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## ORIGINAL ARTICLE

# Asymptotic behavior of an anti-competitive system of second-order difference equations

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**Abstract**  In this paper, we study the boundedness and persistence, existence and uniqueness of positive equilibrium, local and global behavior of positive equilibrium point, and rate of convergence of positive solutions of following system of rational difference equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 y_n},$$

where the parameters $\alpha_i$, $\beta_i$, $a_i$, $b_i$ for $i \in \{1, 2\}$ and initial conditions $x_0$, $x_{-1}$, $y_0$, $y_{-1}$ are positive real numbers. Some numerical examples are given to verify our theoretical results.

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## 1. Introduction

Systems of nonlinear difference equations of higher-order are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of systems differential and delay differential equations which model various phenomena in biology, ecology, physiology, physics, engineering and economics. For applications and basic theory of rational difference equations we refer to [1–3]. It is very interesting to investigate the behavior of

solutions of a system of nonlinear difference equations and to discuss the local asymptotic stability of their equilibrium points. Competitive and anti-competitive systems of rational difference equations are very important in population dynamics. The theory of these systems has remarkable applications in biological sciences.

Gibbons et al. [4] investigated the qualitative behavior of the following second-order rational difference equations

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\gamma + x_n}.$$

Recently, Din *et al.* [5] studied the qualitative behavior of the following competitive system of rational difference equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_{n-1}}{a_1 + b_1 y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 y_{n-1}}{a_2 + b_2 x_n}.$$

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Motivated by above study, our aim in this paper was to investigate the qualitative behavior of positive solutions of following second-order system of rational difference equations:

$$x_{n+1} = \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 y_n}, \tag{1}$$

where the parameters $\alpha_i$, $\beta_i$, $a_i$, $b_i$ for $i \in \{1, 2\}$ and initial conditions $x_0$, $x_{-1}$, $y_0$, $y_{-1}$ are positive real numbers.

More precisely, we investigate the boundedness character, persistence, existence and uniqueness of positive steady-state, local asymptotic stability and global behavior of unique positive equilibrium point, and rate of convergence of positive solutions of system (1) which converge to its unique positive equilibrium point.

## 2. Boundedness and persistence

In the following theorem we show the boundedness and persistence of the positive solutions of system (1). We refer to [5–8] for similar methods to prove boundedness and persistence.

**Theorem 1.** *Assume that $\beta_1 \beta_2 < a_1 a_2$, then every positive solution $\{(x_n, y_n)\}$ of system (1) is bounded and persists.*

**Proof.** For any positive solution $\{(x_n, y_n)\}$ of system (1), one has

$$x_{n+1} \leqslant A_1 + B_1 y_{n-1}, \quad y_{n+1} \leqslant A_2 + B_2 x_{n-1}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $A_i = \frac{\alpha_i}{a_i}$ and $B_i = \frac{\beta_i}{a_i}$ for $i \in \{1, 2\}$. Consider the following system of difference equations

$$u_{n+1} = A_1 + B_1 v_{n-1}, \quad v_{n+1} = A_2 + B_2 u_{n-1}, \quad n = 0, 1, 2, \ldots. \tag{3}$$

Solution of system (3) is given by

where $c_i$ for $i \in \{1, 2, 3, 4\}$ depend upon initial conditions $u_{-1}$, $u_0$, $v_{-1}$, $v_0$. Assume that $\beta_1 \beta_2 < a_1 a_2$, then the sequences $\{u_n\}$ and $\{v_n\}$ are bounded, which implies that the sequences $\{x_n\}$ and $\{y_n\}$ are also bounded. Suppose that $u_{-1} = x_{-1}$, $u_0 = x_0$, $v_{-1} = y_{-1}$ and $v_0 = y_0$, then by comparison we have

$$x_n \leqslant \frac{\alpha_1 a_2 + \alpha_2 \beta_1}{a_1 a_2 - \beta_1 \beta_2} = U_1, \quad y_n \leqslant \frac{\alpha_2 a_1 + \alpha_1 \beta_2}{a_1 a_2 - \beta_1 \beta_2} = U_2, \quad n = 1, 2, \ldots. \tag{4}$$

