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ORIGINAL ARTICLE

Cardinal functions for Legendre pseudo-spectral method for solving the integro-differential equations

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Abstract In this article, we present a new numerical method to solve the integro-differential equations (IDEs). The proposed method uses the Legendre cardinal functions to express the approximate solution as a finite series. In our method the operational matrix of derivatives is used to reduce IDEs to a system of algebraic equations. To demonstrate the validity and applicability of the proposed method, we present some numerical examples. We compare the obtained numerical results from the proposed method with some other methods. The results show that the proposed algorithm is of high accuracy, more simple and effective.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 65G99; 65K10; 35E99; 68U20

1. Introduction

The integro-differential equation is an equation that involves both integrals and derivatives of an unknown function. Mathematical modeling of real-life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro-differential equations and stochastic equations. Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in many fields like fluid dynamics, biological models and chemical kinetics [1].

Legendre polynomials occur in the solution of Laplace equation of the potential, $\nabla^2\Phi(x) = 0$, in a charge-free region of space, using the method of separation of variables, where the boundary conditions have axial symmetry, the solution for the potential will be

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta),$$

A_l and B_l are to be determined according to the boundary conditions of each problem. They also appear when solving

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Schrödinger equation in three dimensions for a central force. In recent years, there has been a growing interest in IDEs which are combination of differential and Fredholm–Volterra integral equations. This is an important branch of modern mathematics and arise frequently in many applied areas which include engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, etc. [2–6]. The mentioned IDEs are usually difficult to solve analytically, so approximate and numerical methods are required [7–13]. The concept of IDEs has motivated a huge size of research work in recent years. Several numerical methods were used, such as the successive approximations [14], homotopy perturbation method [15,16], Chebyshev and Taylor collocation [17,18], Haar function methods [19], variational iteration method [20–24], etc., [25–30]. Moreover, the following methods for IDEs have been presented the Monte Carlo method by Farnoosh and Ebrahimi [31] and the direct method based on Fourier and block-pulse functions by Asady et al. [32].

In this paper, by means of the matrix relations between the Legendre cardinal functions and their derivatives, the above mentioned methods are modified and developed for solving the mth order linear and non-linear integro-differential equations with variable coefficients

$$\sum_{k=0}^{m} p_k(x)y^{(k)}(x) = f(x) + \gamma y(x) + \int_{-1}^{x}[g(t)y(t) + h(t)\Theta(y(t))]dt, \quad -1 \leqslant x, t \leqslant 1, \tag{1}$$

under the mixed-boundary conditions

$$\sum_{k=0}^{m-1}(a_{ik}y^{(k)}(-1)+b_{ik}y^{(k)}(1)+c_{ik}y^{(k)}(c))=\lambda_i, \; i=0,1,\dots,m-1, \tag{2}$$

where a_{ik}, b_{ik}, c_{ik} and λ_i are suitable constants; $-1 \leqslant c \leqslant 1$.

The organization of this paper is as follows. In the next section, the definition of Legendre cardinal functions is introduced, in Section 3, the procedure of the numerical solution of the proposed problem is given, in Section 4, two numerical examples are introduced, in the last Section 5, conclusions and discussion are presented.

2. Legendre cardinal functions

In this section, to construct the so called Legendre cardinal functions for the set of orthogonal Legendre polynomials $\{P_i(x)\}_{i=0}^{\infty}$, we will use the Taylor expansion of $P_{N+1}(x)$ in neighborhood the jth root of $P_{N+1}(x)$, which gives

$$P_{N+1}(x) \simeq P_{N+1}(x_j) + P_{N+1,x}(x - x_j) + O(x - x_j)^2,$$

from this relation, since the first term in the right hand side vanishes, then we can define the cardinal function of degree N in $[-1, 1]$ as follows [33]

$$C_j(x) = \frac{P_{N+1}(x)}{P_{N+1,x}(x_j)(x - x_j)}, \tag{3}$$

where the subscript x denotes x-differentiation and x_j $(j = 1, 2, \dots, N + 1)$ are the zeros of $P_{N+1}(x)$. Now any function $f(x)$ on $[-1, 1]$ can be approximated as follows

$$f(x) \approx \sum_{j=1}^{N+1} f(x_j)C_j(x) = F^T\Phi_N(x), \tag{4}$$

where

$$F = [f(x_1), f(x_2), \dots, f(x_{N+1})]^T, \quad \text{and}$$
$$\Phi_N(x) = [C_1(x), C_2(x), \dots, C_{N+1}(x)]^T. \tag{5}$$

Note that, we can use the expansion of the form (4) on any interval $[a, b]$ if we use the change of variable $t = \frac{(b-a)}{2}(x + 1) + a$. For more detail about these functions and its properties see [27,34].

