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ORIGINAL ARTICLE

On Jordan *-mappings in rings with involution

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Abstract The objective of this paper is to study Jordan $*$-mappings in rings with involution $*$. In particular, we prove that if R is a prime ring with involution $*$, of characteristic different from 2 and D is a nonzero Jordan $*$-derivation of R such that $[D(x), x] = 0$, for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative. Further, we also prove a similar result in the setting of Jordan left $*$-derivation. Finally, we prove that any symmetric Jordan triple $*$-biderivation on a 2-torsion free semiprime ring with involution $*$ is a symmetric Jordan $*$-biderivation.

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1. Introduction

This paper deals with the study of Jordan $*$-derivations, Jordan left $*$-derivations and symmetric Jordan triple $*$-biderivations in rings with involution. Throughout this article, R will represent an associative ring with center $Z(R)$. A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a prime ring if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$, and is called a semiprime ring in case $aRa = (0)$ implies $a = 0$. We write $[x, y]$ for $xy - yx$ and $x \circ y$ for $xy + yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$, and a Lie ideal U is called a square-closed if $u^2 \in U$ for all $u \in U$. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in R$. A ring equipped with an involution is known as ring with involution or $*$-ring. An element x in a ring with involution $*$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. Note that in this case x is normal i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a normal ring. An example is the ring of quaternions. A description of such rings can be found in [1], where further references are given.

An additive map $D : R \to R$ is said to be a derivation (resp. Jordan derivation) of a ring R if

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$D(xy) = D(x)y + xD(y)$ (resp. $(D(x^2) = D(x)x + xD(x))$ holds for all $x, y \in R$. An additive map $D: R \to R$ is said to be Jordan triple derivation if $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ for all $x, y \in R$. Obviously every Jordan derivation is a Jordan triple derivation, but the converse is not true in general. A classical result due to Brešar [2, Theorem 4.3], asserts that a Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. An additive map D of R into itself is said to be a left derivation (resp. Jordan left derivation) of R if $D(xy) = xD(y) + yD(x)$ (resp. $D(x^2) = 2xD(x)$) holds for all $x, y \in R$. A map f of R into itself is called centralizing if $[f(x), x] \in Z(R)$ holds for all $x \in R$; in the special case when $[f(x), x] = 0$ holds for all $x \in R$, the map f is said to be commuting. The history of commuting and centralizing maps goes back to 1955 when Divinsky [3], proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. Two years later, Posner [4] has proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Several authors have proved commutativity theorems for prime and semiprime rings admitting automorphisms or derivations which are commuting or centralizing on an appropriate subset of the ring (see [5] for partial bibliography). In [6], Brešar and Vukman showed that a prime ring must be commutative if it admits a nonzero left derivation. Further in [7], the first named author together with Ashraf and Rehman proved that if R is a 2-torsion free prime ring and $D: R \to R$ is an additive map, such that $D(x^2) = 2xD(x)$ for all x in a square closed Lie ideal U of R, then either $U \subseteq Z(R)$ or $D(U) = (0)$. The above mentioned result was extended for semiprime rings in [8]. During the last few decades, there has been ongoing interest concerning the relationship between the left derivation and Jordan left derivation on prime and semiprime rings (cf.; [9–12] for further references).

Let R be a ring with involution $*$. According to [13], an additive map $D: R \to R$ is said to be a $*$-derivation (resp. Jordan $*$-derivation) of R if $D(xy) = D(x)y^* + xD(y)$ (resp. $D(x^2) = D(x)x^* + xD(x)$) holds for all $x, y \in R$. Notice that the map $x \mapsto x - x^*$ is a Jordan $*$-derivation of R. An additive map $D: R \to R$ is said to be Jordan triple $*$-derivation if $D(xyx) = D(x)y^*x^* + xD(y)x^* + xyD(x)$ for all $x, y \in R$. Clearly every Jordan $*$-derivation is a Jordan triple $*$-derivation, but the converse is not true in general. However, Fošner and Iliševič [14, Theorem 5.2] proved that a Jordan triple $*$-derivation on a 2-torsion free semiprime ring with involution $*$ is a Jordan $*$-derivation. By our knowledge the concept of Jordan $*$-derivations appears for the first time in the work of Brešar and Vukman [15]. The notion of Jordan $*$-derivations arises naturally in the theory of representability of quadratic functionals with sesquilinear functionals (see [16], where further references can be found). For results concerning this theory we refer the reader to [17–19]. Inspired by the definition of $*$-derivation (resp. Jordan $*$-derivation), we introduce the notion of left $*$-derivation (resp. Jordan left $*$-derivation) as follows: an additive map D of R into itself is called a left $*$-derivation (resp. Jordan left $*$-derivation) of R if $D(xy) = y^*D(x) + xD(y)$ (resp. $D(x^2) = x^*D(x) + xD(x)$) holds for all $x, y \in R$. It is to remark that, in case of a commutative ring with involution, any Jordan $*$-derivation is a Jordan left $*$-derivation and vice versa. However, the above statement need not be true for arbitrary rings. The following example justifies this fact:

Example 1.1. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}$. Then R is a noncommutative ring under usual matrix operations. Define the maps $D: R \longrightarrow R$, and $*: R \longrightarrow R$ as follows:

$$D\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, it is easy to verify that D is a Jordan left $*$-derivation on R, but D is not a Jordan $*$-derivation on R.

It can be easily proved that every left $*$-derivation on a noncommutative prime $*$-ring is zero (see Proposition 2.2). In [15, Theorem 3], Brešar and Vukman proved that a noncommutative prime ring of characteristic different from 2 is normal if and only if there exists a nonzero commuting Jordan $*$-derivation on R. Motivated by this result, in Section 3 we prove that if a prime ring R with involution, of characteristic different from 2 admits a nonzero Jordan $*$-derivation such that $[D(x), x] = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative. Further, we establish a similar result in the setting of Jordan left $*$-derivations in prime rings.

A symmetric biadditive map $B: R \times R \to R$ is called a symmetric biderivation if $B(xy, z) = B(x, z)y + xB(y, z)$ is fulfilled for all $x, y, z \in R$. The concept of a symmetric biderivation was introduced by Maksa in [20] (see also [21], where an example can be found). A symmetric biadditive map $B: R \times R \to R$ is said to be a symmetric Jordan biderivation if $B(x^2, z) = B(x, z)x + xB(x, z)$ holds for all $x, z \in R$. Following [22], a symmetric biadditive map $B: R \times R \to R$ is called a symmetric $*$-biderivation if $B(xy, z) = B(x, z)y^* + xB(y, z)$ holds for all $x, y, z \in R$, where R is a ring with involution $*$. Motivated by the definitions of Jordan $*$-derivation and Jordan triple $*$-derivation in rings with involution, we introduce the concept of symmetric Jordan $*$-biderivation and symmetric Jordan triple $*$-biderivation as follows: A symmetric biadditive map $D: R \times R \to R$ is said to be a symmetric Jordan $*$-biderivation if $D(x^2, z) = D(x, z)x^* + xD(x, z)$ holds for all $x, z \in R$. A symmetric biadditive map $D: R \times R \to R$ is called a symmetric Jordan triple $*$-biderivation if $D(xyx, z) = D(x, z)y^*x^* + xD(y, z)x^* + xyD(x, z)$ holds for all $x, y, z \in R$. It is obvious to see that every symmetric Jordan $*$-biderivation on 2-torsion free ring with involution is a symmetric Jordan triple $*$-biderivation. But the converse need not be true in general. In the last section of the present paper, our aim is to establish set of conditions under which every symmetric Jordan triple $*$-biderivation on a ring with involution $*$ is a symmetric Jordan $*$-biderivation. More precisely, we prove that on a 2-torsion free semiprime ring with involution, every symmetric Jordan triple $*$-biderivation is a symmetric Jordan $*$-biderivation.

2. Preliminaries

Throughout this paper, we make an extensive use of the basic commutator identities $[x, yz] = y[x, z] + [x, y]z$ and $[xy, z] = [x, z]y + x[y, z]$. Moreover, we need the following lemma to develop the proof of our main results:

Lemma 2.1. *Let R be a prime ring with involution $*$, of characteristic different from 2. If $S(R) \subseteq Z(R)$, then R is commutative.*

Proof. By the assumption, we have $S(R) \subseteq Z(R)$. This gives $[k, x] = 0$ for all $k \in S(R)$ and $x \in R$. If $h \in H(R)$, $k \in S(R)$, then $hk + kh \in S(R)$ and hence $0 = [hk + kh, x] = [h, x]k + k[h, x] = 2k[h, x]$, for all $x \in R$ and $h \in H(R), k \in S(R)$. Since $\mathrm{char}(R) \neq 2$, we get $k[h, x] = 0$, for all $x \in R$ and $h \in H(R)$, $k \in S(R)$. Using the primeness of R we get either $K = (0)$ or $H(R) \subseteq Z(R)$. If $K = (0)$, then for every $x \in R$, $2x \in H(R)$. Therefore, we obtain $(2x2y)^* = (2y)^*(2x)^*$, for all $x, y \in R$. This implies that $2x2y = 2y2x$. That is, $4xy = 4yx$, for all $x, y \in R$. Since $\mathrm{char}(R) \neq 2$, we get $xy = yx$, for all $x, y \in R$, what proves that R is commutative. On the other hand, suppose $H(R) \subseteq Z(R)$. Since R is a prime ring with involution, of characteristic different from 2, every $x \in R$ can be represented as $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. This gives $2R \subseteq Z(R)$. Since $\mathrm{char}(R) \neq 2$, we get $R \subseteq Z(R)$. Hence, R is commutative. Therefore the proof is completed. □