Furthermore, from system (1) and (4) we obtain that

$$x_{n+1} \geqslant \frac{\alpha_1}{a_1 + b_1 x_n} \geqslant \frac{\alpha_1(a_1 a_2 - \beta_1 \beta_2)}{a_1(a_1 a_2 - \beta_1 \beta_2) + b_1(\alpha_1 a_2 + \alpha_2 \beta_1)} = L_1, \tag{5}$$

$$y_{n+1} \geqslant \frac{\alpha_2}{a_2 + b_2 y_n} \geqslant \frac{\alpha_2(a_1 a_2 - \beta_1 \beta_2)}{a_2(a_1 a_2 - \beta_1 \beta_2) + b_2(\alpha_2 a_1 + \alpha_1 \beta_2)} = L_2. \tag{6}$$

From (4)–(6), it follows that

$$L_1 \leqslant x_n \leqslant U_1, \quad L_2 \leqslant y_n \leqslant U_2, \quad n = 1, 2, \ldots.$$

Hence, theorem is proved. $\square$

**Lemma 1.** *Let $\{(x_n, y_n)\}$ be a positive solution of system (1). Then, $[L_1, U_1] \times [L_2, U_2]$ is invariant set for system (1).*

**Proof.** The proof follows by induction. $\square$

## 3. Stability analysis

To construct corresponding linearized form of system (1) we consider the following transformation:

$$(x_n, y_n, x_{n-1}, y_{n-1}) \mapsto (f, g, f_1, g_1), \tag{7}$$

where $f = x_{n+1}, g = y_{n+1}, f_1 = x_n$ and $g_1 = y_n$. The linearized system of (1) about $(\bar{x}, \bar{y})$ is given by

$$Z_{n+1} = F_J(\bar{x}, \bar{y}) Z_n,$$

$$u_n = \frac{A_1 + A_2 B_1}{1 - B_1 B_2} - \frac{\sqrt[4]{B_1} c_1 \left( \left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - i\left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + i\left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n \right)}{4 B_2^{3/4}}$$

$$+ \frac{\sqrt{B_1} c_2 \left( \left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n \right)}{4\sqrt{B_2}}$$

$$+ \frac{1}{4} c_3 \left( \left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n \right)$$

$$- \frac{c_4 \left( \left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - i\left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + i\left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n \right)}{4\sqrt[4]{B_1}\sqrt[4]{B_2}}, \quad n = 1, 2, \cdots,$$

$$v_n = \frac{A_1 B_2 + A_2}{1 - B_1 B_2} - \frac{\sqrt[4]{B_2} c_3 \left( \left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - i\left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + i\left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n \right)}{4 B_1^{3/4}}$$

$$+ \frac{\sqrt{B_2} c_4 \left( \left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n \right)}{4\sqrt{B_1}}$$

$$+ \frac{1}{4} c_2 \left( \left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + \left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n \right)$$

$$- \frac{c_1 \left( \left(-\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - \left(\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n - i\left(-i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n + i\left(i\sqrt[4]{B_1}\sqrt[4]{B_2}\right)^n \right)}{4\sqrt[4]{B_1}\sqrt[4]{B_2}}, \quad n = 1, 2, \ldots,$$

where $Z_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$ and the Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (7) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} -\frac{b_1 \bar{x}}{a_1 + b_1 \bar{x}} & 0 & 0 & \frac{\beta_1}{a_1 + b_1 \bar{x}} \\ 0 & -\frac{b_2 \bar{y}}{a_2 + b_2 \bar{y}} & \frac{\beta_2}{a_2 + b_2 \bar{y}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

**Lemma 2** [3]. *Assume that $X_{n+1} = F(X_n), n = 0, 1, \cdots,$ be a system of difference equations such that $\overline{X}$ be a fixed point of F. If all eigenvalues of the Jacobian matrix $J_F$ about $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\overline{X}$ is unstable.*

Arguing as in [5–11], the following theorem shows the existence and uniqueness of positive equilibrium point of system (1).