The first derivative of vector $\Phi_N(x)$ in Eq. (5) can be expressed into the matrix form

$$\Phi'_N(x) = \boldsymbol{D}^{(1)}\Phi_N(x), \tag{6}$$

where $\boldsymbol{D}^{(1)}$ is $(N + 1) \times (N + 1)$ operational matrix of derivative for Legendre cardinal functions. The matrix $\boldsymbol{D}$ can be obtained by the following process.

Let $\Phi'_N(x) = [C'_1(x), C'_2(x), \dots, C'_{N+1}(x)]^T$. Using Eq. (4), any function $C'_j(x)$ can be approximated as

$$C'_j(x) = \sum_{k=1}^{N+1} C'_j(x_k)C_k(x), \tag{7}$$

comparing Eqs. (6) and (7) we get

$$\boldsymbol{D}^{(1)} = \begin{bmatrix} C'_1(x_1) & \cdots & C'_1(x_{N+1}) \\ \vdots & \ddots & \vdots \\ C'_{N+1}(x_1) & \cdots & C'_{N+1}(x_{N+1}) \end{bmatrix}. \tag{8}$$

By the same procedure we can write the nth derivative of vector $\Phi_N(x)$ in the following matrix form

$$\Phi_N^{(n)}(x) = \boldsymbol{D}^{(n)}\Phi_N(x) \text{ where } \boldsymbol{D}^{(n)} = \begin{bmatrix} C_1^{(n)}(x_1) & \cdots & C_1^{(n)}(x_{N+1}) \\ \vdots & \ddots & \vdots \\ C_{N+1}^{(n)}(x_1) & \cdots & C_{N+1}^{(n)}(x_{N+1}) \end{bmatrix}. \tag{9}$$

3. Procedure of the numerical solution

In this section, we are going to construct the fundamental matrix equation corresponding to Eq. (1). We use Eq. (4) to approximate the function $y(x)$ as

$$y(x) = Y^T\Phi_N(x), \tag{10}$$

where Y is $(N + 1)$ unknown vector as $Y = [y_1, y_2, \dots, y_{N+1}]^T$ and should be found. Now using Eqs. (9) and (10) we can write

$$y'(x) = Y^T\Phi'_N(x) = Y^T\boldsymbol{D}^{(1)}\Phi_N(x), \tag{11}$$
$$y''(x) = Y^T\boldsymbol{D}^{(1)}\Phi'_N(x) = Y^T\boldsymbol{D}^{(2)}\Phi_N(x), \tag{12}$$

and

$$y^{(n)}(x) = Y^T\boldsymbol{D}^{(n)}\Phi_N(x). \tag{13}$$

Using Eqs. (10)–(13) in Eq. (1) we get

$$\sum_{k=0}^{m} p_k(x)Y^T\boldsymbol{D}^{(k)}\Phi_N(x) = f(x) + \gamma Y^T\Phi_N(x) + \int_{-1}^{x}[g(t)Y^T\Phi_N(t) + h(t)\Theta(Y^T\Phi_N(t))]dt. \tag{14}$$

Collocation of Eq. (14) at some points $\tau_j (j = 1, 2, \dots, N - m + 1)$ in the interval $[-1, 1]$ gives

$$\sum_{k=0}^{m} p_k(\tau_j)Y^T\boldsymbol{D}^{(k)}\Phi_N(\tau_j) - f(\tau_j) - \gamma Y^T\Phi_N(\tau_j) - \int_{-1}^{\tau_j}[g(t)Y^T\Phi_N(t) + h(t)\Theta(Y^T\Phi_N(t))]dt = 0. \tag{15}$$

