different from two and $T: M \to M$ an additive mapping which satisfies $T(x\alpha x\alpha x) = I(x)\alpha T(x)\alpha I(x)$ for all $x \in M$ and $\alpha \in \Gamma$. Then, $T(x\alpha y) = T(y)\alpha I(x) = I(y)\alpha T(x)$ for all $x, y \in M$ and $\alpha \in \Gamma$, that is, T is a reverse I-centralizer on M.*

Proof. By the given hypothesis, we have $T(x\alpha x\alpha x) = I(x)\alpha T(x)\alpha I(x)$ for all $x \in M$ and $\alpha \in \Gamma$. Applying involution I on both sides to the above expression, we get $I(T(x\alpha x\alpha x)) = x\alpha I(T(x))\alpha x$ for all $x \in M$ and $\alpha \in \Gamma$. Define a new map $S : M \to M$ such that $S(x) = I(T(x))$ for all $x \in M$. Then, we see that $S(x\alpha x\alpha x) = I(I(T(x\alpha x\alpha x))) = I(I(x)\alpha T(x)\alpha I(x)) = x\alpha I(T(x))\alpha x = x\alpha S(x)\alpha x$ for all $x \in M$ and $\alpha \in \Gamma$. Hence, we conclude that $S(x\alpha x\alpha x) = x\alpha S(x)\alpha x$ for all $x \in M$ and $\alpha \in \Gamma$. Thus, S is an additive mapping such that $S(x\alpha x\alpha x) = x\alpha S(x)\alpha x$. In view of [5], we are forced to conclude that S is a two sided centralizer that is, $S(x\alpha y) = x\alpha S(y) = S(x)\alpha y$ for all $x, y \in M$ and $\alpha \in \Gamma$. This implies that $I(T(x\alpha y)) = x\alpha I(T(y)) = I(T(x))\alpha y$ for all $x, y \in M$ and $\alpha \in \Gamma$. Again applying involution both sides to the last relation, we find that $T(x\alpha y) = T(y)\alpha I(x) = I(y)\alpha T(x)$ for all $x, y \in M$ and $\alpha \in \Gamma$. □

Lemma 2.5. *Let M be a prime Γ-ring with involution I satisfying the condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, and let $S, T : M \to M$ be Jordan left-I-centralizers. Suppose that $[S(x), T(x)]_\alpha = 0$ holds for all $x \in M$ and $\alpha \in \Gamma$. If $T = 0$, then there exists $p \in C(M)$ such that $S = pT$.*

Proof. By Theorem 2.3 we conclude that S and T are reverse left-I-centralizers on M. In view of the hypothesis, we have

$$[S(x), T(x)]_\alpha = 0, \tag{1}$$

for all $x \in M$ and $\alpha \in \Gamma$. Linearizing (1) and using it, we get

$$[S(x), T(y)]_\alpha + [S(y), T(x)]_\alpha = 0, \tag{2}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Replacing x by $z\beta x$ in (2), we obtain

$$[S(x), T(y)]_\alpha \beta I(z) + S(x)\beta[I(z), T(y)]_\alpha + [S(y), T(x)]_\alpha \beta I(z) + T(x)\beta[S(y), I(z)]_\alpha = 0. \tag{3}$$

Application of (2) yields that

$$S(x)\beta[I(z), T(y)]_\alpha + T(x)\beta[S(y), I(z)]_\alpha = 0. \tag{4}$$

Replacing x by $w\delta x$ in (4), we get

$$S(x)\beta I(w)\delta[I(z), T(y)]_\alpha + T(x)\beta I(w)\delta[S(y), I(z)]_\alpha = 0. \tag{5}$$

Replacing w by $I(w)$ and z by $I(z)$ in (5), we obtain

$$S(x)\beta w\delta[z, T(y)]_\alpha + T(x)\beta w\delta[S(y), z]_\alpha = 0. \tag{6}$$

It follows from Lemma 2.2 that there exists $y, z \in M$ and $\alpha \in \Gamma$ such that $[T(y), I(z)]_\alpha = 0$, since $T \neq 0$. In view of Lemma 2.1 and from relation (6) we conclude that $S(x) = pT(x)$, where p is from $C(M)$. Thus, the relation (6) forces that for some $p, q \in C(M)$, $0 = pT(x)\beta w\delta[T(y), z]_\alpha - T(x)\beta w\delta[qT(y), z]_\alpha = p\,T(x)\beta w\delta[T(y), z]_\alpha - T(x)\beta w\delta q[T(y), z]_\alpha = (p - q)T(x)\beta w\delta[T(y), z]_\alpha$, for all $y, z \in M$ and $\alpha, \beta, \delta \in \Gamma$. Since M is a prime Γ-ring, the above expression yields that either $(p - q)T(x) = 0$ or $[T(y), z]_\alpha = 0$. Since $[T(y), z]_\alpha \neq 0$, we have $(p - q)T(x) = 0$ for all $x \in M$ and $\alpha \in \Gamma$. This implies that $pT(x) = qT(x)$ for all $x \in M$. This gives $S(x) = pT(x)$ for all $x \in M$ as desired. If we replace the commutator by anti-commutator in Lemma 2.5, the corresponding result also holds. □