**Theorem 2.** *Assume that $\beta_1 \beta_2 < a_1 a_2$, then there exists unique positive equilibrium point of system (1) in $[L_1, U_1] \times [L_2, U_2]$, if the following conditions are satisfied:*

$$L_1(a_1 + b_1 L_1)(\beta_1 a_2 + b_2 L_1(a_1 + b_1 L_1))$$
$$+ \alpha_1^2 < \beta_1^2(\alpha_2 + \beta_2 L_1) + \alpha_1 \beta_1 a_2$$
$$+ \alpha_1 L_1(a_1 + b_1 L_1)(1 + b_2), \tag{8}$$
$$U_1(a_1 + b_1 U_1)(\beta_1 a_2 + b_2 U_1(a_1 + b_1 U_1))$$
$$+ \alpha_1^2 > \beta_1^2(\alpha_2 + \beta_2 U_1) + \alpha_1 \beta_1 a_2$$
$$+ \alpha_1 U_1(a_1 + b_1 U_1)(1 + b_2), \tag{9}$$
$$2b_2 \alpha_1 + \frac{\beta_1^2 \beta_2}{a_1 + 2b_1 L_1} < \beta_1 a_2 + 2b_2 L_1(a_1 + b_1 L_1). \tag{10}$$

**Proof.** Consider the following system of equations:

$$x = \frac{\alpha_1 + \beta_1 y}{a_1 + b_1 x}, \quad y = \frac{\alpha_2 + \beta_2 x}{a_2 + b_2 y}. \tag{11}$$

Assume that $(x, y) \in [L_1, U_1] \times [L_2, U_2]$, then it follows from (11) that

$$x = \frac{y(a_2 + b_2 y) - \alpha_2}{\beta_2}, \quad y = \frac{x(a_1 + b_1 x) - \alpha_1}{\beta_1}.$$

Taking

$$F(x) = \frac{f(x)(a_2 + b_2 f(x)) - \alpha_2}{\beta_2} - x,$$

where $f(x) = \frac{x(a_1 + b_1 x) - \alpha_1}{\beta_1}$ and $x \in [L_1, U_1]$. Assume that condition (8) holds, then it follows that

$$F(L_1) = \frac{f(L_1)(a_2 + b_2 f(L_1)) - \alpha_2}{\beta_2} - L_1 < 0.$$

Furthermore, under the condition (9) we have

$$F(U_1) = \frac{f(U_1)(a_2 + b_2 f(U_1)) - \alpha_2}{\beta_2} - U_1 > 0.$$

Hence, $F(x) = 0$ has at least one positive solution in $[L_1, U_1]$.

Furthermore, assume that condition (10) is satisfied, then one has

$$F'(x) = \frac{f'(x)(a_2 + 2b_2 f(x))}{\beta_2} - 1$$
$$= \frac{(a_1 + 2b_1 x)(a_2 + 2b_2 f(x))}{\beta_1 \beta_2} - 1$$
$$> \frac{(a_1 + 2b_1 L_1)(a_2 + 2b_2 f(L_1))}{\beta_1 \beta_2} - 1 > 0.$$

Hence, $F(x) = 0$ has a unique positive solution in $[L_1, U_1]$. The proof is therefore completed. $\square$

**Theorem 3.** *The unique positive equilibrium point $(\bar{x}, \bar{y})$ of system (1) is locally asymptotically stable if*

$$\frac{b_1 U_1}{a_1 + b_1 L_1} + \frac{b_2 U_2}{a_2 + b_2 L_2} + \frac{b_1 b_2 U_1 U_2}{(a_1 + b_1 L_1)(a_2 + b_2 L_2)}$$
$$+ \frac{\beta_1 \beta_2}{(a_1 + b_1 L_1)(a_2 + b_2 L_2)} < 1.$$

