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Local Lie groups and local top spaces

N. Ebrahimi

Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran

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Abstract In this paper a generalization of local Lie groups, using the concept of top spaces, is given and some theorems about the relation between this generalization and local Lie groups are provided.

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1. Local Lie group

The concept of Lie group was local at first. Local Lie groups defined as open subsets of Euclidean spaces and the group multiplication and inversion operators only being defined for elements sufficiently near the identity. Lie groups, in the way that we know today, are formed when the definition of manifold constructed. In this paper we generalize the concept of local Lie group, using top spaces.

The notion of generalized group was introduced by Molaei when working on constructing a geometric unified theory by using Santilli's iso theory, which is applied in mathematical physics. After that he introduced top spaces as a generalization of Lie groups by using generalized group. Let us recall the definition of generalized group and top space.

Definition 1.1 [1]. A generalized group is a non-empty set G admitting an operation called multiplication, subject to the set of rules given below:

- $(xy)z = x(yz)$, for all $x, y, z \in G$;
- For each x in G there exists a unique z in G such that $xz = zx = x$ (we denote z by $e(x)$);
- For each $x \in T$ there exists $y \in T$ such that $xy = yx = e(x)$ (we denote y by x^{-1}).

We recall that T is a topological generalized group if:

- T is a generalized group.
- T is a hausdorff topological space.
- The mappings, $m_1 : T \to T,\ m_1(x) = x^{-1}$ and $m_2 : T\times T \to T,\ m_2(x, y) = xy$ are continuous maps.

Definition 1.2 [2]. A topological generalized group $(T, \cdot)$ is called a top space if:

- T is a smooth manifold.
- The mapping $m_1 : T \to T$ is defined by $m_1(x) = x^{-1}$ and the mapping $m_2 : T \times T \to T$ is defined by $m_2(x, y) = xy$ are smooth maps.

E-mail address: n_ebrahimi@mail.uk.ac.ir

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A top space T is called a normal top space if $e(xy) = e(x)e(y)$ for all $x, y \in T$.

Suppose that a group G and two sets Λ and I are given. If $p : I \times \Lambda \to G$ is a mapping, then $\Lambda \times G \times I$ with the product $(\lambda, g, i)(\lambda_1, g_1, i_1) = (\lambda, gp(i, \lambda_1)g_1, i_1)$ is a generalized group, which is called Rees matrix semigroup denoted by $M(G, I, \Lambda, p)$ [2].

Theorem 1.1 [2]. *If I and Λ are smooth manifolds, G is a Lie group and $p : I \times \Lambda \to G$ is a smooth map, then $M(G, I, \Lambda, p)$ is a top space.*

It is also proved in [3] that every top space with finite number of identities is diffeomorphic with $M(G, I, \Lambda, p)$, for some Lie group G and two finite subsets I and Λ. See [2–8] for more information about top spaces.

In the remaining of this section we recall the concept of local Lie groups and some basic definitions that we need in the next section. While the definition of a global Lie group is standard, the precise definition of a local Lie group varies from author to author. The following one is from [9].

Definition 1.3 [9]. A smooth manifold L is called a local Lie group if there exists

- a distinguished element $e \in L$, the identity element,
- a smooth product map $\mu : U \to L$ defined on an open subset $(\{e\} \times L) \cup (L \times \{e\}) \subset U \subset (L \times L)$,
- a smooth inversion map $i : V \to L$ defined on an open subset $e \in V \subset L$ such that $V \times i(V) \subset U$, and $i(V) \times V \subset U$,

all satisfying the following properties:

(i) Identity: $\mu(e, x) = x = \mu(x, e)$ for all $x \in L$.
(ii) Inverse: $\mu(i(x), x) = \mu(x, i(x)) = e$ for all $x \in V$.
(iii) Associativity: If $(x, y), (y, z), (\mu(x, y), z)$ and $(x, \mu(y, z))$ all belongs to U, then

$$\mu(\mu(x, y), z) = \mu(x, \mu(y, z)).$$

Example 1.1 [9]. The most basic example of a local Lie group is provided by any neighborhood $e \in N \subset G$ of the identity element in a global Lie group. Indeed we set U to be any open subset of $N \times N$ such that $(\{e\} \times N) \cup (N \times \{e\}) \subset U \subset (N \times N) \cap \mu^{-1}(N)$, and V to be any open subset of N such that $\{e\} \subset V \subset N \cap i^{-1}(N)$, and $(V \times i(V)) \cup (i(V) \times V) \subset U$. The group multiplication μ and inversion i on G then restricted to define local group multiplication and inversion maps on N.