Lemma 2.6. *Let M be a prime Γ-ring with involution I satisfying the condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, and let $S, T : M \to M$ be Jordan left-I-centralizers. Suppose that $\langle S(x), T(x)\rangle_\alpha = 0$ holds for all $x \in M$ and $\alpha \in \Gamma$. If $T \neq 0$, then there exists $p \in C(M)$ such that $S = pT$.*

Proof. By the assumption, we have

$$\langle S(x), T(x)\rangle_\alpha = 0, \tag{7}$$

for all $x \in M$ and $\alpha \in \Gamma$. Replacing x by $x + y$ in (7), we obtain

$$\langle S(x), T(x)\rangle_\alpha + \langle S(x), T(y)\rangle_\alpha + \langle S(y), T(x)\rangle_\alpha + \langle S(y), T(y)\rangle_\alpha = 0. \tag{8}$$

Using (7) in (8), we get

$$\langle S(x), T(y)\rangle_\alpha + \langle S(y), T(x)\rangle_\alpha = 0. \tag{9}$$

Substituting $z\beta y$ for y in (9) and using the fact that S and T are reverse left-I-centralizers, we find that $0 = \langle S(x), T(z\beta y)\rangle_\alpha + \langle S(z\beta y), T(x)\rangle_\alpha = \langle S(x), T(y)\beta I(z)\rangle_\alpha + \langle T(x), S(y)\beta I(z)\rangle_\alpha = \langle S(x), T(y)\rangle_\alpha \beta I(z) - T(y)\beta[S(x), I(z)]_\alpha + \langle T(x), S(y)\rangle_\alpha \beta I(z) - S(y)\beta[T(x), I(z)]_\alpha$. Application of (9) yields that

$$T(y)\beta[S(x), I(z)]_\alpha + S(y)\beta[T(x), I(z)]_\alpha = 0. \tag{10}$$

Replacing y by $w\delta y$ in (10), we obtain

$$T(y)\beta I(w)\delta[S(x), I(z)]_\alpha + S(y)\beta I(w)\delta[T(x), I(z)]_\alpha = 0. \tag{11}$$

Replacing w by $I(w)$ and z by $I(z)$ in (11), we get

$$T(y)\beta w\delta[S(x), z]_\alpha + S(y)\beta w\delta[T(x), z]_\alpha = 0. \tag{12}$$

Henceforth using similar approach as we have used after Eq. (6) in the proof of Lemma 2.5, we get the required result. This finishes the proof of the lemma. □

3. Main results

The main result of the present paper is the following theorem which is inspired by [1].

Theorem 3.1. *Let M be a 2-torsion free semiprime Γ-ring with involution I satisfying the condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, and $S, T : M \to M$ be Jordan left-I-centralizers. Suppose that $\langle S(x), T(x)\rangle_\alpha \beta S(x) - S(x)\beta\langle S(x), T(x)\rangle_\alpha = 0$ holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then, $[S(x), T(x)]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$. Moreover, if M is a prime Γ-ring and $S \neq 0$ ($T \neq 0$), then there exists $p \in C(M)$ such that $T = pS$ ($S = qT$, $q \in C(M)$).*

Proof. In view of Lemma 2.3, we conclude that S and T are reverse left-I-centralizers. By the hypothesis, we have

$$\langle S(x), T(x)\rangle_\alpha \beta S(x) - S(x)\beta\langle S(x), T(x)\rangle_\alpha = 0, \tag{13}$$

for all $x \in M$ and $\alpha, \beta \in \Gamma$. Linearization of the relation (13) yields that

$$\langle S(x),T(x)\rangle_\alpha\beta S(y)+\langle S(x),T(y)\rangle_\alpha\beta S(x)+\langle S(x),T(y)\rangle_\alpha\beta S(y)$$
$$+\langle S(y),T(x)\rangle_\alpha\beta S(x)+\langle S(y),T(x)\rangle_\alpha\beta S(y)$$
$$+\langle S(y),T(x)\rangle_\alpha\beta S(x)-S(y)\beta\langle S(x),T(x)\rangle_\alpha$$
$$-S(x)\beta\langle S(x),T(y)\rangle_\alpha-S(y)\beta\langle S(x),T(y)\rangle_\alpha$$
$$-S(x)\beta\langle S(y),T(x)\rangle_\alpha-S(y)\beta\langle S(y),T(x)\rangle_\alpha$$
$$-S(x)\beta\langle S(y),T(y)\rangle_\alpha=0, \quad (14)$$

for all $x,y\in M$ and $\alpha,\beta\in\Gamma$. Replacing x by $-x$ in (14), we get

