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ORIGINAL ARTICLE

E-Bayesian estimation for the Lomax distribution based on type-II censored data

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Abstract This paper is concerned with using the E-Bayesian method for computing estimates of the unknown parameter and some survival time parameters e.g. reliability and hazard functions of Lomax distribution based on type-II censored data. These estimates are derived based on a conjugate prior for the parameter under the balanced squared error loss function. A comparison between the new method and the corresponding Bayes and maximum likelihood techniques is conducted using the Monte Carlo simulation.

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1. Introduction

The two-parameter Lomax distribution, denoted by Lomax(α, β), with probability density function (pdf) is defined as:

$$f(x;\alpha,\beta) = \alpha\beta(1+\beta x)^{-(\alpha+1)}, \quad x > 0, \quad (\alpha,\beta > 0), \tag{1}$$

and hence the cumulative distribution function (cdf)

$$F(x;\alpha,\beta) = 1-(1+\beta x)^{-\alpha}, \quad x > 0, \quad (\alpha,\beta > 0), \tag{2}$$

where α and β are the shape and scale parameters, respectively. This version of the Lomax distribution separates the two parameters and often simplifies the algebric in the subsequent Bayesian techniques. From (1) and (2) the reliability function $R(t)$, and the hazard (instantaneous failure rate) function $h(t)$ at mission time t for the Lomax distribution are

$$R(t) = (1+\beta t)^{-\alpha}, \qquad t > 0, \tag{3}$$

and

$$h(t) = \frac{\alpha\beta}{(1+\beta t)}, \qquad t > 0. \tag{4}$$

The Lomax (Pareto of the second kind or Pareto type-II) distribution can be considered as a mixture of the exponential gamma distribution. Lomax [10] used this distribution for analysis of the business failure data. Marshall and Olkin [11] have shown that the Lomax distribution can be applied as a lifetime distribution. Bryson [5] argued that Lomax distribution provides a very good alternative to common lifetime

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distribution such as exponential, Weibull, or gamma distributions when the experimenter presumes that the population distribution may be heavy-tailed. Details on Pareto distributions as well as areas of application can be found in Arnold [2]. Balkema and De Haan [4] showed that this distribution arises as a limit distribution of the residual lifetime at great age. Monotonicity of the hazard rate is well presented by Lomax distribution. Also, it has been shown its utilities for modeling and analyzing lifetime data in medical and biological sciences, engineering, etc. So, it has been received greatest attention from theoretical and statisticians primarily due to its use in reliability and lifetime testing studies. For many references and historical notes on this subject, we refer the interested reader to Balakrishnan and Aggarwala [3]. For more details, see Cramer and Schmiedit [6].

For estimating the parameter α, the reliability and the hazard functions of Lomax distribution based on balanced loss function (BLF), which is introduced by Zellner [13], we shall use the following form introduced by Ahmadi et al. [1]:

$$L^q_{\rho,\omega,\delta_0}(\Lambda(\alpha),\delta) = \omega q(\alpha)\rho(\delta_0,\delta) + (1-\omega)q(\alpha)\rho(\Lambda(\alpha),\delta), \quad (5)$$

where $q(.)$ is a suitable positive weight function and $\rho(\Lambda(\alpha),\delta)$ is an arbitrary loss function when estimating $\Lambda(\alpha)$ by δ. The parameter δ_0 is a chosen priori estimator of $\Lambda(\alpha)$, obtained for instance from the criterion of maximum likelihood, least squares or unbiasedness, among others. They give a general Bayesian connection between the case of $\omega > 0$ and $\omega = 0$ where $0 \leqslant \omega < 1$. By choosing $\rho(\Lambda(\alpha),\delta) = (\delta - \Lambda(\alpha))^2$ and $q(\alpha) = 1$, the BLF reduced to the balanced squared error loss (BSEL) function, used by Ahmadi et al. [1], in the form

$$L_{\omega,\delta_0}(\Lambda(\alpha),\delta) = \omega(\delta-\delta_0)^2 + (1-\omega)\rho(\delta-\Lambda(\alpha))^2, \quad (6)$$

and the corresponding Bayes estimate of the function $\Lambda(\alpha)$ is given by

$$\delta_{\omega,\Lambda,\delta_0}(\mathbf{x}) = \omega\delta_0 + (1-\omega)E(\Lambda(\alpha)|\mathbf{x}). \quad (7)$$

In this article, we consider type-II censored data from a two-parameter Lomax distribution. E-Bayes and Bayes approaches have been used for obtaining the estimates of the unknown parameter, and some other lifetime characteristics such as the reliability and hazard functions. Bayes estimators have been developed under BSEL function in Section 2. E-Bayes estimates are derived based on a conjugate prior for the parameter of interest and the balanced squared error loss function in Section 3. Properties of E-Bayesian estimation are carried out in Section 4. Finally, comparisons between the new method and the corresponding Bayes and maximum likelihood techniques are made using the Monte Carlo Simulation in Section 5.