One can define the right and left multiplication by

$$l_x(y) = \mu(x, y), \quad r_x(y) = \mu(y, x).$$

Definition 1.4 [9]. Let (L, μ, U, i, V) and $(\widetilde{L}, \widetilde{\mu}, \widetilde{U}, \widetilde{i}, \widetilde{V})$ be local Lie groups. A smooth map $\varphi : L \to \widetilde{L}$ is called a local group homomorphism if

- $\varphi \times \varphi(U) \subset \widetilde{U}$, $\varphi(V) \subset \widetilde{V}$, $\varphi(e) = \widetilde{e}$,
- $\varphi(\mu(g, h)) = \widetilde{\mu}(\varphi(g), \varphi(h))$ for $(g, h) \in U$,
- $\varphi(i(g)) = \widetilde{i}(\varphi(g))$ for $g \in V$.

A local group homomorphism is called a homeomorphism if it is one-to-one and onto with smooth inverse.

Definition 1.5 [9]. A local group is

- associative with order n if, for every $3 \leqslant m \leqslant n$, and every ordered m-tuple of group elements $(x_1, x_2, \ldots, x_m) \in L^{\times m}$, all corresponding well defined m-fold products are equal. A local group is called globally associative if it is associative with every order $n \geqslant 3$.
- globalizable if there exists a local group homeomorphism $\Phi : L \to N$ mapping L on to a neighborhood $e \in N \subset G$ of the identity of a global Lie group G.
- globally inversional if the inversion map i is defined everywhere, so that $V = L$.
- regular if, for each $x \in L$, the maps l_x and r_x are diffeomorphisms on their respective domains of definition.

Theorem 1.2 [9]. *Every inversional local Lie group is regular.*

Theorem 1.3 [9]. *A local Lie group L is globalizable if and only if it is globally associative.*

2. Local top spaces

In this section we use top space to generalize the concept of local Lie group.

Definition 2.1. A smooth manifold H is called a local top space if there exists

- a set $e(H) \subset H$, the identity elements,
- a smooth product map $\mu : U \to H$ defined on an open subset $(e(H) \times H) \cup (H \times e(H)) \subset U \subset (H \times H)$,
- a smooth inversion map $i : V \to H$ defined on an open subset $e(H) \subset V \subset H$ such that $V \times i(V) \subset U$, and $i(V) \times V \subset U$,

all satisfying the following properties:

(i) Identity: For each $x \in H$ there is a unique element $e(x)$ such that $\mu(e(x), x) = x = \mu(x, e(x))$.
(ii) Inverse: $\mu(i(x), x) = \mu(x, i(x)) = e(x)$ for all $x \in V$.
(iii) Associativity: If $(x, y), (y, z), (\mu(x, y), z)$ and $(x, \mu(y, z))$ all belongs to U, then

$$\mu(\mu(x, y), z) = \mu(x, \mu(y, z)),$$

(iv) $\mu(e(x), e(y)) = e(\mu(x, y))$, for each $x, y \in H$.
(v) $e : H \to H$ is a smooth map.

One can use the symbol (H, μ, U, i, V) for a local top space H with the functions μ, i, U and V as the above definition.

Remark 2.1

(1) Using (i), (iii) and (iv) in the above definition, we have

$$\mu(x, \mu(e(x), e(x))) = \mu(\mu(x, e(x)), e(x)) = \mu(x, e(x)) = x.$$

Hence uniqueness of the identity element implies that $\mu(e(x), e(x)) = e(x)$ and consequently $e(e(x)) = e(x)$, for every $x \in H$.

(2) $\mu(\mu(\mu(e(x),e(y)),e(x)),e(x)) = \mu(\mu(e(x),e(y)),\mu(e(x),e(x))) = \mu(\mu(e(x),e(y)),e(x))$.
Using (i) and (iv) in Definition 2.1 implies that $\mu(\mu(e(x),e(y)),e(x)) = e(x)$.

(3) $e(i(x)) = e(\mu(i(x),\mu(x,i(x)))) = \mu(e(i(x)),e(x)) = e(\mu(i(x)),x)) = e(e(x)) = e(x)$. Hence $e(i(x)) = e(x)$.