**Proof.** The characteristic polynomial of Jacobian matrix $F_J(\bar{x}, \bar{y})$ about $(\bar{x}, \bar{y})$ is given by

$$P(\lambda) = \lambda^4 + \left( \frac{b_1 \bar{x}}{a_1 + b_1 \bar{x}} + \frac{b_2 \bar{y}}{a_2 + b_2 \bar{y}} \right) \lambda^3$$
$$+ \frac{b_1 b_2 \bar{x} \bar{y}}{(a_1 + b_1 \bar{x})(a_2 + b_2 \bar{y})} \lambda^2$$
$$- \frac{\beta_1 \beta_2}{(a_1 + b_1 \bar{x})(a_2 + b_2 \bar{y})}. \tag{12}$$

Let $\Phi(\lambda) = \lambda^4$, and $\Psi(\lambda) = -\left( \frac{b_1 \bar{x}}{a_1 + b_1 \bar{x}} + \frac{b_2 \bar{y}}{a_2 + b_2 \bar{y}} \right) \lambda^3 - \frac{b_1 b_2 \bar{x} \bar{y}}{(a_1 + b_1 \bar{x})(a_2 + b_2 \bar{y})} \lambda^2 + \frac{\beta_1 \beta_2}{(a_1 + b_1 \bar{x})(a_2 + b_2 \bar{y})}$. Assume that $\frac{b_1 U_1}{a_1 + b_1 L_1} + \frac{b_2 U_2}{a_2 + b_2 L_2} + \frac{b_1 b_2 U_1 U_2}{(a_1 + b_1 L_1)(a_2 + b_2 L_2)} + \frac{\beta_1 \beta_2}{(a_1 + b_1 L_1)(a_2 + b_2 L_2)} < 1$, and $|\lambda| = 1$, then one has

$$|\Psi(\lambda)| < \left( \frac{b_1 \bar{x}}{a_1 + b_1 \bar{x}} + \frac{b_2 \bar{y}}{a_2 + b_2 \bar{y}} \right) + \frac{b_1 b_2 \bar{x} \bar{y}}{(a_1 + b_1 \bar{x})(a_2 + b_2 \bar{y})}$$
$$+ \frac{\beta_1 \beta_2}{(a_1 + b_1 \bar{x})(a_2 + b_2 \bar{y})}$$
$$< \frac{b_1 U_1}{a_1 + b_1 L_1} + \frac{b_2 U_2}{a_2 + b_2 L_2} + \frac{b_1 b_2 U_1 U_2}{(a_1 + b_1 L_1)(a_2 + b_2 L_2)}$$
$$+ \frac{\beta_1 \beta_2}{(a_1 + b_1 L_1)(a_2 + b_2 L_2)} < 1.$$

Then, by Rouche's Theorem $\Phi(\lambda)$ and $\Phi(\lambda) - \Psi(\lambda)$ have same number of zeroes in an open unit disk $|\lambda| < 1$. Hence, all the roots of (12) satisfies $|\lambda| < 1$, and it follows from Lemma 2 that the unique positive equilibrium point $(\bar{x}, \bar{y})$ of the system (1) is locally asymptotically stable. $\square$

Arguing as in [2], we have following result for global behavior of (1).

**Lemma 3.** *Assume that $f : (0, \infty) \times (0, \infty) \to (0, \infty)$ and $g : (0, \infty) \times (0, \infty) \to (0, \infty)$ be continuous functions and a, b, c, d are positive real numbers with $a < b, c < d$. Moreover, suppose that $f : [a, b] \times [c, d] \to [a, b]$ and*

$g : [a, b] \times [c, d] \rightarrow [c, d]$ *such that following conditions are satisfied:*

    *(i) $f(x, y)$ is decreasing in $x$ and increasing in $y$, $g(x, y)$ is increasing in $x$ and decreasing in $y$.*
    *(ii) Let $m_1$, $M_1$, $m_2$, $M_2$ are real numbers such that*

$m_1 = f(M_1, m_2), M_1 = f(m_1, M_2), m_2 = g(m_1, M_2)$ *and* $M_2 = g(M_1, m_2)$, *then* $m_1 = M_1$ *and* $m_2 = M_2$. *Then the system of difference equations* $x_{n+1} = f(x_n, y_{n-1})$, $y_{n+1} = g(x_{n-1}, y_n)$ *has a unique positive equilibrium point* $(\bar{x}, \bar{y})$ *such that* $\lim_{n\to\infty}(x_n, y_n) = (\bar{x}, \bar{y})$.