2. Bayesian estimation

In this section, Bayes estimators of the parameters, reliability functions, and hazard rate functions are obtained by considering balanced squared error loss function. Based on type-II censored samples of size r obtained from a life test of n items from the Lomax(α,β) distribution, the likelihood function can be written as

$$L(\alpha,\beta|\underline{x}) = \frac{n!}{(n-r)!} \quad \alpha^r v(\beta;\underline{x})e^{-T\alpha}, \quad (8)$$

where

$$\underline{x} = (x_1, x_2, \ldots, x_r), \qquad v(\beta;\underline{x}) = \frac{\beta^r}{\prod_{i=1}^{r}(1+\beta x_i)}, \quad (9)$$

and

$$T \equiv T(\beta;\underline{x}) = \sum_{i=1}^{r}\ln(1+\beta x_i) + (n-r)\ln(1+\beta x_r). \quad (10)$$

When β is known, the maximum likelihood estimate (MLE) of the parameter α, is given by

$$\hat{\alpha}_{ML} = \frac{r}{T}. \quad (11)$$

By Eq. (11), the corresponding MLEs of the reliability function $R(t)$ and the hazard rate function $h(t)$ are obtained, respectively, from (3) and (4) after replacing α by its MLE, $\hat{\alpha}_{ML}$.

We use the gamma conjugate prior density for the parameter α as

$$g(\alpha|a,b) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha}, \qquad \alpha > 0, \quad (12)$$

where $a > 0$ and $b > 0$. This prior was first used by Papadopoulos [12]. The posterior density of α given $\underline{x}$ can be obtained from (8) and (12)

$$q(\alpha|\ \underline{x}) = \kappa\alpha^{r+a-1}e^{-(b+T)\alpha}, \qquad \alpha > 0, \quad (13)$$

where

$$\kappa = \frac{(b+T)^{r+a}}{\Gamma(r+a)}. \quad (14)$$

Under the BSEL function, the Bayes estimate of α can be given as:

$$\hat{\alpha}_{BS}(a,b) = \omega\left(\frac{r}{T}\right) + (1-w)\left(\frac{r+a}{b+T}\right). \quad (15)$$

For more details about the BLF, see for example, Zellner [13] and Ahmadi et al. [1].

The Bayes estimate of the reliability, based on the BSEL function is obtained from (3) and (13) as

$$\widehat{R}_{BS}(t) = \omega(1+\beta t)^{-\frac{r}{T}} + (1-w)\left(\frac{b+T}{b+T+\tau}\right)^{r+a}, \quad (16)$$

where

$$\tau \equiv \tau(\beta;t) = \ln(1+\beta t). \quad (17)$$

Similarly, the Bayes estimate of the hazard rate, based on the BSEL function is obtained from (4) and (13) as

$$\hat{h}_{BS}(t) = \left(\frac{\beta}{1+\beta t}\right)\left[\omega\left(\frac{r}{T}\right) + (1-w)\left(\frac{r+a}{b+T}\right)\right]. \quad (18)$$

3. E-Bayesian estimation

According to Han [8], the prior parameters a and b should be selected to guarantee that the prior $g(\alpha|a,b)$ in (12) be a decreasing function of α. The derivative of $g(\alpha|a,b)$ with respect to α is

$$\frac{dg(\alpha|a,b)}{d\alpha}=\frac{b^a}{\Gamma(a)}\alpha^{a-2}e^{-b\alpha}[(a-1)-b\alpha].$$

Thus, for $0<a<1, b>0$, the prior $g(\alpha|a,b)$ is a decreasing function of α.

Assuming that the hyperparameters a and b in (12) are independent and

$$\pi(a,b)=\pi_1(a)\pi_2(b),$$

the E-Bayesian estimate of α (expectation of the Bayesian estimate of α) is

$$\hat{\alpha}_{EB}=E(\alpha|\underline{X})=\int_{\varrho}\int\hat{\alpha}_{BS}(a,b)\pi(a,b)dadb, \tag{19}$$

where ϱ is the domain of a and b for which the prior density is decreasing in α. $\hat{\alpha}_B(a,b)$ is the Bayes estimate of α given by (15). For more details, see Han [9].

3.1. E-Bayesian estimate of α

E-Bayesian estimate of α is obtained based on three different distributions of the hyperparameters a and b. These distributions are used to investigate the influence of the different prior distributions on the E-Bayesian estimation of α.