Lemma 2.1. *Let H be a local top space with finite number of identities then $e^{-1}(e(x))$ is both open and closed.*

Proof. The proof is a direct consequence of (v) in Definition 2.1. □

Lemma 2.2. *Let (H,μ,U,i,V) be a local top space with finite number of identities, $e(H) = \{e_1,e_2,\ldots,e_n\}$. Then $e^{-1}(e_i)$ is a local Lie group, for $i \in \{1,2,\ldots,n\}$.*

Proof. Let $U_i = e^{-1}(e_i) \times e^{-1}(e_i) \cap U$ and $V_i = V \cap e^{-1}(e_i)$. By restriction of the multiplication and inversion operator on H, it is easy to check that $e^{-1}(e_i)$ is a local Lie group, for $i = \{1,2,\ldots,n\}$. □

Example 2.1. Let $H \subseteq T$ be a neighborhood of $e(T)$ in the normal top space T and U be any open subset of $H \times H$ such that $(e(T) \times H) \cup (H \times e(T)) \subset U \subset (H \times H) \cap \mu^{-1}(H)$. In addition let V be any open subset of H such that $e(T) \subset V \subset H \cap i^{-1}(H)$, and $(V \times i(V)) \cup (i(V) \times V) \subset U$. The group multiplication μ and inversion i on T then restricted to define local top space multiplication and inversion maps on H.

Example 2.2. Suppose that λ and I are finite sets and G is a Lie group. Let $p : I \times \Lambda \to G$ be defined by $p = e$ and $(N \subseteq G,\mu,U,i,V)$ be a local Lie group (see Example 1.1). $\Lambda \times N \times I \subseteq M(G,I,\Lambda,p)$ is a local top space by choosing $\widetilde{U} = \{(\Lambda,u_1,I) \times (\Lambda,u_2,I) : (u_1,u_2) \in U\}$ and $\widetilde{V} = \Lambda \times V \times I$.

Definition 2.2. Let (H,μ,U,i,V) and $(\widetilde{H},\widetilde{\mu},\widetilde{U},\widetilde{i},\widetilde{V})$ be local top spaces. A smooth map $\varphi : H \to \widetilde{H}$ is called a local top space homomorphism if

- $\varphi \times \varphi(U) \subset \widetilde{U},\ \varphi(V) \subset \widetilde{V},\ \varphi(e(t)) = \widetilde{e}(\varphi(t))$,
- $\varphi(\mu(g,h)) = \widetilde{\mu}(\varphi(g),\varphi(h))$ for $(g,h) \in U$,
- $\varphi(i(g)) = \widetilde{i}(\varphi(g))$ for $g \in V$.

A local top space homomorphism is called a local top space homeomorphism if it is one-to-one and onto with smooth inverse.

Lemma 2.3. *Let $\varphi : H \to \widetilde{H}$ be a local top space homomorphism between two local top spaces with finite number of identities. Then $\varphi|_{e^{-1}(e(x))} : e^{-1}(e(x)) \to e^{-1}(e(\varphi(x)))$ is a local Lie group homomorphism, for every $x \in H$.*

Lemma 2.4. *Let H be a local top space with finite number of identities, $e(H) = \{e_1,e_2,\ldots,e_n\}$. Then $e^{-1}(e_i)$ is diffeomorphic with $e^{-1}(e_j)$, for $i,j \in \{1,2,\ldots,n\}$.*

Proof. Let the map $\varphi_{ij} : e^{-1}(e_i) \to e^{-1}(e_j)$ be defined by $\varphi_{ij}(s) = \mu(\mu(e_j,s),e_j)$, for $s \in e^{-1}(e_i)$. If $\varphi_{ij}(s_1) = \varphi_{ij}(s_2)$ then we have:

$$\mu(\mu(e_j,\mu(\mu(e_j,s_1),e_j)),e_j) = \mu(\mu(e_j,\mu(\mu(e_j,s_2),e_j)),e_j),$$

$$\mu(e_i,\mu(\mu(e_j,\mu(\mu(e_j,s_1),e_j)),e_j)) = \mu(e_i,\mu(\mu(e_j,\mu(\mu(e_j,s_2),e_j)),e_j)),$$

$$\mu(\mu(e_i,\mu(\mu(e_j,\mu(\mu(e_j,s_1),e_j)),e_j)),e_i) = \mu((\mu(\mu(e_j,\mu(\mu(e_j,s_2),e_j)),e_j)),e_i).$$