$$\langle S(x),T(x)\rangle_\alpha\beta S(y)+\langle S(x),T(y)\rangle_\alpha\beta S(x)-\langle S(x),T(y)\rangle_\alpha\beta S(y)$$
$$+\langle S(y),T(x)\rangle_\alpha\beta S(x)-\langle S(y),T(x)\rangle_\alpha\beta S(y)$$
$$-\langle S(y),T(y)\rangle_\alpha\beta S(x)-S(y)\beta\langle S(x),T(x)\rangle_\alpha$$
$$-S(x)\beta\langle S(x),T(y)\rangle_\alpha+S(y)\beta\langle S(x),T(y)\rangle_\alpha$$
$$-S(x)\beta\langle S(y),T(x)\rangle_\alpha+S(y)\beta\langle S(y),T(x)\rangle_\alpha$$
$$+S(x)\beta\langle S(y),T(y)\rangle_\alpha=0. \quad (15)$$

Combining (14) and (15), we obtain $2\langle S(x),T(x)\rangle_\alpha\beta S(y)+2\langle S(x),T(y)\rangle_\alpha\beta S(x)+2\langle S(y),T(x)\rangle_\alpha\beta S(x)-2S(y)\beta\langle S(x),T(x)\rangle_\alpha-2S(x)\beta\langle S(x),T(y)\rangle_\alpha-2S(x)\beta\langle S(y),T(x)\rangle_\alpha=0$. Since M is 2-torsion free, the above relation reduces to

$$\langle S(x),T(x)\rangle_\alpha\beta S(y)+\langle S(x),T(y)\rangle_\alpha\beta S(x)+\langle S(y),T(x)\rangle_\alpha\beta S(x)$$
$$-S(y)\beta\langle S(x),T(x)\rangle_\alpha-S(x)\beta\langle S(x),T(y)\rangle_\alpha$$
$$-S(x)\beta\langle S(y),T(x)\rangle_\alpha=0. \quad (16)$$

Replacing y by $y\delta x$ in (16), we obtain $\langle S(x),T(x)\rangle_\alpha\beta S(x)\delta I(y)+\langle S(x),T(x)\delta y\rangle_\alpha\beta S(x)+\langle S(x)\delta T(y),T(x)\rangle_\alpha\beta S(x)-S(x)\delta I(y)\beta\langle S(x),T(x)\rangle_\alpha-S(x)\beta\langle S(x),T(x)\delta I(y)\rangle_\alpha-S(x)\beta\langle T(x),S(x)\delta I(y)\rangle_\alpha=0$. By using anti-commutator identity, the above relation can be written as

$$\langle S(x),S(x)\rangle_\alpha\delta S(x)\beta I(y)+\langle S(x),T(x)\rangle_\alpha\delta I(y)\beta S(x)$$
$$-T(x)\delta[S(x),I(y)]_\alpha\beta S(x)+\langle T(x),S(x)\rangle_\alpha\delta I(y)\beta S(x)$$
$$-S(x)\delta[T(x),I(y)]_\alpha\beta S(x)-S(x)\beta I(y)\delta\langle S(x),T(x)\rangle_\alpha$$
$$-S(x)\beta\langle S(x),T(x)\rangle_\alpha\delta I(y)+S(x)\beta T(x)\delta[S(x),I(y)]_\alpha$$
$$-S(x)\beta\langle T(x),S(x)\rangle_\alpha\delta I(y)+S(x)\beta S(x)\delta[T(x),I(y)]_\alpha=0. \quad (17)$$

In view of (13) and (17) reduces to

$$\langle S(x),T(x)\rangle_\alpha\delta I(y)\beta S(x)-T(x)\delta[S(x),I(y)]_\alpha\beta S(x)$$
$$+\langle S(x),T(x)\rangle_\alpha\delta I(y)\beta S(x)-S(x)\delta[T(x),I(y)]_\alpha\beta S(x)$$
$$-S(x)\delta I(y)\beta\langle S(x),T(x)\rangle_\alpha-S(x)\beta\langle S(x),T(x)\rangle_\alpha\delta I(y)$$
$$+S(x)\beta T(x)\delta[S(x),I(y)]_\alpha+S(x)\beta S(x)\delta[T(x),I(y)]_\alpha=0. \quad (18)$$