**Theorem 4.** *The unique positive equilibrium point of system* (1) *is global attractor if* $a_1 a_2 \neq \beta_1 \beta_2$.

**Proof.** Let $f(x, y) = \frac{\alpha_1 + \beta_1 y}{a_1 + b_1 x}$ and $g(x, y) = \frac{\alpha_2 + \beta_2 x}{a_2 + b_2 y}$. Then, it is easy to see that $f(x, y)$ is decreasing in $x$ and increasing in $y$. Moreover, $g(x, y)$ is increasing in $x$, and decreasing in $y$. Let $(m_1, M_1, m_2, M_2)$ be a solution of the system

$$m_1 = f(M_1, m_2), \quad M_1 = f(m_1, M_2)$$
$$m_2 = g(m_1, M_2), \quad M_2 = g(M_1, m_2).$$

Then, one has

$$m_1 = \frac{\alpha_1 + \beta_1 m_2}{a_1 + b_1 M_1}, \quad M_1 = \frac{\alpha_1 + \beta_1 M_2}{a_1 + b_1 m_1}, \tag{13}$$

and

$$m_2 = \frac{\alpha_2 + \beta_2 m_1}{a_2 + b_2 M_2}, \quad M_2 = \frac{\alpha_2 + \beta_2 M_1}{a_2 + b_2 m_2}. \tag{14}$$

Furthermore, from (13) we have

$$m_1(a_1 + b_1 M_1) = \alpha_1 + \beta_1 m_2, \tag{15}$$

and

$$M_1(a_1 + b_1 m_1) = \alpha_1 + \beta_1 M_2. \tag{16}$$

On subtracting (16) from (15) we obtain

$$a_1(M_1 - m_1) = \beta_1(M_2 - m_2). \tag{17}$$

Similarly, from (14) we obtain

$$a_2(M_2 - m_2) = \beta_2(M_1 - m_1). \tag{18}$$

Furthermore, from (17) and (18) we obtain

$$\left( \frac{a_1 a_2 - \beta_1 \beta_2}{\beta_1 a_2} \right)(M_1 - m_1) = 0. \tag{19}$$

Finally, from (19) it follows that $m_1 = M_1$. Similarly, it is easy to see that $m_2 = M_2$. $\quad\square$

**Lemma 4.** *Under the conditions of* Theorems 3 and 4 *the unique positive equilibrium of* (1) *is globally asymptotically stable.*

## 4. Rate of convergence

In this section we will determine the rate of convergence of a solution that converges to the unique positive equilibrium point of the system (1). Similar methods can be found in [5,7,8,11–13].

The following result gives the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = (A + B(n))X_n, \tag{20}$$

where $X_n$ is an $m$-dimensional vector, $A \in C^{m \times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \rightarrow C^{m \times m}$ is a matrix function satisfying

$$\|B(n)\| \rightarrow 0 \tag{21}$$

as $n \rightarrow \infty$, where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

**Proposition 1** (*Perron's Theorem [14]*). *Suppose that condition* (21) *holds. If $X_n$ is a solution of* (20), *then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{1/n} \tag{22}$$

*exists and is equal to the modulus of one of the eigenvalues of matrix A.*

**Proposition 2** [14]. *Suppose that condition* (21) *holds. If $X_n$ is a solution of* (20), *then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{23}$$