We approximate the integral term in Eq. (15) using Newton–Cotes integration rule as

$$\int_{-1}^{\tau_j}[g(t)Y^T\Phi_N(t)+h(t)\Theta(Y^T\Phi_N(t))]dt \cong \sum_{r=0}^{M} w_r\Omega(t_r), \quad (16)$$

with

$$\Omega(t)=g(t)Y^T\Phi_N(t)+h(t)\Theta(Y^T\Phi_N(t)), \quad (17)$$

where w_r and t_r, $r=0,1,\ldots,M$ are the weights and nodes of Newton–Cotes integration technique, respectively. Substituting Eq. (16) in Eq. (15) we have the following equations

$$\sum_{k=0}^{m} p_k(\tau_j)Y^T\boldsymbol{D}^{(k)}\Phi_N(\tau_j)-f(\tau_j)-\gamma Y^T\Phi_N(\tau_j)-\sum_{r=0}^{M} w_r\Omega(t_r)=0,$$
$$j=1,2,\ldots,N-m+1. \quad (18)$$

We can obtain the corresponding matrix forms for the conditions (2) as

$$\sum_{k=0}^{m-1}(a_{ik}+b_{ik}+c_{ik})(\boldsymbol{D}^{(k)})Y=\lambda_i, \quad i=0,1,\ldots,m-1. \quad (19)$$

Eqs. (18) together with Eqs. (19) give a system of $N+1$ linear or non-linear algebraic equations, which can be solved for y_k, $k=1,2,\ldots,N+1$, so the unknown function $y(x)$ can be found using a suitable numerical method.

4. Numerical examples

In this section, to achieve the validity, the accuracy and support our theoretical discussion in this paper of the proposed method, we give some computational results of numerical examples.

Example 1. Consider Eq. (1) with the following functions and coefficients

$$p_i(x)=0,\ (i=0,1,2,3),\ \ p_4(x)=1,\ \ f(x)=x+(x+3)e^x,\ \ \gamma=1,$$

$$g(x)=-1, \quad h(x)=0, \quad \Theta(y)=y^2(x),$$

subject to the boundary conditions

$$y(-1)=1-e^{-1},\ \ y''(-1)=e^{-1},\ \ y(1)=1+e,\ \ y''(1)=3e, \quad (20)$$

i.e., Eq. (1) takes the form

$$y^{(iv)}(x)=f(x)+y(x)-\int_{-1}^{x}y(t)dt, \quad -1<x<1. \quad (21)$$

We apply the suggested method with $N=6$, and approximate the solution $y(x)$ as follows

$$y_6(x)\cong\sum_{i=1}^{7}y_iC_i(x)\equiv Y^T\Phi_6(x). \quad (22)$$

Using Eq. (15) we have

$$Y^T\boldsymbol{D}^{(4)}\Phi_6(\tau_j)-f(x)-Y^T\Phi_6(\tau_j)-\int_{-1}^{\tau_j}Y^T\Phi_6(t)dt=0,$$
$$j=1,2,3. \quad (23)$$

We approximate the integral term in Eq. (23) using Newton–Cotes integration rule as the formula (16) we have

$$Y^T\boldsymbol{D}^{(4)}\Phi_6(\tau_j)-f(x)-Y^T\Phi_6(\tau_j)-\sum_{r=0}^{M}w_r\Omega(t_r)=0,\ \ j=1,2,3, \quad (24)$$

where $\Omega(t)=Y^T\Phi_6(t)$, also, the matrix equations of the mixed-boundary conditions are

$$Y^T\Phi_6(-1)=1-e^{-1}, \quad Y^T\Phi_6(1)=1+e, \quad (25)$$
$$Y^T\boldsymbol{D}^{(2)}\Phi_6(-1)=e^{-1}, \quad Y^T\boldsymbol{D}^{(2)}\Phi_6(1)=3e. \quad (26)$$

Eqs. (24)–(26) represent linear system of algebraic equations. By solving it we obtain

$$y_1=1.0209,\quad y_2=3.4532,\quad y_3=0.6350,\quad y_4=2.5631,$$
$$y_5=0.6581,\quad y_6=1.6236,\quad y_7=0.7496. \quad (27)$$

Therefore, the approximate solution of this example can be obtained using (22) as

$$y(x)\cong 1.021C_1(x)+3.453C_2(x)+0.635C_3(x)+2.563C_4(x)$$
$$+0.658C_5(x)+1.624C_6(x)+0.750C_7(x). \quad (28)$$

Now, we compare the approximate solution using the proposed method with the well-known approximate variational iteration method (VIM) as follows.