The following distributions of $0<a<1$ and $0<b<s$ may be used

$$\left.\begin{aligned}\pi_1(a,b)&=\tfrac{1}{sB(u,v)}a^{u-1}(1-a)^{v-1},\\ \pi_2(a,b)&=\tfrac{2}{s^2B(u,v)}(s-b)a^{u-1}(1-a)^{v-1},\\ \pi_3(a,b)&=\tfrac{2b}{s^2B(u,v)}a^{u-1}(1-a)^{v-1},\end{aligned}\right\} \tag{20}$$

where $B(u,v)$ is the beta function. For $\pi_1(a,b)$, the E-Bayesian estimate of the parameter α is obtained from (15), (19) and (20) as

$$\begin{aligned}\hat{\alpha}_{EBS1}&=\int_{\varrho}\int\hat{a}_{BS}(a,b)\pi_1(a,b)dbda,\\ &=\frac{1}{sB(u,v)}\int_0^1\int_0^s\left[\omega\left(\frac{r}{T}\right)+(1-w)\left(\frac{r+a}{b+T}\right)\right]\\ &\quad\times a^{u-1}(1-a)^{v-1}dbda.\\ &=\frac{\omega r}{T}+\frac{1-w}{s}\left(r+\frac{u}{u+v}\right)\ln\left(\frac{s+T}{T}\right).\end{aligned} \tag{21}$$

Similarly, the E-Bayesian estimates of α based on $\pi_2(a,b)$ and $\pi_3(a,b)$ are computed and given, respectively, by

$$\begin{aligned}\hat{\alpha}_{EBS2}&=\frac{\omega r}{T}+\frac{2(1-w)}{s}\left(r+\frac{u}{u+v}\right)\\ &\quad\times\left[\frac{s+T}{s}\ln\left(\frac{s+T}{T}\right)-1\right],\end{aligned} \tag{22}$$

and

$$\begin{aligned}\hat{\alpha}_{EBS3}&=\frac{\omega r}{T}+\frac{2(1-w)}{s}\left(r+\frac{u}{u+v}\right)\\ &\quad\times\left[1-\frac{T}{s}\ln\left(\frac{s+T}{T}\right)\right].\end{aligned} \tag{23}$$

3.2. E-Bayesian estimation for the reliability

Adopting the BSEL function, the E-Bayesian estimates of the reliability function is computed with respect to the three different distributions of the hyperparameters a and b given by (20). For $\pi_1(a,b)$, the E-Bayesian estimate of the reliability is obtained from (16), (19) and (20) as

$$\begin{aligned}\widehat{R}_{EBS1}&=\int_{\varrho}\int\widehat{R}_{BS}(t)\pi_1(a,b)dbda\\ &=\frac{1}{sB(u,v)}\int_0^1\int_0^s\Big\{\omega(1+\beta t)^{-\frac{r}{T}}+(1-w)\\ &\quad\times\left(\frac{b+T}{b+T+\tau}\right)^{r+a}\Big\}a^{u-1}(1-a)^{v-1}dbda,\\ &=\omega(1+\beta t)^{-\frac{r}{T}}+\frac{1-w}{sB(u,v)}\int_0^s\left(\frac{b+T}{b+T+\tau}\right)^r\\ &\quad\times\left\{\int_0^1 e^{a\ln\left(\frac{b+T}{b+T+\tau}\right)}a^{u-1}(1-a)^{v-1}da\right\}db,\\ &=\omega(1+\beta t)^{-\frac{r}{T}}+\frac{1-w}{s}\int_0^s\left(\frac{b+T}{b+T+\tau}\right)^r\\ &\quad\times F_{1:1}\left(u,u+v;\ln\left(\frac{b+T}{b+T+\tau}\right)\right)db,\end{aligned} \tag{24}$$

where, $F_{1:1}(.,.;.)$ is the generalized hypergeometric function. [see, Gradshteyn and Ryzhik [7], (formula 3.383(1))]. Similarly, the E-Bayesian estimates of the reliability based on $\pi_2(a,b)$ and $\pi_3(a,b)$ are computed and given, respectively, by

$$\begin{aligned}\widehat{R}_{EBS2}&=\omega(1+\beta t)^{-\frac{r}{T}}+\frac{2(1-w)}{s^2}\int_0^s(s-b)\left(\frac{b+T}{b+T+\tau}\right)^r\\ &\quad\times F_{1:1}\left(u,u+v;\ln\left(\frac{b+T}{b+T+\tau}\right)\right)db,\end{aligned} \tag{25}$$

and

$$\begin{aligned}\widehat{R}_{EBS3}&=\omega(1+\beta t)^{-\frac{r}{T}}+\frac{2(1-w)}{s^2}\int_0^s b\left(\frac{b+T}{b+T+\tau}\right)^r\\ &\quad\times F_{1:1}\left(u,u+v;\ln\left(\frac{b+T}{b+T+\tau}\right)\right)db.\end{aligned} \tag{26}$$

The double integrals in (24)–(26) cannot be computed analytically, therefore, it may be derived numerically using mathematical packages such as Maple[12].