*exists and is equal to the modulus of one of the eigenvalues of matrix A.*

Let $\{(x_n, y_n)\}$ be an arbitrary solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$, where $\bar{x} \in [L_1, U_1]$ and $\bar{y} \in [L_2, U_2]$. To find the error terms, one has from the system (1)

$$x_{n+1} - \bar{x} = \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 x_n} - \frac{\alpha_1 + \beta_1 \bar{y}}{a_1 + b_1 \bar{x}}$$
$$= -\frac{b_1 \bar{x}(x_n - \bar{x})}{a_1 + b_1 x_n} + \frac{\beta_1(y_{n-1} - \bar{y})}{a_1 + b_1 x_n},$$

and

$$y_{n+1} - \bar{y} = \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 y_n} - \frac{\alpha_2 + \beta_2 \bar{x}}{a_2 + b_2 \bar{y}}$$
$$= -\frac{b_2 \bar{y}(y_n - \bar{y})}{a_2 + b_2 y_n} + \frac{\beta_2(x_{n-1} - \bar{x})}{a_2 + b_2 y_n}.$$

Let $e_n^1 = x_n - \bar{x}$, and $e_n^2 = y_n - \bar{y}$, then one has

$$e_{n+1}^1 = a_n e_n^1 + b_n e_{n-1}^2,$$

and

$$e_{n+1}^2 = c_n e_n^2 + d_n e_{n-1}^1,$$

where

$$a_n = -\frac{b_1 \bar{x}}{a_1 + b_1 x_n}, \quad b_n = \frac{\beta_1}{a_1 + b_1 x_n},$$

$$c_n = -\frac{b_2 \bar{y}}{a_2 + b_2 y_n}, \quad d_n = \frac{\beta_2}{a_2 + b_2 y_n}.$$

Moreover,

$$\lim_{n\to\infty} a_n = -\frac{b_1\bar{x}}{a_1 + b_1\bar{x}}, \quad \lim_{n\to\infty} b_n = \frac{\beta_1}{a_1 + b_1\bar{x}},$$

$$\lim_{n\to\infty} c_n = -\frac{b_2\bar{y}}{a_2 + b_2\bar{y}}, \quad \lim_{n\to\infty} d_n = \frac{\beta_2}{a_2 + b_2\bar{y}}.$$

Now the limiting system of error terms can be written as

$$\begin{bmatrix} e_{n+1}^1 \\ e_{n+1}^2 \\ e_n^1 \\ e_n^2 \end{bmatrix} = \begin{bmatrix} -\frac{b_1\bar{x}}{a_1+b_1\bar{x}} & 0 & 0 & \frac{\beta_1}{a_1+b_1\bar{x}} \\ 0 & -\frac{b_2\bar{y}}{a_2+b_2\bar{y}} & \frac{\beta_2}{a_2+b_2\bar{y}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_n^1 \\ e_n^2 \\ e_{n-1}^1 \\ e_{n-1}^2 \end{bmatrix},$$

which is similar to linearized system of (1) about the equilibrium point $(\bar{x}, \bar{y})$. Using Proposition 1, one has following result.

**Theorem 5.** *Assume that $\{(x_n, y_n)\}$ be a positive solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$, where $\bar{x} \in [L_1, U_1]$ and $\bar{y} \in [L_2, U_2]$. Then, the error vector*

$$e_n = \begin{pmatrix} e_n^1 \\ e_n^2 \\ e_{n-1}^1 \\ e_{n-1}^2 \end{pmatrix}$$ *of every solution of (1) satisfies both of the following asymptotic relations*

$$\lim_{n\to\infty} (\|e_n\|)^{\frac{1}{n}} = |\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})|, \quad \lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = |\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})|,$$

*where $\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})$ are the characteristic roots of Jacobian matrix $F_J(\bar{x}, \bar{y})$.*