Since $\mu(e_i,s_1) = s_1$ and $\mu(e_i,s_2) = s_2$, we have:

$$\mu(\mu(e_i,\mu(\mu(e_j,\mu(\mu(e_j,\mu(e_i,s_1)),e_j)),e_j)),e_i) = \mu((\mu(\mu(e_j,\mu(\mu(e_j,\mu(e_i,s_2)),e_j)),e_j)),e_i).$$

Using associativity, we have:

$$\mu(\mu(\mu(\mu(\mu(e_i,\mu(e_j,e_i)),s_1),e_j),e_j),e_i) = \mu(\mu(\mu(\mu(\mu(e_i,\mu(e_j,e_i)),s_2),e_i),e_j),e_i).$$

Now since by using Remark 2.1 we have

$$\mu(\mu((e_i,\mu(e_j,e_i)),s_2) = s_2, \quad \mu(\mu((e_i,\mu(e_j,e_i)),s_1) = s_1,$$

$$\mu(\mu(\mu(s_1,e_j),e_j),e_i) = \mu(\mu(\mu(s_2,e_j),e_j),e_i).$$

Remark 2.1 implies that:

$$\mu(\mu(\mu(\mu(s_1,e_i),e_j),e_j),e_i) = \mu(\mu(\mu(\mu(s_2,e_i),e_j),e_j),e_i),$$

$$\mu(s_1,\mu(\mu(e_i,e_j),e_i)) = \mu(s_2,\mu(\mu(e_i,e_j),e_i)),$$

and consequently $s_1 = s_2$. This implies that φ_{ij} is one to one. Since $\varphi_{ij}^{-1} = \varphi_{ji}$ and φ_{ij} is smooth, it is a diffeomorphism. □

Theorem 2.1. *Let (H,μ,U,i,V) be a local top space with finite number of identities, $e(H) = \{e_1,e_2,\ldots,e_n\}$. If $(u_1,u_2) \in U$ implies that*

$$(\mu(e(H),\mu(u_1,e(H))),\mu(e(H),\mu(u_2,e(H))) \in U, \tag{1}$$

and $v \in V$ implies that

$$\mu(\mu(e(H),V),e(H)) \in V, \tag{2}$$

then $e^{-1}(e_i)$ is homeomorphic with $e^{-1}(e_j)$, for $i,j \in \{1,2,\ldots,n\}$.

Proof. Based on (1) and (2), we have $\varphi_{ij} \times \varphi_{ij}(U_i) \subset U_j$ and $\varphi_{ij}(V_i) \subseteq V_j$. In addition $\varphi_{ij}(e_i) = e_j$.

$$\varphi_{ij}(gh) = \mu(e_j,\mu(\mu(g,h),e_j)) = \mu(e_j,\mu(g,\mu(h,e_j))) = \mu(e_j,\mu(\mu(g,\mu(\mu(e_i,e_j),e_i))),\mu(h,e_j)))$$

since $e(g) = e_i$ and $\mu(e_i,\mu(e_j,e_i)) = e_i$. In addition we have $\mu(e_j,e_j) = e_j$. Hence:

$$\mu(e_j,\mu(\mu(g,\mu(\mu(e_i,\mu(e_j,e_j)),e_i))),\mu(h,e_j))).$$

Consequently, associativity and the assumption imply that:

$$\mu(\mu(\mu(e_j,g),e_j),\mu(\mu(e_j,h),e_j)) = \mu(\varphi_{ij}(g),\varphi_{ij}(h)),$$

and the second condition is satisfied.

$\varphi_{ij}(i(g)) = \mu(e_j, \mu(i(g), e_j))$.