3.3. E-Bayesian estimation for the failure rate

Based on the BSEL function, the E-Bayesian estimates of the failure rate function is computed for the three different distributions of the hyperparameters a and b given by (20). For $\pi_1(a,b)$, $\pi_2(a,b)$ and $\pi_3(a,b)$, the E-Bayesian estimates of the failure rate are obtained from (18)–(20) as

$$\begin{aligned}\hat{h}_{EBS1}&=\int_{\varrho}\int\hat{h}_{BS}(t)\pi_1(a,b)dbda\\ &=\frac{1}{sB(u,v)}\int_0^1\int_0^s\left(\frac{\beta}{1+\beta t}\right)\left[\omega\left(\frac{r}{T}\right)+(1-w)\right.\\ &\quad\left.\times\left(\frac{r+a}{b+T}\right)\right]a^{u-1}(1-a)^{v-1}dbda,\\ &=\left(\frac{\beta}{1+\beta t}\right)\left[\left(\frac{\omega r}{T}\right)+(1-w)\int_0^1\int_0^s\left(\frac{r+a}{b+T}\right)\right.\\ &\quad\left.\times a^{u-1}(1-a)^{v-1}dbda\right],\\ &=\left(\frac{\beta}{1+\beta t}\right)\left[\frac{\omega r}{T}+\frac{1-w}{s}\left(r+\frac{u}{u+v}\right)\ln\left(\frac{s+T}{T}\right)\right],\end{aligned} \tag{27}$$

$$\hat{h}_{EBS2} = \left(\frac{\beta}{1+\beta t}\right)\left[\frac{\omega r}{T} + \frac{2(1-w)}{s}\left(r + \frac{u}{u+v}\right) \times \left(\frac{s+T}{s}\ln\left(\frac{s+T}{T}\right) - 1\right)\right], \quad (28)$$

and

$$\hat{h}_{EBS3} = \left(\frac{\beta}{1+\beta t}\right)\left[\frac{\omega r}{T} + \frac{2(1-w)}{s}\left(r + \frac{u}{u+v}\right) \times \left(1 - \frac{T}{s}\ln\left(\frac{s+T}{T}\right)\right)\right], \quad (29)$$

respectively.

4. Properties of E-Bayesian estimation

Now, we discuss the relations among $\hat{\alpha}_{EBSi}, \widehat{R}_{EBSi}$ and $\hat{h}_{EBSi}$ $(i = 1, 2, 3)$.

1. Relations among $\hat{\alpha}_{EBSi}$ $(i = 1, 2, 3)$

Proposition 4.1. *Let $0 < s < T$ and $\hat{\alpha}_{EBSi}$ $(i = 1, 2, 3)$ be given by Eqs. (21)–(23). Then*

(i) $\hat{\alpha}_{EBS2} < \hat{\alpha}_{EBS1} < \hat{\alpha}_{EBS3}$.
(ii) $\lim_{T\to\infty}\hat{\alpha}_{EBS1} = \lim_{T\to\infty}\hat{\alpha}_{EBS2} = \lim_{T\to\infty}\hat{\alpha}_{EBS3}$.

Proof. See, Appendix A. □

2. Relations among $\widehat{R}_{EBSi}$ $(i = 1, 2, 3)$

Proposition 4.2. *Let $0 < s < T$ and $\widehat{R}_{EBSi}$ $(i = 1, 2, 3)$ be given by Eqs. (24)–(26). Then*

$$\lim_{T\to\infty}\widehat{R}_{EBS1} = \lim_{T\to\infty}\widehat{R}_{EBS2} = \lim_{T\to\infty}\widehat{R}_{EBS3}.$$

Proof. See, Appendix A. □

From (24)–(26), we have

$$\widehat{R}_{EBS3} - \widehat{R}_{EBS1} = \widehat{R}_{EBS1} - \widehat{R}_{EBS2} = \frac{1-\omega}{s^2}\int_0^s (2b - s)\left(\frac{b+T}{b+T+\tau}\right)^r \times F_{1:1}\left(u, u+v; \ln\left(\frac{b+T}{b+T+\tau}\right)\right) db > 0, \quad (30)$$

The integral (30) cannot be computed analytically in a simple closed form. Using the mathematical package Maple[12], we found that this integral is positive. It follows that

$\widehat{R}_{EBS2} < \widehat{R}_{EBS1} < \widehat{R}_{EBS3}$.

3. Relations among $\hat{h}_{EBSi}$ $(i = 1, 2, 3)$

Proposition 4.3. *Let $0 < s < T$ and $\hat{h}_{EBSi}$ $(i = 1, 2, 3)$ be given by Eqs. (27)–(29). Then*

(i) $\hat{h}_{EBS3} < \hat{h}_{EBS1} < \hat{h}_{EBS2}$.
(ii) $\lim_{T\to\infty}\hat{h}_{EBS1} = \lim_{T\to\infty}\hat{h}_{EBS2} = \lim_{T\to\infty}\hat{h}_{EBS3}$.

Proof. See, Appendix A. □

5. Monte-Carlo simulation and comparisons

In this section, a Monte Carlo simulation was conducted for comparing the Bayes and E-Bayes techniques of estimation. The following steps were considered.