VIM gives the possibility to write the solution of Eq. (21) with the aid of the correction functionals

$$y_{n+1}(x)=y_n(x)+\int_0^x\lambda(\tau)\left[y_n^{(iv)}(\tau)-f(\tau)-y_n(\tau)+\int_{-1}^{\tau}\tilde{y}_n(s)ds\right]d\tau,\ \ n\geqslant 0, \quad (29)$$

where λ is a general Lagrange multiplier. Making the above correction functional stationary

$$\delta y_{n+1}(x)=\delta y_n(x)+\delta\int_0^x\lambda(\tau)\left[y_n^{(iv)}(\tau)-f(\tau)-\tilde{y}(\tau)\right.$$
$$\left.+\int_{-1}^{\tau}\tilde{y}_n(s)ds\right]d\tau=\delta y_n+\left[\lambda(\tau)\delta y_n'''-\lambda'\delta y_n''\right.$$
$$\left.+\lambda''(\tau)\delta y_n'-\lambda'''\delta y_n\right]_{\tau=x}+\int_0^x\left[\lambda^{(iv)}(\tau)\delta y_n\right]d\tau=0, \quad (30)$$

where $\delta\tilde{y}_n$ is considered as a restricted variation, i.e., $\delta\tilde{y}_n=0$, yields the following stationary conditions (by comparison the two sides in the above equation)

$$\lambda^{(iv)}(\tau)=0, \quad \lambda(\tau)|_{\tau=x}=\lambda'(\tau)|_{\tau=x}=\lambda''(\tau)|_{\tau=x}=0,$$
$$1-\lambda'''(\tau)|_{\tau=x}=0. \quad (31)$$

The equations in (31) are called Lagrange–Euler equation and the natural boundary conditions, respectively, the Lagrange multiplier can be obtained by solving this equation as follows

$$\lambda(\tau)=\frac{1}{3!}(\tau-x)^3. \quad (32)$$

Now, by substituting from (32) in (29), the following variational iteration formula can be obtained [20]

$$y_{n+2}(x)=y_{n+1}(x)+\int_0^x\frac{1}{3!}(\tau-x)^3\left[-f(\tau)-(y_{n+1}(\tau)-y_n(\tau))\right.$$
$$\left.+\int_{-1}^{\tau}(y_{n+1}(s)-y_n(s))ds\right]d\tau, \quad n\geqslant 0. \quad (33)$$

We start with initial approximation $y_0(x)=ax^3+bx^2+cx+d$, for some constants a,b,c and d which will determine later, and by using the above iteration formula (33), we can directly obtain the components of the solution. Now, the first two approximations of the solution $y(x)$ of Eq. (21) by using (33) are

$$y_0(x) = ax^3 + bx^2 + cx + d,$$

$$y_1(x) = y_0(x) + (1/6)(6 - 3x^2 - 2x^3 + e^x(-6 + 6x) - 0.0009ax^8 + x^7(0.0072a - 0.0024b) + x^6(0.01667b - 0.00833c) + x^4(0.1840 + 0.0625a - 0.08333b + 0.1250c) + x^5(0.05 + 0.05c - 0.05d)).$$

Now, to find the constants a, b, c, and d we impose the boundary conditions (20) on the n-term approximation $y_3(x)$, we obtain

$$a = 0.5001107, \quad b = 0.995759, \quad c = 0.999917, \quad d = 1.0031507.$$

The exact solution of this problem is $y(x) = 1 + xe^x$.

The behavior of the numerical solutions using the proposed cardinal function method, with $N = 6$, compared with the approximate solution using VIM, y_{VIM}, with three components ($n = 3$) are presented in Fig. 1. Also, in Table 1 to show the effect of the numbers of terms of the series (22), N, we introduced the absolute error of our approximate solution with different values of $N = 3, 5, 7$ at some values of x. From this figure, it is clear that the proposed method can be considered as an efficient method.

Example 2. Consider Eq. (1) with the following functions and coefficients

$$p_i(x) = 0 (i = 0, 1, 2, 3), \quad p_4(x) = 1, \quad f(x) = e^{-1}, \quad \gamma = 0, \quad g(x) = 0, \quad h(x) = e^{-x}, \quad \Theta(y) = y^2(x),$$

subject to the boundary conditions

$$y(-1) = e^{-1}, \quad y''(-1) = e^{-1}, \quad y(1) = e, \quad y''(1) = e, \tag{34}$$

i.e., Eq. (1) takes the form

$$y^{(iv)}(x) = e^{-1} + \int_{-1}^{x} e^{-t} y^2(t)dt, \quad -1 < x < 1. \tag{35}$$

We apply the suggested method with $N = 6$, and approximate the solution $y(x)$ as follows