Proposition 2.2. *Let R be a noncommutative prime ring with involution $*$ and let $D : R \to R$ be a left $*$-derivation on R. Then $D = 0$.*

Proof. Computing $D(xyz)$ in two different ways and comparing the so obtained two expressions we get $[x, z^*]D(y) = 0$, for all $x, y, z \in R$. Replacing x by xt, we get $0 = [xt, z^*]D(y) = [x, z^*]tD(y) + x[t, z^*]D(y) = [x, z^*]tD(y)$, for all $x, y, z, t \in R$. This implies that $[x, z^*]RD(y) = (0)$, for all $x, y, z \in R$. The primeness of R yields either $[x, z^*] = 0$, for all $x, y \in R$, or $D(y) = 0$, for all $y \in R$. Since R is noncommutative, we are forced to conclude that $D(y) = 0$, for all $y \in R$. Hence $D = 0$. □

3. On Jordan $*$-derivations in prime rings

In [15, Theorem 3] Brešar and Vukman proved the following result:

Theorem 3.1. *Let R be a noncommutative prime ring with involution $*$, of characteristic different from 2. Then R is normal if and only if there exists a nonzero commuting Jordan $*$-derivation $D : R \to R$.*

This result motivated us to prove the following result:

Theorem 3.2. *Let R be a prime ring with involution $*$, of characteristic different from 2. Let D be a nonzero Jordan $*$-derivation of R such that $[D(x), x] = 0$, for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$. Then R is commutative.*

Proof. We are given that $D : R \to R$ is a Jordan $*$-derivation and hence in view of [[23], Theorem 3.2], we conclude that $D(x) = px + \lambda(x)$ for all $x \in R$, where $p \in C$ and $\lambda : R \to C$. Suppose $p \neq 0$. Since D is a nonzero Jordan $*$-derivation, the above relation yields that $px^2 + \lambda(x^2) = pxx^* + px^2 + \lambda(x)x^* + \lambda(x)x$. This implies that

$$0 = pxx^* + \lambda(x)x^* + \lambda(x)x - \lambda(x^2)$$

for all $x \in R$. If $x = k \in S(R)$ skew symmetric element, then we arrive at $0 = pk^2 + \lambda(k^2)$. Therefore $[pk^2, y] = 0$ for all $y \in R$ and $k \in S(R)$. It is easy to verify that $p[k^2, y] = 0$ for all $y \in R$ and $k \in S(R)$. In case $p \neq 0$, we get $k^2 \in Z(R)$ for all $x \in K$. On the other hand, if $p = 0$, then $D(x) = \lambda(x)$ and hence $[D(x), y] = 0$ for all $x, y \in R$. Replacing x by x^2 and using the fact D is a Jordan $*$-derivation, we get $0 = [D(x^2), y] = [D(x)x^* + xD(x), y] = D(x)[x^*, y] + [x, y]D(x)$. This further implies that $D(x)[x + x^*, y] = 0$, for all $x, y \in R$. Replacing y by yz in the last expression, we obtain $D(x)y[x + x^*, z] = 0$, for all $x, y, z \in R$. Thus for each $x \in R$, by the primeness of R either $D(x) = 0$ or $[x + x^*, z] = 0$. Now let $A = \{x \in R | D(x) = 0, \}$ and $B = \{x \in R | [x + x^*, z] = 0, \mathrm{for all} z \in R\}$. Thus A and B are additive subgroups of R and $R = A \cup B$. But a group cannot be a union of two of its proper subgroups and hence either $R = A$ or $R = B$. Since we have assumed $D \neq 0$, we have $R = B$, that is $[x + x^*, z] = 0$, for all $x, z \in R$. Replacing x by $h + k$, where $h \in H(R)$ and $k \in S(R)$, we get $2[h, z] = 0$. Since $char(R) \neq 2$, we obtain $[h, z] = 0$, for all $h \in H(R)$ and $z \in R$. That is, $h \in Z(R)$, for all $h \in H(R)$. This further implies that $k^2 \in Z(R)$, for all $k \in S(R)$. Thus in both cases $k^2 \in Z(R)$ for all $k \in S(R)$. Now since $S(R) \cap Z(R) \neq (0)$, let $0 \neq k_0 \in S(R) \cap Z(R)$ and let k be an arbitrary element of $S(R)$. Then k^2, k_0^2 and $(k + k_0)^2 = k^2 + k_0^2 + 2kk_0$ are all in $Z(R)$; it follows that $2kk_0 \in Z(R)$ and hence $k \in Z(R)$ for all $k \in S(R)$. This implies that R is commutative in view of Lemma 2.1, thereby completing the proof of the theorem. □