Upon substituting $I(S(x))\mu y$ for y in (18), we get $\langle S(x),T(x)\rangle_\alpha\delta I(y)\mu S(x)\beta S(x)-T(x)\delta[S(x),I(y)\mu S(x)]_\alpha\beta S(x)+\langle T(x),S(x)\rangle_\alpha\delta I(y)\mu S(x)\beta S(x)-S(x)\delta[T(x),I(y)\mu S(x)]_\alpha\beta S(x)-S(x)\delta I(y)\mu S(x)\beta S(x),T(x)\rangle_\alpha-S(x)\beta\langle S(x),T(x)\rangle_\alpha\delta I(y)\mu S(x)+S(x)\beta T(x)\delta[S(x),I(y)\mu S(x)]_\alpha+S(x)\mu S(x)\delta[T(x),I(y)\mu S(x)]_\alpha=0$. This implies that

$$\langle S(x),T(x)\rangle_\alpha\delta I(y)\mu S(x)\beta S(x)-T(x)\delta[S(x),I(y)]_\alpha\mu S(x)$$
$$+\langle S(x),T(x)\rangle_\alpha\delta I(y)\mu S(x)\beta S(x)-S(x)\beta[T(x),I(y)]_\alpha\mu S(x)\beta S(x)$$
$$-S(x)\delta I(y)\beta[T(x),S(x)]_\alpha\mu S(x)-S(x)\delta I(y)\mu S(x)\beta\langle S(x),T(x)\rangle_\alpha$$
$$-S(x)\beta\langle S(x),T(x)\rangle_\alpha\delta I(y)\mu S(x)+S(x)\beta T(x)\delta[S(x),I(y)]_\alpha\mu S(x)$$
$$+S(x)\beta S(x)\delta[T(x),I(y)]_\alpha\mu S(x)+S(x)\beta S(x)\delta I(y)\mu[T(x),S(x)]_\alpha=0. \quad (19)$$

Application of (18) yields that

$$S(x)\delta I(y)\beta[T(x),S(x)]_\alpha\mu S(x)-S(x)\beta S(x)\delta I(y)\mu[T(x),S(x)]_\alpha=0. \quad (20)$$

Replacing y by $y\gamma I(T(x))$ in (20), we obtain

$$S(x)\delta T(x)\gamma I(y)\beta[S(x),T(x)]_\alpha\mu S(x)$$
$$-S(x)\beta S(x)\delta T(x)\gamma I(y)\mu[S(x),T(x)]_\alpha=0. \quad (21)$$

Left multiplying (20) by $T(x)$ gives

$$T(x)\gamma S(x)\delta I(y)\beta[T(x),S(x)]_\alpha\mu S(x)$$
$$-T(x)\gamma S(x)\beta S(x)\delta I(y)\mu[T(x),S(x)]_\alpha=0. \quad (22)$$

On combining (21) and (22), we obtain

$$[S(x),T(x)]_\alpha\gamma I(y)\delta[S(x),T(x)]_\alpha\mu S(x)$$
$$-[S(x)S(x),T(x)]_\alpha\gamma I(y)\mu[S(x),T(x)]_\alpha=0. \quad (23)$$

By our hypothesis, we have

$0=\langle S(x),T(x)\rangle_\alpha S(x)-S(x)\beta\langle S(x),T(x)\rangle_\alpha=S(x)\alpha T(x)\beta S(x)+T(x)\alpha S(x)\beta S(x)-S(x)\beta S(x)\alpha T(x)-S(x)\beta T(x)\alpha S(x)=T(x)\alpha S(x)\beta S(x)-S(x)\beta S(x)\alpha T(x)$. The above expression can be further written as

$$[S(x)\beta S(x),T(x)]_\alpha=0. \quad (24)$$

Using (24) in (23), we get

$$[S(x),T(x)]_\alpha\gamma I(y)\delta[S(x),T(x)]_\alpha\mu S(x)=0. \quad (25)$$

Replacing y by $y\gamma I(S(x))$ in (25), we obtain

$$[S(x),T(x)]_\alpha\gamma S(x)\mu I(y)\delta[S(x),T(x)]_\alpha\mu S(x)=0. \quad (26)$$

Since M is a semiprime Γ-ring it follows from relation (26) that

$$[S(x),T(x)]_\alpha\mu S(x)=0. \quad (27)$$

In view of relation (24) and (27), we have

$$S(x)\mu[S(x),T(x)]_\alpha=0. \quad (28)$$

Replacing x by $x+y$ in (28) and using the same techniques as we used to obtain (16) from (13), we get

$$S(y)\mu[S(x),T(x)]_\alpha+S(x)\mu[S(y),T(x)]_\alpha$$
$$+S(x)\mu[S(x),T(y)]_\alpha=0. \quad (29)$$

Substituting $y\beta x$ for y in (29), we obtain $S(x)\beta I(y)\mu[S(x),T(x)]_\alpha+S(x)\mu S(x)\beta[I(y),T(x)]_\alpha+S(x)\mu[S(x),T(x)]_\alpha\beta I(y)+S(x)\mu[S(x),T(x)]_\alpha\beta I(y)+S(x)\mu T(x)\beta[S(x),I(y)]_\alpha=0$. This implies that

$$S(x)\beta I(y)\mu[S(x),T(x)]_\alpha+S(x)\mu S(x)\beta[I(y),T(x)]_\alpha$$
$$+S(x)\mu T(x)\beta[S(x),I(y)]_\alpha=0. \quad (30)$$