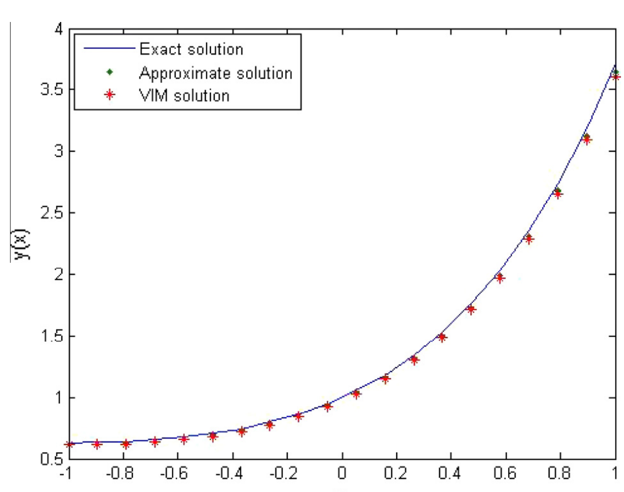

Fig. 1 The behavior of the approximate solution and the exact solution at $N = 6$ and comparison with the solution using VIM.

Table 1 The absolute error of the numerical solution with different values of $N = 3, 5, 7$.

x	$Err_{N=3}$	$Err_{N=5}$	$Err_{N=7}$
−1.0	0.25781e−03	0.74215e−05	0.36341e−07
−0.6	0.75850e−03	0.36987e−05	0.45447e−07
−0.2	0.26897e−03	0.21587e−05	0.25874e−07
0.2	0.97542e−03	0.12354e−05	0.21578e−07
0.6	0.67894e−03	0.21589e−05	0.25478e−07
1	0.25988e−03	0.36981e−05	0.32548e−07

$$y_6(x) \cong \sum_{i=1}^{7} y_i C_i(x) \equiv Y^T \Phi_6(x). \tag{36}$$

Using Eq. (15) we have

$$Y^T \boldsymbol{D}^{(4)} \Phi_6(\tau_j) - e^{-1} - \int_{-1}^{\tau_j} e^{-t}(Y^T \Phi_6(t))^2 dt = 0, \; j = 1, 2, 3. \tag{37}$$

We approximate the integral term in Eq. (37) using Newton–Cotes integration rule as the formula (16) we have

$$Y^T \boldsymbol{D}^{(4)} \Phi_6(\tau_j) - e^{-1} - \sum_{r=0}^{M} w_r \Omega(t_r) = 0, \quad j = 1, 2, 3, \tag{38}$$

with $\Omega(t) = e^{-t}(Y^T \Phi_6(t))^2$, also, the matrix equations of the mixed-boundary conditions are

$$Y^T \Phi_6(-1) = e^{-1}, \quad Y^T \Phi_6(1) = e, \tag{39}$$

$$Y^T \boldsymbol{D}^{(2)} \Phi_6(-1) = e^{-1} \quad Y^T \boldsymbol{D}^{(2)} \Phi_6(1) = e. \tag{40}$$

Eqs. (38)–(40) represent non-linear system of algebraic equations. By solving it we obtain

$$y_1 = 1.0044, \quad y_2 = 2.5837, \quad y_3 = 0.3875, \quad y_4 = 2.1007, \quad y_5 = 0.4785, \quad y_6 = 1.5039, \quad y_7 = 0.6703. \tag{41}$$

Therefore, the approximate solution of this example can be obtained using (36) as

$$y(x) \cong 1.004C_1(x) + 2.584C_2(x) + 0.388C_3(x) + 2.102C_4(x) + 0.479C_5(x) + 1.504C_6(x) + 0.670C_7(x). \tag{42}$$

Now, we compare the approximate solution using the proposed method with the well-known VIM as follows.

VIM gives the possibility to write the solution of Eq. (35) with the aid of the correction functionals

$$y_{n+2}(x) = y_{n+1}(x) + \int_0^x \lambda(\tau)\left[y_n^{(iv)}(\tau) - e^{-1} - \int_{-1}^{\tau} e^{-s} \tilde{y}_n^2(s)ds\right]d\tau, \; n \geqslant 0, \tag{43}$$

By the same procedure in the previous example, the Lagrange multiplier is

$$\lambda(\tau) = \frac{1}{3!}(\tau - x)^3. \tag{44}$$

Now, by substituting from (44) in (43), the following variational iteration formula can be obtained [20]

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{3!}(\tau - x)^3 \left[-e^{-1} - \int_{-1}^{\tau} e^{-s}(y_{n+1}^2(s) - y_n^2(s))ds\right]d\tau, \quad n \geqslant 0. \tag{45}$$

We start with initial approximation $y_0(x) = ax^3 + bx^2 + cx + d$, for some constants a, b, c, and d which will determine later, and by using the above iteration formula (45), we can directly obtain the approximations of the solution.