- For given values of the prior parameters (u, v) and $(0, s)$, we generate samples a and b from the beta and uniform priors (20), respectively.
- For given values of (a, b) we generate α from the gamma prior density (12).
- For known values of α, type-II censored of different sizes are generated samples from the Lomax(α, β) with pdf (1). The codes of Maple[12] are used to generate from the gamma, beta and uniform distributions.
- Based on the BSEL function, the estimates $\hat{\alpha}_{EB}, \hat{\alpha}_{EBS1}, \hat{\alpha}_{EBS2}$ and $\hat{\alpha}_{EBS3}$ of α are computed from (15), (21), (22) and (23).
- Based on the BSEL function, the estimates $\widehat{R}_{BS}, \widehat{R}_{EBS1}, \widehat{R}_{EBS2}$ and $\widehat{R}_{EBS3}$ of R are computed from (16), (24), (25) and (26).
- Based on the BSEL, the estimates $\hat{h}_{BS}, \hat{h}_{EBS1}, \hat{h}_{EBS2}$ and $\hat{h}_{EBS3}$ of h are computed from (18), (27), (28) and (29).
- The quantities $(\hat{\phi} - \phi)^2$ are computed where $\hat{\phi}$ stands for an estimate of ϕ.
- The above steps were repeated 10000 times and the estimated risks (ER) of the estimates are computed by averaging the squared deviations over 10000 repetitions:

$$ER(\hat{\phi}) = \frac{1}{10000}\sum(\hat{\phi} - \phi)^2.$$

- The computational results are displayed in Tables 1.

Table 1 Estimated risks (ER) of the estimates of $\hat{\alpha}_{BS}$, $\hat{\alpha}_{EBS1}$, $\hat{\alpha}_{EBS2}$, and $\hat{\alpha}_{EBS3}$ $(u = 4, v = 5, s = 0.1, t = 2, \omega = 0.5)$.

n	r	$\hat{\alpha}_{BS}$	$\hat{\alpha}_{EBS1}$	$\hat{\alpha}_{EBS2}$	$\hat{\alpha}_{EBS3}$
25	20	0.569327	0.559041	0.558573	0.559511
	25	0.383843	0.386466	0.386642	0.386294
30	20	0.654114	0.643296	0.642795	0.643798
	25	0.532800	0.523685	0.523267	0.524103
	30	0.350833	0.352175	0.352276	0.352078
35	20	0.702136	0.691569	0.691078	0.692060
	30	0.501973	0.493870	0.493498	0.494243
	35	0.327858	0.328662	0.328728	0.328598
50	20	0.774577	0.764737	0.764279	0.765194
	35	0.630382	0.623325	0.622997	0.623653
	30	0.692678	0.685118	0.684768	0.685468
	40	0.548875	0.542136	0.541824	0.542448
	45	0.430836	0.424560	0.424271	0.424850
	50	0.271340	0.270934	0.270930	0.270930
70	30	0.774652	0.767880	0.767567	0.768194
	35	0.746078	0.739969	0.739687	0.740251
	50	0.620926	0.615756	0.615516	0.615995
	60	0.478206	0.473181	0.472948	0.473414
	65	0.368753	0.363998	0.363778	0.364217
	70	0.230064	0.229470	0.229452	0.229489

6. Concluding remarks

In this paper, E-Bayes and Bayes methods are used for estimating the parameter, the reliability and hazard functions of the Lomax distribution based on type-II censored samples [Monte-Carlo simulation and comparisons are used for computing E-Bayes and Bayes estimates.] It has been noticed, from the Tables, that the estimated risks of the estimates decrease as the sample size increases.

- Generally, the estimated risk of the E-Bayes estimate of α, R and h have the smallest estimated risks. On the other hand, the estimated risk of the E-Bayes estimates of α, R and h based on the BSEL are less than the estimated risk of their corresponding Bayes estimates.
- It has been noticed, from Tables 1–6, that the E-Bayes estimates, in most cases, tend to be more efficient than the Bayes estimates in the sense of having smaller estimated risks of the estimates. Also, the estimated risks of the estimates decrease

Table 2 Estimated risks (ER) of the estimates of $\widehat{R}_{BS}$, $\widehat{R}_{EBS1}$, $\widehat{R}_{EBS2}$, and $\widehat{R}_{EBS3}$ $(u = 4, v = 5, s = 0.1, t = 2, \omega = 0.5)$.

n	r	$\widehat{R}_{BS}$	$\widehat{R}_{EBS1}$	$\widehat{R}_{EBS2}$	$\widehat{R}_{EBS3}$
25	20	0.059045	0.057408	0.057334	0.057483
	25	0.026109	0.025196	0.025155	0.025237
30	20	0.072902	0.071129	0.071049	0.071210
	25	0.034091	0.033405	0.033374	0.033437
	30	0.022632	0.021899	0.021866	0.021932
35	20	0.081183	0.079346	0.079262	0.079429
	30	0.045557	0.044489	0.044441	0.044538
	35	0.020166	0.019558	0.019531	0.019586
50	20	0.094765	0.092843	0.092756	0.092930
	30	0.074046	0.072779	0.072722	0.072837
	35	0.062239	0.061187	0.061139	0.061235
	40	0.049407	0.048539	0.048499	0.048578
	45	0.034091	0.033405	0.033374	0.033437
	50	0.015371	0.014977	0.014959	0.014995
70	30	0.089170	0.087842	0.087781	0.087902
	35	0.081921	0.080787	0.080735	0.080839
	50	0.057760	0.057011	0.056977	0.057045
	60	0.037919	0.037362	0.037336	0.037387
	65	0.026089	0.025638	0.025617	0.025659
	70	0.012107	0.011850	0.011838	0.011862