Thus we have the relation

$$S(x)\beta I(y)\mu[S(x),T(x)]_\alpha+S(x)\mu S(x)\beta[I(y),T(x)]_\alpha$$
$$+S(x)\mu T(x)\beta[S(x),I(y)]_\alpha=0,$$

which can be further written in the form

$$S(x)\beta I(y)\mu[S(x),T(x)]_\alpha+S(x)\mu S(x)\beta I(y)\alpha T(x)$$
$$-S(x)\mu T(x)\beta I(y)\alpha S(x)+S(x)\beta[T(x),S(x)]_\alpha\beta I(y)=0.$$

Application of (28) forces that

$$S(x)\beta I(y)\mu[S(x),T(x)]_\alpha + S(x)\mu S(x)\beta I(y)\alpha T(x) - S(x)\mu T(x)\beta I(y)\alpha S(x) = 0. \quad (31)$$

Left multiplication of (31) by $T(x)$ gives

$$T(x)\alpha S(x)\beta I(y)\mu[S(x),T(x)]_\alpha + T(x)\alpha S(x)\mu S(x)\beta I(y)\alpha T(x) - T(x)\alpha S(x)\mu T(x)\beta I(y)\alpha S(x) = 0. \quad (32)$$

On substituting $y\alpha I(T(x))$ for y in (31), we have

$$S(x)\mu T(x)\beta I(y)\alpha[S(x),T(x)]_\alpha + S(x)\mu S(x)\beta T(x)\alpha I(y)\alpha T(x) - S(x)\mu T(x)\beta S(x)\alpha I(y)\alpha S(x) = 0. \quad (33)$$

Combining (32) and (33), we obtain

$$[S(x),T(x)]_\alpha\mu I(y)\beta[S(x),T(x)]_\alpha + [S(x)\alpha S(x),T(x)]_\alpha\beta I(y)\mu T(x) + [T(x),S(x)]_\alpha\beta T(x)\mu I(y)\alpha S(x) = 0. \quad (34)$$

Using (24), the above expression reduces to

$$[S(x),T(x)]_\alpha\beta I(y)\mu[S(x),T(x)]_\alpha + [T(x),S(x)]_\alpha\beta T(x)\mu I(y)\alpha S(x) = 0. \quad (35)$$

Substituting $z\beta S(x)I(y)$ for y in (35), we get

$$[S(x),T(x)]_\alpha\beta I(y)\mu S(x)\beta I(z)\delta[S(x),T(x)]_\alpha + [T(x),S(x)]_\alpha\beta T(x)\delta I(y)\mu S(x)\beta I(z)\delta S(x) = 0. \quad (36)$$

On the other hand right multiplying to (35) by $I(z)\delta S(x)$, we get

$$[S(x),T(x)]_\alpha\beta I(y)\beta[S(x),T(x)]_\alpha\delta I(z)\mu S(x) + [T(x),S(x)]_\alpha\beta T(x)\mu I(y)\alpha S(x)\delta I(z)\delta S(x) = 0. \quad (37)$$

On comparing (36) and (37), we obtain

$$[S(x),T(x)]_\alpha\beta I(y)\delta A(x,z) = 0, \quad (38)$$

where
$A(x,z) = [S(x),T(x)]_\alpha\beta I(z)\mu S(x) - S(x)\mu I(z)\beta[S(x),T(x)]_\alpha$.
Substituting $y\delta I(S(x))\mu z$ for y in (38) gives

$$[S(x),T(x)]_\alpha\beta I(z)\delta S(x)\mu I(y)\delta A(x,z) = 0. \quad (39)$$

Left multiplying to (38) by $S(x)\mu I(z)$, we get

$$S(x)\beta I(z)\delta[S(x),T(x)]_\alpha\beta I(y)\mu A(x,z) = 0. \quad (40)$$

From (39) and (40), we arrive at $A(x,z)\mu y\delta A(x,z) = 0$. That is, $A(x,z)\Gamma M\Gamma A(x,z) = 0$. The semiprimeness of M forces that $A(x,z) = 0$. In other words, we have

$$[S(x),T(x)]_\alpha\beta I(z)\delta S(x) = S(x)\delta I(z)\mu[S(x),T(x)]_\alpha. \quad (41)$$

Replacing z by $y\mu I(T(x))$ in (41), we obtain

$$[S(x),T(x)]_\alpha\beta T(x)\mu I(y)\delta S(x) = S(x)\delta T(x)\beta I(y)\mu[S(x),T(x)]_\alpha. \quad (42)$$