$i(\mu(e_j, \mu(g, e_j))) = \mu(e_j, \mu(i(g), e_j))$ since

$$\begin{aligned}&\mu(\mu(e_j, \mu(i(g), e_j)), \mu(e_j, \mu(g, e_j)))\\ &= \mu(\mu(e_j, \mu(\mu(i(g), e_i), e_j)), \mu(e_j, \mu(\mu(e_i, g), e_j)))\\ &= \mu(\mu(e_j, \mu(i(g), \mu(e_i, e_j)), \mu(\mu(e_j, \mu(e_i, \mu(g, e_j))\\ &= \mu(e_j, \mu(e_i, e_j)) = e_j.\end{aligned}$$

In a similar way one can prove that $\mu(\mu(e_j, \mu(g, e_j)), \mu(e_j, \mu(i(g), e_j))) = e_j$. Consequently, the third condition is satisfied and φ_{ij} is a homeomorphism. □

Definition 2.3. Let H be a local top space and L be a local Lie group. An onto local homomorphism $\varphi : H \to L$ is a local covering map if:

(i) $e^{-1}(e(x))$ is a local Lie group, for every $x \in H$.
(ii) $\varphi|_{e^{-1}(e(x))}$ is an onto local Lie group homeomorphism, for every $x \in H$.

Definition 2.4. A local top space H is globalizable if there exists a homeomorphism $\varphi : H \to N$ mapping H on to a neighborhood $e(T) \subset N \subset T$ of a normal top space T.

Lemma 2.5. *let (H, μ_H, U, i_H, V) be a local top space with finite number of identities, satisfying (1) and (2). The Local top space H is globalizable if and only if there is a local covering map $\varphi_0 : H \to (M, \mu, U_M, i, V_M)$, where M is a neighborhood of the identity element of a Lie group G.*

Proof. Let H be a globalizable local top space and the map $\varphi : H \to N$ be as in Definition 2.4. Let $\{e_1, e_2, \ldots, e_n\}$ be the set of identity elements of H. Since φ is a homeomorphism, $card(e(H)) = card(e(T))$. Let $\{e'_1, e'_2, \ldots, e'_n\}$ be the set of identity elements of T. We set $M = e^{-1}(e'_1) \subset N$. Using Theorem 2.1, $\varphi_{i1}|_{e^{-1}(e'_i)} : e^{-1}(e'_i) \to e^{-1}(e\prime_1)$ is a homeomorphism. Now we define $\varphi_0(y) = \varphi_{i1}(\varphi(y))$, where $e(\varphi(y)) = e'_i$.

Let $e(\varphi(y_1)) = e'_l, e(\varphi(y_2)) = e'(k)$. Then $\mu(e'_l, e'_k) = e(\mu(\varphi(y_1), \varphi(y_2)))$. We have:

$$\begin{aligned}\mu(\mu(e'_l, e'_k), \mu(\mu(e'_l, e'_1), e'_k)) &= \mu(\mu(\mu(e'_l, e'_k), e'_l), \mu(e'_1, e'_k))\\ &= \mu(\mu(e'_l, e'_1), e'_k).\end{aligned}$$

In a similar way one can prove that $\mu(\mu(\mu(e'_l, e'_1), e_k ime), \mu(e'_l, e'_k)) = \mu(\mu(e'_l, e'_1), e'_k)$ and, consequently $\mu(e'_l, e'_k) = \mu(\mu(e'_l, e'_1), e'_k)$.

$$\begin{aligned}\varphi_0(\mu_H(y_1, y_2)) &= \varphi_{i1}(\varphi(\mu_H(y_1, y_2)))\\ &= \mu(\mu(e'_1, \varphi(\mu_H(y_1, y_2))), e'_1)\\ &= \mu(\mu(e'_1, \mu(\varphi(y_1), \varphi(y_2))), e'_1)\\ &= \mu(\mu(e'_1, \mu(\mu(\varphi(y_1), e'_l))), \mu(\mu(e'_k, \varphi(y_2)), e'_1))\\ &= \mu(\mu(e'_1, \mu(\varphi(y_1), e'_l))), \mu(\mu(e'_k, \varphi(y_2)), e'_1))\\ &= \mu(\mu(\mu(e'_1, \varphi(y_1)), \mu(e'_l, e'_k)), \mu(\varphi(y_2, e'_1)))\\ &= \mu(\mu(\mu(e'_1, \varphi(y_1)), \mu(\mu(e'_l, e'_1), e'_k)), \mu(\varphi(y_2, e'_1)))\\ &= \mu(\mu(\mu(e'_1, \varphi(y_1)), e'_1), \mu(e'_1, \mu(\varphi(y_2), e'_1))\\ &= \mu(\varphi_0(y_1), \varphi_0(y_2))\end{aligned}$$

since $\mu(e'_1, e'_1) = e'_1$.