Table 3 Estimated risks (ER) of the estimates of $\hat{h}_{BS}$, $\hat{h}_{EBS1}$, $\hat{h}_{EBS2}$, and $\hat{h}_{EBS3}$ $(u = 4, v = 5, s = 0.1, t = 2, \omega = 0.5)$.

n	r	$\hat{h}_{BS}$	$\hat{h}_{EBS1}$	$\hat{h}_{EBS2}$	$\hat{h}_{EBS3}$
25	20	0.253034	0.248463	0.248255	0.248672
	25	0.170597	0.171763	0.171841	0.171686
30	20	0.290717	0.285909	0.285687	0.286132
	25	0.236800	0.232749	0.232563	0.232935
	30	0.155926	0.156522	0.156567	0.156479
35	20	0.243944	0.240949	0.240811	0.241088
	30	0.223099	0.219498	0.219332	0.219664
	35	0.145715	0.146072	0.146101	0.146043
50	20	0.344257	0.339883	0.339680	0.340086
	30	0.307857	0.304497	0.304341	0.304653
	40	0.243944	0.240949	0.240811	0.241088
	45	0.191483	0.188693	0.188565	0.188822
	50	0.120595	0.120415	0.120414	0.120417
70	30	0.344290	0.341280	0.341141	0.341419
	50	0.275967	0.273669	0.273563	0.273776
	60	0.212536	0.210303	0.210199	0.210406
	65	0.163890	0.161777	0.161679	0.161874
	70	0.102251	0.101987	0.101979	0.101995

Table 4 Estimated risks (ER) of the estimates of $\hat{\alpha}_{BS}$, $\hat{\alpha}_{EBS1}$, $\hat{\alpha}_{EBS2}$, and $\hat{\alpha}_{EBS3}$ $(u = 4, v = 5, s = 0.1, t = 2, \omega = 0.0)$.

n	r	$\hat{\alpha}_{BS}$	$\hat{\alpha}_{EBS1}$	$\hat{\alpha}_{EBS2}$	$\hat{\alpha}_{EBS3}$
25	20	0.570403	0.549869	0.548960	0.550782
	25	0.379977	0.385231	0.385659	0.384816
30	20	0.655081	0.633478	0.632490	0.634468
	25	0.534177	0.515974	0.515156	0.516795
	30	0.348209	0.350913	0.351172	0.350664
35	30	0.503420	0.487234	0.486503	0.487967
	35	0.325865	0.327494	0.327672	0.327324
50	40	0.550101	0.536631	0.536010	0.537252
	45	0.432339	0.419797	0.419224	0.420371
	50	0.270610	0.269819	0.269839	0.269804
70	60	0.479364	0.469319	0.468855	0.469783
	65	0.370084	0.360579	0.360144	0.361014
	70	0.229780	0.228607	0.228586	0.228630

Table 5 Estimated risks (ER) of the estimates of $\widehat{R}_{BS}$, $\widehat{R}_{EBS1}$, $\widehat{R}_{EBS2}$, and $\widehat{R}_{EBS3}$ $(u = 4, v = 5, s = 0.1, t = 2, \omega = 0.0)$.

n	r	$\widehat{R}_{BS}$	$\widehat{R}_{EBS1}$	$\widehat{R}_{EBS2}$	$\widehat{R}_{EBS3}$
25	20	0.064011	0.060663	0.060511	0.060815
	25	0.029124	0.027179	0.027092	0.027266
30	20	0.078156	0.074555	0.074392	0.074719
	25	0.055466	0.052820	0.052700	0.052941
	30	0.025083	0.023520	0.023450	0.023591
35	30	0.048892	0.0466719	0.046620	0.046818
	35	0.022216	0.020921	0.020862	0.020979
50	40	0.049407	0.048539	0.048499	0.048578
	45	0.036291	0.034900	0.034836	0.034963
	50	0.016725	0.015885	0.015847	0.015923
70	60	0.036291	0.034900	0.034836	0.034963
	65	0.027563	0.026651	0.026609	0.026693
	70	0.013006	0.012455	0.012430	0.012480

Table 6 Estimated risks (ER) of the estimates of $\hat{h}_{BS}$, $\hat{h}_{EBS1}$, $\hat{h}_{EBS2}$, and $\hat{h}_{EBS3}$ $(u = 4, v = 5, s = 0.1, t = 2, \omega = 0.0)$.