We now prove another result in the spirit of above theorem in the setting of Jordan left $*$-derivations.

Theorem 3.3. *Let R be a prime ring with involution $*$, of characteristic different from 2. Let D be a nonzero Jordan left $*$-derivation of R such that $[D(x), x] = 0$, for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$. Then R is commutative.*

Proof. By the assumption, we have $D(x) = px + \lambda x$ for all $x \in R$, where $p \in C$ and $\lambda : R \to C$. Suppose $p \neq 0$. Since D is a non zero Jordan left $*$-derivation, so the above relation yields that $px^2 + \lambda(x^2) = px^*x + px^2 + x^*\lambda(x) + x\lambda(x)$. This implies that

$$px^*x + x^*\lambda(x) + x\lambda(x) - \lambda(x^2) = 0$$

for all $x \in R$. Now using the same technique with necessary variations as we have used in the proof of Theorem 3.2, we get the required result. □

4. On Jordan $*$-biderivations in semiprime rings

In this section, we study a biadditive mapping $D : R \times R \to R$ satisfying

$$D(x^2, z) = D(x, z)x^* + xD(x, z),$$

$$D(xyx, z) = D(x, z)y^*x^* + xD(y, z)x^* + xyD(x, z)$$

for all $x, y, z \in R$. Since we investigate these mappings in the setting of a 2-torsion free semiprime ring with involution, the first identity implies second identity (see Lemma 4.2), but not conversely. The purpose of the present section is to establish set of conditions under which second identity implies first identity. We begin our discussion with the following lemma whose proof can be found at the beginning of [[24], Section 2].

Lemma 4.1. *Let R is a semiprime ring with involution $*$, if $x \in R$, then $yxy^* = 0$ for all $y \in R$ implies that $x = 0$.*

Lemma 4.2. *Let R be a 2-torsion free ring with involution $*$. If $D : R \times R \to R$ is a symmetric Jordan $*$-biderivation, then the following hold:*

(i) $D(xy + yx, z) = D(x, z)y^* + D(y, z)x^* + xD(y, z) + yD(x, z)$ *for all* $x, y, z \in R$;
(ii) $D(xyx, z) = D(x, z)y^*x^* + xD(y, z)x^* + xyD(x, z)$ *for all* $x, y, z \in R$;
(iii) $D(xyt + tyx, z) = D(x, z)y^*t^* + xD(y, z)t^* + xyD(t, z) + D(t, z)y^*x^* + tD(y, z)x^* + tyD(x, z)$ *for all* $x, y, z, t \in R$.

Proof.

(i) For any $x, y \in R$, we have

$$\begin{aligned} D(xy + yx, z) &= D((x + y)^2, z) - D(x^2, z) - D(y^2, z) \\ &= D(x, z)y^* + D(y, z)x^* + xD(y, z) + yD(x, z). \end{aligned}$$

(ii) Replacing y by $xy + yx$ in (i), we get

$$\begin{aligned} &D(x(xy + yx) + (xy + yx)x, z) \\ &= D(x, z)y^*x^* + D(x, z)x^*y^* + D(x, z)y^*x^* \\ &\quad + D(y, z)(x^*)^2 + xD(y, z)x^* + yD(x, z)x^* \\ &\quad + xD(x, z)y^* + xD(y, z)x^* + x^2D(y, z) + xyD(x, z) \\ &\quad + xyD(x, z) + yxD(x, z) \end{aligned} \tag{4.1}$$

for all $x, y, z \in R$. On the other hand, we have

$$\begin{aligned} &D(x(xy + yx) + (xy + yx)x, z) \\ &= D(x^2y + yx^2, z) + 2D(xyx, z) \\ &= D(x, z)x^*y^* + xD(x, z)y^* + D(y, z)(x^*)^2 \\ &\quad + x^2D(y, z) + yD(x, z)x^* + yxD(x, z) + 2D(xyx, z) \end{aligned} \tag{4.2}$$

for all $x, y, z \in R$. Comparing (4.1) and (4.2) and using the fact that R is 2-torsion free, we get the required result.