Combining (35) and (42), we obtain $[S(x),T(x)]_\alpha\beta I(y)\delta[S(x),T(x)]_\alpha - S(x)\delta T(x)\beta I(y)\mu[S(x),T(x)]_\alpha = 0$. This further reduces to

$$T(x)\delta S(x)\beta I(y)\mu[S(x),T(x)]_\alpha = 0. \quad (43)$$

If we substitute $y\alpha I(T(x))$ for y in (43), we find that

$$T(x)\delta S(x)\beta T(x)\alpha I(y)\mu[S(x),T(x)]_\alpha = 0. \quad (44)$$

Multiplying (43) from the left side by $T(x)$, we get

$$T(x)\delta T(x)\mu S(x)\beta I(y)\alpha[S(x),T(x)]_\alpha = 0. \quad (45)$$

Subtracting (45) from (44), we get

$$T(x)\delta[S(x),T(x)]_\alpha\mu I(y)\beta[S(x),T(x)]_\alpha = 0. \quad (46)$$

Replacing $I(T(x))\delta y$ for y in (46), we obtain

$$T(x)\delta[S(x),T(x)]_\alpha\beta I(y)\delta T(x)\mu[S(x),T(x)]_\alpha = 0. \quad (47)$$

That is, $T(x)\Gamma[S(x),T(x)]_\Gamma\Gamma M\Gamma T(x)[S(x),T(x)]_\Gamma = 0$ for all $x \in M$. The semiprimeness of M yields that

$$T(x)\Gamma[S(x),T(x)]_\Gamma = 0. \quad (48)$$

Replacing y by $I(T(x))\alpha y$ in (42) gives, because of (48)

$$[S(x),T(x)]_\alpha\beta I(y)\delta T(x)\mu S(x) = 0. \quad (49)$$

Substituting $x+y$ for x in (27) and using the same approach as we used to obtain (16) from (13), we get

$$[S(x),T(x)]_\alpha\beta S(y) + [S(x),T(y)]_\alpha\beta S(x) + [S(y),T(x)]_\alpha\beta S(x) = 0. \quad (50)$$

On substituting y x for y in (50), we obtain $[S(x),T(x)]_\alpha \beta S(x)\beta I(y) + T(x)\beta[S(x),I(y)]_\alpha\beta S(x) + [S(x),T(x)]_\alpha\beta I(y)\beta S(x) + [S(x),T(x)]_\alpha\beta I(y)\beta S(x) + S(x)\beta[I(y),T(x)]_\alpha\beta S(x) = 0$. Application of (27) yields that

$$[S(x),T(x)]_\alpha\beta I(y)\beta S(x) + T(x)\beta[S(x),I(y)]_\alpha\beta S(x) + [S(x),T(x)]_\alpha\beta I(y)\beta S(x) + S(x)\beta[I(y),T(x)]_\alpha\beta S(x) = 0. \quad (51)$$

This implies that

$$2[S(x),T(x)]_\alpha\beta I(y)\beta S(x) + T(x)\beta[S(x),I(y)]_\alpha\beta S(x) + S(x)\beta[I(y),T(x)]_\alpha\beta S(x) = 0. \quad (52)$$

This can be further written as

$2[S(x),T(x)]_\alpha\beta I(y)\beta S(x) + T(x)\beta S(x)\beta I(y)\alpha S(x) - T(x)\alpha I(y)\beta S(x)\beta S(x) + S(x)\alpha I(y) \quad \beta T(x)\beta S(x) - S(x)\beta T(x)\alpha I(y)\beta S(x) = 0$, which reduces to

$$[S(x),T(x)]_\alpha\beta I(y)\beta S(x) + S(x\beta)I(y)\alpha T(x)\beta S(x) - T(x)\beta I(y)\alpha S(x)\beta S(x) = 0. \quad (53)$$

Using (41) in (53), we obtain $0 = S(x)\beta I(y)\beta[S(x),T(x)]_\alpha + S(x)\beta I(y)\alpha T(x)\beta S(x) - Tx)\beta I(y)\beta S(x)\alpha S(x) = S(x)\beta I(y)\alpha S(x)\alpha T(x) - T(x)\beta I(y)\alpha S(x)\beta S(x)$.The above expression yields that

$$S(x)\alpha I(y)\beta S(x)\beta T(x) = T(x)\beta I(y)\alpha S(x)\beta S(x). \quad (54)$$

Substituting $y\alpha I(T(x))$ for y in (54), we have

$$S(x)\alpha T(x)\beta I(y)\beta S(x)\alpha T(x) = T(x)\beta T(x)\alpha I(y)\alpha S(x)\beta S(x). \quad (55)$$

Left multiplication to (54) by $T(x)$ leads to

$$T(x)\alpha S(x)\beta I(y)\alpha S(x)\beta T(x) = T(x)\alpha T(x)\beta I(y)\beta S(x)\alpha S(x). \quad (56)$$

By combining (55) and (56), we arrive at

$$[S(x), T(x)]_\alpha \beta I(y)\beta S(x)\alpha T(x) = 0. \quad (57)$$

From (49) and (57), we obtain $[S(x), T(x)]_\alpha \beta I(y)\beta[S(x), T(x)]_\alpha = 0$. That is, $[S(x), T(x)]_\Gamma \Gamma M \Gamma [S(x), T(x)]_\Gamma = 0$. The semiprimeness of M yields that $[S(x), T(x)]_\alpha = 0$ for all$x \in M$ and $\alpha \in \Gamma$. If M is prime, then in view of Lemma 2.5 we get the required result. Thereby the proof of theorem is completed. □