n	r	$\hat{h}_{BS}$	$\hat{h}_{EBS1}$	$\hat{h}_{EBS2}$	$\hat{h}_{EBS3}$
25	20	0.253512	0.244386	0.243982	0.244792
	25	0.168879	0.171214	0.171404	0.171029
30	20	0.291147	0.281546	0.281106	0.281986
	20	0.291147	0.281546	0.281106	0.281986
	25	0.237412	0.229322	0.228958	0.229687
	30	0.154760	0.155961	0.156076	0.155851
35	30	0.223742	0.216548	0.216223	0.216874
	35	0.144829	0.145553	0.145632	0.145477
50	40	0.244489	0.238503	0.238227	0.238779
	45	0.223099	0.219498	0.219332	0.219664
	50	0.120271	0.119919	0.119928	0.119913
70	60	0.213051	0.208586	0.208380	0.208792
	65	0.164482	0.160257	0.160064	0.160451
	70	0.102125	0.101603	0.101594	0.101613

Table 7 Estimated risks (ER) of the estimates of $\hat{\alpha}_i, \widehat{R}_i, \hat{h}_i$, $i = BS, EBS1, EBS2, EBS3$ $(r = 20, u = 4, v = 5, s = 0.1, t = 2, \omega = 0.5)$.

n	25	30	35	50	70
P.					
$\hat{\alpha}_{BS}$	0.56933	0.65411	0.70214	0.77458	0.81596
$\hat{\alpha}_{EBS1}$	0.55904	0.64330	0.69157	0.76474	0.80665
$\hat{\alpha}_{EBS2}$	0.55857	0.64280	0.69108	0.76428	0.80621
$\hat{\alpha}_{EBS3}$	0.55951	0.64378	0.69206	0.76519	0.80708
$\widehat{R}_{BS}$	0.05905	0.07290	0.08118	0.09477	0.10305
$\widehat{R}_{EBS1}$	0.05741	0.07113	0.07934	0.09284	0.10108
$\widehat{R}_{EBS2}$	0.05733	0.07105	0.07926	0.09276	0.10099
$\widehat{R}_{EBS3}$	0.05748	0.07121	0.07942	0.09293	0.10117
$\hat{h}_{BS}$	0.25303	0.29072	0.31206	0.34426	0.36265
$\hat{h}_{EBS1}$	0.24846	0.28591	0.30736	0.33988	0.35851
$\hat{h}_{EBS2}$	0.24826	0.28569	0.30715	0.33968	0.35832
$\hat{h}_{EBS3}$	0.24867	0.28613	0.30758	0.34009	0.35870

P.≡ Parameters ≡ $\hat{\alpha}_r, \widehat{R}_r, \hat{h}_r, r = BS, EBS1, EBS2, EBS3$.

as n (and r) increases and the E-Bayes estimates have the smallest estimated risks as compared with their corresponding Bayes estimates. By increasing n (and r), the computations in Tables 1–6 show that the E-Bayes estimates (based on BSEL) are better than the Bayes in the sense of comparing the estimated risks of the estimates.

- From Table 7, the estimated risks of the estimates decrease as n (and fixed r) decrease. the E-Bayes estimates have the smallest estimated risks as compared with their corresponding Bayes estimates. By increasing n (and fixed r), the computations in Table 7 show that the E-Bayes estimates (based on BSEL) are better than the Bayes in the sense of comparing the estimated risks of the estimates.
- The computations in Tables 4–6 show that E-Bayes and Bayes estimates based on squared error loss function which is a special case of BSEL function.
- Different values of the prior parameters u, v rather than those appearing in the above Tables have been considered but did not change the previous conclusion. If the prior parameters are unknown, the empirical Bayes approach may be used to estimate such parameters.
- The author suggests take beta and uniform distribution as the priors of the hyperparameters a and b, respectively. The work in this paper showed that the E-Bayesian estimation method is both efficient and easy to perform.

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Appendix A. Proofs of Proposition

Proof of Proposition 4.1. **(i)** From (21)–(23), we have

$$\hat{\alpha}_{EBS2} - \hat{\alpha}_{EBS1} = \hat{\alpha}_{EBS1} - \hat{\alpha}_{EBS3} = \frac{1-\omega}{s}\left(r + \frac{u}{u+v}\right)\left[\frac{s+2T}{s}\ln\left(\frac{T+s}{T}\right) - 2\right]. \quad \text{(A.1)}$$

For $-1 < x < 1$, we have: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$. Let $x = \frac{s}{T}$, when $0 < s < T, 0 < \frac{s}{T} < 1$, we get:

$$\left.\begin{aligned}
&\left[\tfrac{s+2T}{s}\ln\left(\tfrac{T+s}{T}\right) - 2\right] \\
&= \tfrac{s+2T}{s}\left[\left(\tfrac{s}{T}\right) - \tfrac{1}{2}\left(\tfrac{s}{T}\right)^2 + \tfrac{1}{3}\left(\tfrac{s}{T}\right)^3 - \tfrac{1}{4}\left(\tfrac{s}{T}\right)^4 + \tfrac{1}{5}\left(\tfrac{s}{T}\right)^5 - \cdots\right] - 2 \\
&= \left[\left(\tfrac{s}{T}\right) - \tfrac{1}{2}\left(\tfrac{s}{T}\right)^2 + \tfrac{1}{3}\left(\tfrac{s}{T}\right)^3 - \tfrac{1}{4}\left(\tfrac{s}{T}\right)^4 + \tfrac{1}{5}\left(\tfrac{s}{T}\right)^5 - \cdots\right] - 2 \\
&+ \left(2 - \left(\tfrac{s}{T}\right) + \tfrac{2}{3}\left(\tfrac{s}{T}\right)^2 - \tfrac{2}{4}\left(\tfrac{s}{T}\right)^3 + \tfrac{2}{5}\left(\tfrac{s}{T}\right)^4 - \cdots\right) \\
&= \left(\tfrac{s^2}{6T^2} - \tfrac{s^3}{6T^3}\right) + \left(\tfrac{3s^4}{6T^4} - \tfrac{2s^5}{15T^5}\right) + \cdots \\
&= \tfrac{s^2}{6T^2}\left(1 - \tfrac{s}{T}\right) + \tfrac{s^4}{60T^4}\left(9 - \tfrac{8s}{T}\right) + \cdots \\
&> 0.
\end{aligned}\right\} \quad \text{(A.2)}$$

According to (A.1) and (A.2), we have

$\hat{\alpha}_{EBS2} - \hat{\alpha}_{EBS1} = \hat{\alpha}_{EBS1} - \hat{\alpha}_{EBS3} > 0,$

that is

$\hat{\alpha}_{EBS3} < \hat{\alpha}_{EBS1} < \hat{\alpha}_{EBS2}.$

(ii) From (A.1) and (A.2), we get

$$\lim_{T\to\infty}(\hat{\alpha}_{EBS2} - \hat{\alpha}_{EBS1}) = \lim_{T\to\infty}(\hat{\alpha}_{EBS1} - \hat{\alpha}_{EBS3}) = \frac{1-\omega}{s}\left(r + \frac{u}{u+v}\right)\lim_{T\to\infty}\left\{\frac{s^2}{6T^2}\left(1 - \frac{s}{T}\right) + \frac{s^4}{60T^4}\left(9 - \frac{8s}{T}\right) + \cdots\right\} = 0.$$

That is, $\lim_{T\to\infty}\hat{\alpha}_{EBS1} = \lim_{T\to\infty}\hat{\alpha}_{EBS2} = \lim_{T\to\infty}\hat{\alpha}_{EBS3}$.

Thus, the proof is complete. □

Proof of Proposition 4.2. From (24)–(26), we get

$$\lim_{T\to\infty}(\widehat{R}_{EBS3} - \widehat{R}_{EBS1}) = \lim_{T\to\infty}(\widehat{R}_{EBS1} - \widehat{R}_{EBS2}) = \lim_{T\to\infty}\left\{\frac{1-\omega}{s^2}\int_0^s (2b - s)\left(\frac{b+T}{b+T+\tau}\right)^r \times F_{1:1}\left(u, u+v; \ln\left(\frac{b+T}{b+T+\tau}\right)\right)db\right\} = 0.$$

That is,

$\lim_{T\to\infty}\widehat{R}_{EBS1} = \lim_{T\to\infty}\widehat{R}_{EBS2} = \lim_{T\to\infty}\widehat{R}_{EBS3}.$

Thus, the proof is complete. □

Proof of Proposition 4.3. **(i)** From (27)–(29), we have

$$\hat{h}_{EBS2} - \hat{h}_{EBS1} = \hat{h}_{EBS1} - \hat{h}_{EBS3} = \left(\frac{\beta}{1+\beta t}\right)\left(\frac{1-\omega}{s}\right)\left(r + \frac{u}{u+v}\right)\left[\frac{s+2T}{s}\ln\left(\frac{T+s}{T}\right) - 2\right]. \quad \text{(A.3)}$$

According to (A.2) and (A.3), we have

$\hat{h}_{EBS2} - \hat{h}_{EBS1} = \hat{h}_{EBS1} - \hat{h}_{EBS3} > 0,$

that is

$\hat{h}_{EBS3} < \hat{h}_{EBS1} < \hat{h}_{EBS2}.$

(ii) From (A.2) and (A.3), we get

$$\begin{aligned}\lim_{T\to\infty}\left(\hat{h}_{EBS2}-\hat{h}_{EBS1}\right) &= \lim_{T\to\infty}\left(\hat{h}_{EBS1}-\hat{h}_{EBS3}\right)\\ &= \left(\frac{\beta}{1+\beta t}\right)\left(\frac{1-\omega}{s}\right)\left(r+\frac{u}{u+v}\right)\\ &\times \lim_{T\to\infty}\left\{\frac{s^2}{6T^2}\left(1-\frac{s}{T}\right)+\frac{s^4}{60T^4}\left(9-\frac{8s}{T}\right)+\ldots\right\}\\ &= 0.\end{aligned}$$

That is,

$$\lim_{T\to\infty}\hat{h}_{EBS1} = \lim_{T\to\infty}\hat{h}_{EBS2} = \lim_{T\to\infty}\hat{h}_{EBS3}.$$

Thus, the proof is complete. □

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