(iii) Putting $x + t$ instead of x in (ii), we get

$$\begin{aligned} D((x + t)y(x + t), z) &= D(x + t, z)y^*(x^* + t^*) \\ &\quad + (x + t)D(y, z)(x^* + t^*) \\ &\quad + (x + t)yD(x + t, z) \\ &= D(x, z)y^*x^* + D(x, z)y^*t^* \\ &\quad + D(t, z)y^*x^* + D(t, z)y^*t^* \\ &\quad + xD(y, z)x^* + xD(y, z)t^* \\ &\quad + tD(y, z)x^* + tD(y, z)t^* \\ &\quad + xyD(x, z) + xyD(t, z) \\ &\quad + tyD(x, z) + tyD(t, z) \end{aligned}$$

for all $x, y, z, t \in R$. On the other hand, we have

$$\begin{aligned} D((x + t)y(x + t), z) &= D(xyx, z) + D(tyt, z) + D(xyt + tyx, z) \\ &= D(x, z)y^*x^* + xD(y, z)x^* + xyD(x, z) \\ &\quad + D(t, z)y^*t^* + tD(y, z)t^* + tyD(t, z) \\ &\quad + D(xyt + tyx, z) \end{aligned}$$

for all $x, y, z, t \in R$. Comparing so obtained relations we get the desired result. □

We are now ready to prove the main result of the present section:

Theorem 4.3. *Let R be a 2-torsion free semiprime with involution $*$. Then every symmetric Jordan triple $*$-biderivation $D : R \times R \to R$ is a symmetric Jordan $*$-biderivation.*

Proof. By the given assumption, we have

$$D(xyx, z) = D(x, z)y^*x^* + xD(y, z)x^* + xyD(x, z) \tag{4.3}$$

for all $x, y, z \in R$. In view of Lemma 4.2 (iii), we have

$$\begin{aligned} D(xyt + tyx, z) &= D(x, z)y^*t^* + xD(y, z)t^* + xyD(t, z) \\ &\quad + D(t, z)y^*x^* + tD(y, z)x^* + tyD(x, z) \end{aligned}$$

for all $x, y, z, t \in R$. Thus, we obtain

$$\begin{aligned} D((xy)^2, z) &= D(xyxy, z) = D(xy(xy) + (xy)yx - xy^2x, z) \\ &= D(xy(xy) + (xy)yx, z) - D(xy^2x, z) \\ &= D(x, z)(y^*)^2x^* + xD(y, z)y^*x^* + xyD(xy, z) \\ &\quad + D(xy, z)y^*x^* + xyD(y, z)x^* + xy^2D(x, z) \\ &\quad - D(x, z)(y^*)^2x^* - xD(y^2, z)x^* - xy^2D(x, z) \end{aligned}$$

for all $x, y, z \in R$. It follows that

$$\begin{aligned} &D((xy)^2, z) - D(xy, z)y^*x^* - xyD(xy, z) + x(D(y^2, z) \\ &\quad - D(y, z)y^* - yD(y, z)x^*) = 0 \end{aligned} \tag{4.4}$$

for all $x, y, z \in R$. Therefore relation (4.4) can be written as

$$\Delta(xy) + x\Delta(y)x^* = 0 \tag{4.5}$$

for all $x, y \in R$; where

$$\Delta(x) = D(x^2, z) - D(x, z)x^* - xD(x, z)$$

for all $x, z \in R$. In view of relation (4.5), we find that

$$\begin{aligned} 2ty\Delta(x)y^*t^* &= ty\Delta(x)y^*t^* + ty\Delta(x)y^*t^* \\ &= -t\Delta(yx)t^* - \Delta((ty)x) = -t\Delta(yx)t^* - \Delta(tyx) \\ &= \Delta(tyx) - \Delta(tyx) = 0 \end{aligned}$$

for all $x, y, t \in R$. Thus $2ty\Delta(x)y^*t^* = 0$ for all $x, y, t \in R$. Since R is 2-torsion free, the above relation yields that $ty\Delta(x)y^*t^* = 0$ for all $x, y, t \in R$. Hence, the application of Lemma 4.1 twice yields that $\Delta(x) = 0$ for all $x \in R$. That is, $D(x^2, z) - D(x, z)x^* - xD(x, z) = 0$ for all $x, z \in R$. Hence, D is a symmetric Jordan $*$-biderivation on R. Thereby, the proof is completed. □

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