Theorem 3.2. *Let M be a 2-torsion free semiprime Γ-ring with involution I satisfying the condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, and $S, T : M \to M$ be Jordan left-I-centralizers. Suppose that $[S(x), T(x)]_\alpha \beta S(x) - S(x)\beta[S(x), T(x)]_\alpha = 0$ holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then, $[S(x), T(x)]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$. Moreover, if M is a prime Γ-ring and $S \neq 0$ ($T \neq 0$), then there exists $p \in C(M)$ such that $T = pS$ ($S = qT$, $q \in C(M)$).*

Proof. We notice that S and T are reverse left-*I*-centralizers by Lemma 2.3. By the assumption we have the relation

$$[S(x), T(x)]_\alpha \beta S(x) - S(x)\beta[S(x), T(x)]_\alpha = 0, \quad (58)$$

for all $x \in M$ and $\alpha, \beta \in \Gamma$ Replacing x by $x + y$ in (58) and using similar techniques as we used to obtain (16) from (13), we find that

$$[S(x), T(x)]_\alpha \beta S(y) + [S(x), T(y)]_\alpha \beta S(x) + [S(y), T(x)]_\alpha \beta S(x) - S(y)\beta[S(x), T(x)]_\alpha - S(x)\beta[S(x), T(y)]_\alpha - S(x)\beta[S(y), T(x)]_\alpha = 0. \quad (59)$$

Substituting $y\delta x$ for y in (59), we obtain

$$[S(x), T(x)]_\alpha \beta S(x)\delta I(y) + [S(x), T(x)]_\alpha \delta I(y)\beta S(x) + T(x)\delta[S(x), I(y)]_\alpha \beta S(x) + [S(x), T(x)]_\alpha \beta I(y)\delta S(x) + S(x)\beta[I(y), T(x)]_\alpha \delta S(x) - S(x)\delta I(y)\beta[S(x), T(x)]_\alpha - S(x)\beta T(x)\delta[S(x), I(y)]_\alpha - S(x)\beta[S(x), T(x)]_\alpha \delta I(y) - S(x)\beta S(x)\delta[I(y), T(x)]_\alpha - S(x)\beta[S(x), T(x)]_\alpha \delta I(y) = 0. \quad (60)$$

Application of (58) forces that

$$2[S(x), T(x)]_\alpha \beta I(y)\delta S(x) + T(x)\beta[S(x), I(y)]_\alpha \delta S(x) + S(x)\beta[I(y), T(x)]_\alpha \delta S(x) - S(x)\delta I(y)\beta[S(x), T(x)]_\alpha - S(x)\beta T(x)\delta[S(x), I(y)]_\alpha - S(x)\beta[S(x), T(x)]_\alpha \delta I(y) - S(x)\delta S(x)\beta[I(y), T(x)]_\alpha = 0. \quad (61)$$

Substituting $I(S(x))\mu y$ for y in (61), we have

$$2[S(x), T(x)]_\alpha \mu I(y)\beta S(x)\delta S(x) + T(x)\beta[S(x), I(y)]_\alpha \mu S(x)\delta S(x) + S(x)\beta[I(y), T(x)]_\alpha \mu S(x)\delta S(x) + S(x)\beta I(y)\mu[S(x), T(x)]_\alpha \delta S(x) - S(x)\beta I(y)\delta S(x)\mu[S(x), T(x)]_\alpha - S(x)\beta T(x)\delta[S(x), I(y)]_\alpha \mu S(x) - S(x)\beta[S(x), T(x)]_\alpha \delta I(y)\mu S(x) - S(x)\mu S(x)\beta I(y)\delta[S(x), T(x)]_\alpha - S(x)\beta S(x)\delta[I(y), T(x)]_\alpha \mu S(x) = 0. \quad (62)$$

Using (61) in (62), we conclude that

$$S(x)\beta I(y)\delta[S(x), T(x)]_\alpha \mu S(x) - S(x)\beta S(x)\delta I(y)\mu[S(x), T(x)]_\alpha = 0. \quad (63)$$

Substituting $yI(T(x))$ for y in the above relation, we obtain

$$S(x)\beta T(x)\beta I(y)\delta[S(x), T(x)]_\alpha \mu S(x) - S(x)\beta S(x)\delta T(x)\beta I(y)\mu[S(x), T(x)]_\alpha = 0. \quad (64)$$

On the other hand left multiplication of (63) by $T(x)$ gives

$$T(x)\delta S(x)\beta I(y)\delta[S(x), T(x)]_\alpha \beta S(x) - T(x)\beta S(x)\delta S(x)I(y)\beta[S(x), T(x)]_\alpha = 0. \quad (65)$$