## 5. Periodicity nature of solutions of (1)

**Theorem 6.** *Assume that $a_1 a_2 \neq \beta_1 \beta_2$, then system (1) has no prime period-two solutions.*

**Proof.** On contrary suppose that the system (1) has a distinctive prime period-two solutions

$$\cdots, (p_1, q_1), (p_2, q_2), (p_1, q_1), \cdots$$

where $p_1 \neq p_2, q_1 \neq q_2$, and $p_i$, $q_i$ are positive real numbers for $i \in \{1, 2\}$. Then, from system (1) one has

$$p_1 = \frac{\alpha_1 + \beta_1 q_1}{a_1 + b_1 p_2}, \quad p_2 = \frac{\alpha_1 + \beta_1 q_2}{a_1 + b_1 p_1}, \tag{24}$$

and

$$q_1 = \frac{\alpha_2 + \beta_2 p_1}{a_2 + b_2 q_2}, \quad q_2 = \frac{\alpha_2 + \beta_2 p_2}{a_2 + b_2 q_1}. \tag{25}$$

From (24) and (25) we obtain

$$a_1 p_1 + b_1 p_1 p_2 = \alpha_1 + \beta_1 q_1, \tag{26}$$

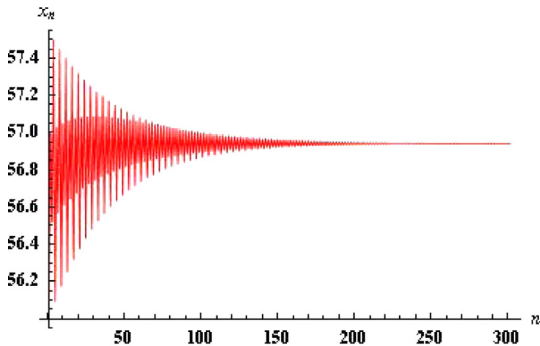$$a_1 p_2 + b_1 p_1 p_2 = \alpha_1 + \beta_1 q_2, \tag{27}$$

$$a_2 q_1 + b_2 q_1 q_2 = \alpha_2 + \beta_2 p_1, \tag{28}$$

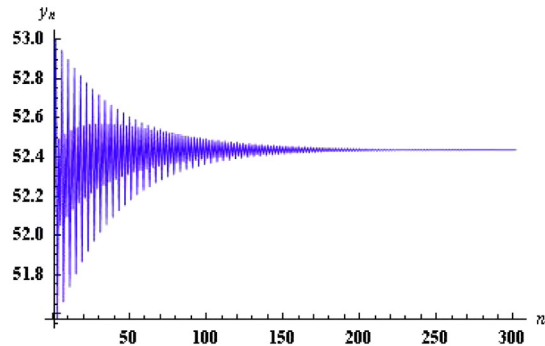$$a_2 q_2 + b_2 q_1 q_2 = \alpha_2 + \beta_2 p_2. \tag{29}$$

On subtracting (26) and (27) it follows that

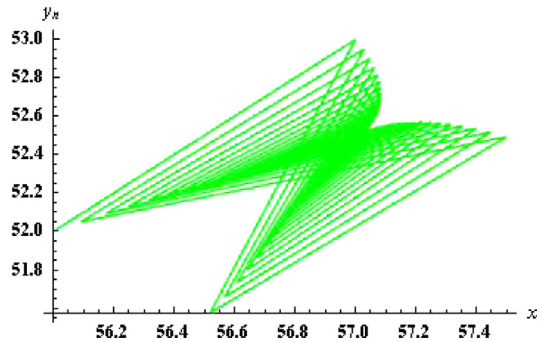$$a_1(p_1 - p_2) = \beta_1(q_1 - q_2). \tag{30}$$