Now, the first two approximations of the solution $y(x)$ of Eq. (35) by using (45) are

$$y_0(x) = ax^3 + bx^2 + cx + d,$$

$$\begin{aligned} y_1(x) = y_0(x) + (1/6)(&0.091x^4 + 0.081a^2x^4 + 0.0001a^2x^{11} \\ &- 0.00006a^2x^{12} + 0.00002a^2x^{13} - 0.186abx^4 \\ &+ 0.0004abx^{10} - 0.0003abx^{11} + 0.00006abx^{12} \\ &+ 0.1095b^2x^4 + 0.0004a^2x^9 - 0.0003b^2x^{10} \\ &+ 0.00005b^2x^{11} + 0.2190acx^4 + 0.0008acx^9 \\ &- 0.0004acx^{10} + 0.0002acx^{11} + 0.0018bcx^8 \\ &- 0.0008bcx^9 + 0.0002bcx^{10} + 0.17083c^3x^4 \\ &- 0.2667bcx^4 + 0.0024c^3x^7 - 0.0009c^3x^8 \\ &- 0.00079adx^9 + 0.00019adx^{10} + 0.3417bdx^4 \\ &- 0.0008adx^9 + 0.0002adx^{10} + 0.3417bdx^4 \\ &+ +0.0002c^2x^9 - 0.26667adx^4 + 0.0018adx^8 \\ &+ 0.0048bdx^7 - 0.0018bdx^8 + 0.0004bdx^9 \\ &- 0.4793cdx^4 + 0.01667cdx^6 - 0.0048cdx^7 \\ &+ 0.0009cdx^8 + 0.41667dx^4 + 0.05d^2x^5 - 0.0083d^2x^6 \\ &+ 0.0013d^4x^7). \end{aligned}$$

Now, to find the constants a, b, c and d we impose the boundary conditions (34) on the n-term approximation $y_3(x)$, we obtain

$$a = 0.166461, \quad b = 0.50217, \quad c = 1.00021, \quad d = 0.998347.$$

The exact solution of this problem is $y(x) = e^x$.

The behavior of the numerical solutions using the proposed cardinal function method, with $N = 6$, compared with the approximate solution using VIM, y_{VIM}, with three components ($n = 3$) are presented in Fig. 2. From this figure, it is clear that the proposed method can be considered as an efficient method.

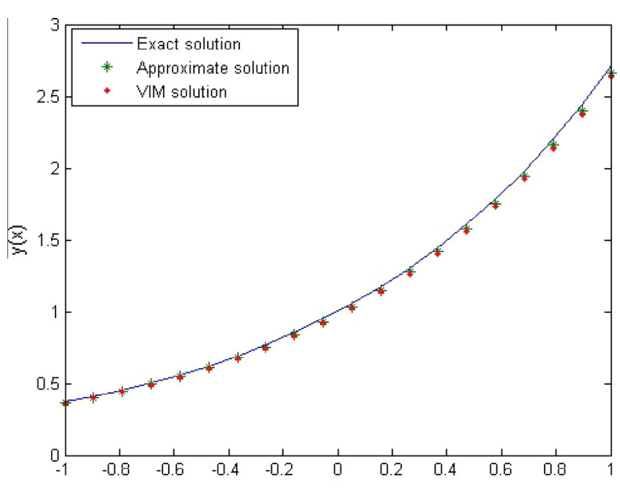

Fig. 2 The behavior of the approximate solution and the exact solution at $N = 6$ and comparison with the solution using VIM.

5. Conclusions and discussion

In this paper, we presented a new highly accurate approximate method for solving the integro-differential equations. In the proposed method we used the cardinal functions with Legendre pseudo-spectral method. Comparison of our obtained results using the proposed method with that obtained by other methods reveals that the presented method is very effective and convenient. The numerical results show that the accuracy improves by increasing N, the number of terms of the series (36). Tables and figures indicate that as N increases the errors decrease more rapidly; hence for better results, using number N is recommended. Also, from the comparison we can conclude that the approximate solution is in excellent agreement with the exact solution. All computations are performed by Matlab 7.1.

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