Conversely suppose that there is a local covering map $\varphi_0 : H \to (M, \mu, U_M, i, V_M)$, where M is a neighborhood of the identity element of a Lie group G. Then one can define a top space T by $T = \cup_{i=1}^n G_i$, where $G_i = (G, i)$ is a Lie group with the product $(g_1, i)(g_2, i) = (g_1 g_2, i)$. It is trivial that G_i is diffeomorphic with G. We define a product on T by $(g_1, i)(g_2, j) = (g_1 g_2, k)$ if $\mu_H(e_i, e_j) = e_k$ in H. Let $M_i = (M, i)$. $(M_0 = \cup M_i, \mu, U_{M_0}, i, V_{M_0})$ with $U_{M_0} = \{((u_1, i), (u_2, j)) : 1 \leqslant i, j \leqslant n : (u_1, u_2) \in U_M\}$ and $V_{M_0} = \{(v, i) : 1 \leqslant i \leqslant n : v \in V_M\}$ is a local top space. The map $\varphi : H \to \cup M_i, \varphi(h) = (\varphi_0(h), j)$, where $e(h) = e_j$, is a local homeomorphism. □

Definition 2.5. A local top space H is called globally inversional if the inversion map i is defined everywhere, so that $V = H$.

Lemma 2.6. *If H is a globally inversional local top space with finite number of identities then $e^{-1}(e(x))$ is a globally inversional local Lie group, for every $x \in H$.*

Definition 2.6. A local top space H is regular if $e^{-1}(e(x))$ is a regular local Lie group, for every $x \in H$.

Lemma 2.7. *Let H be an inversional local top space with finite number of identities then it is regular.*

Proof. The proof is trivial from Theorem 1.2 and Definition 2.6. □

Corollary 2.1. *Let H be a globalizable local top space with finite number of identities then $e^{-1}(e(x))$, for every $x \in H$, is a globalizable local Lie group.*

Theorem 2.2. *The local top space (H, μ_H, U, i_H, V) with finite number of identities, satisfying (1) and (2), is globalizable if and only if it is globally associative.*

Proof. If H is globalizable then it is globally associative since it is homeomorphic with an associative local top space. Conversely, if it is globally associative then $e^{-1}(e_i)$ is associative and consequently it is globalizable. Since it is globalizable, there is a neighborhood, N, of the identity element of a Lie group G and a map $\varphi_0 : e^{-1}(e_i) \to (N, \mu, U_N, i, V_N)$. Now since $e^{-1}(e_i)$ is homeomorphic with $e^{-1}(e_j)$, one can define a covering map from H to N by $\varphi(h) = \varphi_0(\varphi_{ji}(h))$, where $e(h) = e_j$. Using Lemma 2.2, $e^{-1}(e_i)$ is a local Lie group. In addition $\varphi|(e^{-1}(e_i))$ is local homeomorphism, since φ_{ij} and φ_0 are local homeomorphisms.

$$\begin{aligned}\varphi \times \varphi(u_1, u_2) &= (\varphi_0(\varphi_{ji}(u_1)), \varphi_0(\varphi_{ki}(u_2)))\\ &= (\varphi_0(\mu_H(\mu_H(e_i, u_1), e_i)), \varphi_0(\mu_H(\mu_H(e_i, u_2), e_i))) \subseteq U_N.\end{aligned}$$

Let $e(t) = e_j$.

$$\begin{aligned}\varphi_0(\varphi_{ji}(e(t))) &= \varphi_0(\mu_H(\mu_H(e_i, e(t)), e_i))\\ &= \mu(\mu(\varphi_0(e_i), \varphi_0(e(t))), \varphi_0(e_i))\\ &= \mu(\mu(e(\varphi_0(e_i)), e(\varphi_0(t)), e(\varphi_0(e_i)))\\ &= e(\varphi_0(\mu_H(\mu_H((e_i, t), e_i)))) = e(\varphi(t)).\end{aligned}$$

If $e(h) = e_k, e(s) = e_l$ and $\mu(e_k, e_l) = e_j$ then

$$\varphi(\mu(h,s)) = \varphi_0(\varphi_{ji}(\mu_H(h,s))) = \varphi_0(\mu_H(\varphi_{ji}(h), \varphi_{ji}(s)))$$
$$= \mu(\varphi(h), \varphi(s)).$$

In addition we have $\varphi(i_H(v)) = i(\varphi(v))$, which completes the proof. $\square$

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