By comparing (64) and (65), we obtain $0 = [S(x), T(x)]_\alpha \beta I(y)\delta[S(x), T(x)]_\alpha \delta S(x) - [S(x)\beta S(x), T(x)]_\alpha I(y)\beta[S(x), T(x)]_\alpha = [S(x), T(x)]_\alpha \delta I(y)\beta[S(x), T(x)]_\alpha \beta S(x) - ([S(x), T(x)]_\alpha \beta S(x) + S(x)\beta[S(x), T(x)]_\alpha)I(y)\beta[S(x), T(x)]_\alpha$. In view of the hypothesis, the above expression reduces to

$$[S(x), T(x)]_\alpha \delta I(y)\beta[S(x), T(x)]_\alpha S(x) - 2S(x)\delta\beta[S(x), T(x)]_\alpha \beta I(y)\delta[S(x), T(x)]_\alpha. \quad (66)$$

If we multiply (66) by $S(x)$ from left, we get

$$S(x)\delta[S(x), T(x)]_\alpha \delta I(y)\beta[S(x), T(x)]_\alpha S(x) - 2S(x)\delta S(x)\beta[S(x), T(x)]_\alpha I(y)\beta[S(x), T(x)]_\alpha = 0. \quad (67)$$

On the other hand putting $y[S(x), T(x)]_\alpha$ for y in (63), we arrive at

$$S(x)\beta[S(x), T(x)]_\alpha \beta I(y)\beta[S(x), T(x)]_\alpha \beta S(x) - S(x)\beta S(x)\delta[S(x), T(x)]_\alpha \beta I(y)\delta[S(x), T(x)]_\alpha = 0. \quad (68)$$

By combining (67) and (68), we obtain

$$S(x)\beta[S(x), T(x)]_\alpha \delta I(y)\beta[S(x), T(x)]_\alpha \beta S(x) = 0. \quad (69)$$

Using (58) in the above expression, we obtain

$$S(x)\beta[S(x), T(x)]_\alpha \delta I(y)\beta S(x)\beta[S(x), T(x)]_\alpha = 0. \quad (70)$$

Since M is semiprime, it follows that $S(x)\beta[S(x), T(x)]_\alpha = 0$. From (69) and (58), we get $[S(x), T(x)]_\alpha \beta S(x) = 0$. The last two expressions are same as Eqs. (27) and (28) and hence, by using similar approach as we have used after (27) and (28) in the proof of Theorem 3.1, we get the required result. The theorem is thereby proved. □

Corollary 3.3. *Let M be a 2-torsion free semiprime Γ-ring with involution I satisfying the condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, and $T : M \to M$ a Jordan left-I-centralizer.*

(1) Suppose that $\langle T(x), I(x)\rangle_\alpha \beta I(x) - I(x)\beta\langle T(x), I(x)\rangle_\alpha = 0$ holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then, T is a reverse I-centralizer on M.
(2) Suppose that $\langle T(x), I(x)\rangle_\alpha \beta T(x) - T(x)\beta\langle T(x), I(x)\rangle_\alpha = 0$ holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then, T is a reverse I-centralizer on M.
(3) Suppose that $[T(x), I(x)]_\alpha \beta I(x) - I(x)\beta[T(x), I(x)]_\alpha = 0$ holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. In this case, T is a reverse I-centralizer on M.
(4) Suppose that $[T(x), I(x)]_\alpha \beta T(x) - T(x)\beta[T(x), I(x)]_\alpha = 0$ holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. In this case, T is a reverse I-centralizer on M.

Acknowledgment

The authors are highly grateful to referees for their valuable comments and suggestions for improving the paper.

References

[1] S. Ali, N.A. Dar, J. Vukman, Jordan left ★-centralizers of prime and semiprime rings with involution, Beitr. Algebra Geom. (2012), http://dx.doi.org/10.1007/s13366-012-0117-3.
[2] W.E. Barnes, On the Γ-rings of Nobusawa, Pacific J. Math. 18 (1966) 411–422.
[3] M. Bresar, B. Zalar, On the structure of Jordan ★-derivations, Colloq. Math. 63 (2) (1992) 163–171.
[4] Y. Ceven, Jordan left derivations on completely prime Γ-ring, C.U. Fen-Edebiyat Fakultesi Fen Bilimlere Dergisi 23 (2) (2002) 39–43.
[5] M.F. Hoque, A.C. Paul, On centralizers of semiprime gamma rings, Int. Math. Forum 6 (13) (2011) 627–638.
[6] N. Nabusawa, On a generalization of the ring theory, Osaka J. Math. 1 (1964) 65–75.
[7] M.A. Ozturk, Y.B. Jun, On the centroid of the prime gamma rings, Comm. Korean Math. Soc. 15 (3) (2000) 469–479.
[8] M. Soyturk, The commutativity in prime gamma rings with derivation, Turkish J. Math. 18 (2) (1994) 149–155.