Similarly subtracting (28) and (29) we have
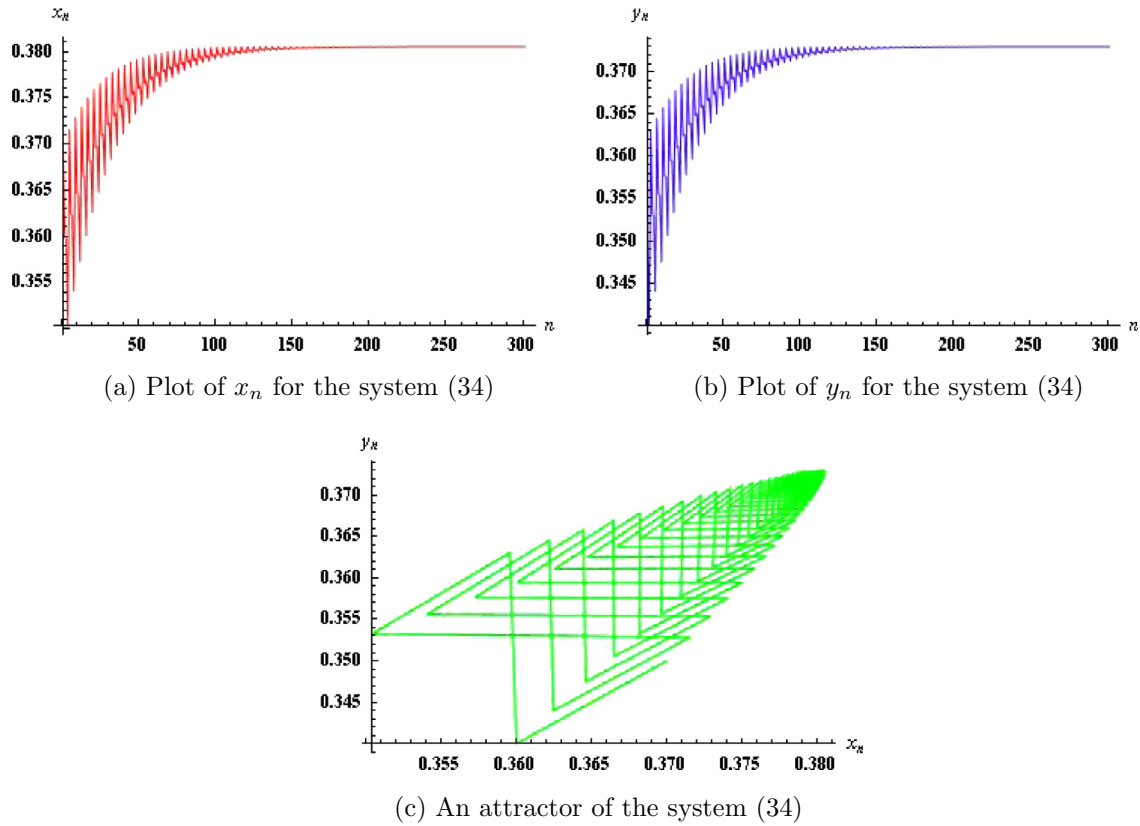


(a) Plot of $x_n$ for the system (33)



(b) Plot of $y_n$ for the system (33)



(c) An attractor of the system (33)

**Figure 1** Plots for the system (33).

(a) Plot of $x_n$ for the system (34)



(b) Plot of $y_n$ for the system (34)



(c) An attractor of the system (34)

**Figure 2**    Plots for the system (34).

$$a_2(q_1 - q_2) = \beta_2(p_1 - p_2). \tag{31}$$

From (30) and (31) one has

$$(a_1 a_2 - \beta_1 \beta_2)(p_1 - p_2) = 0. \tag{32}$$

Assume that $a_1 a_2 \neq \beta_1 \beta_2$, then it follows from (32) that $p_1 = p_2$. Similarly, one can show that $q_1 = q_2$, which is a contradiction. Hence the proof is completed.  □

## 6. Examples

**Example 1.** Let $\alpha_1 = 10$, $\beta_1 = 1.5$, $a_1 = 1.5$, $b_1 = 0.001$, $\alpha_2 = 12$, $\beta_2 = 23$, $a_2 = 25.1$, $b_2 = 0.002$. Then, system (1) can be written as

$$x_{n+1} = \frac{10 + 1.5 y_{n-1}}{1.5 + 0.001 x_n}, \quad y_{n+1} = \frac{12 + 23 x_{n-1}}{25.1 + 0.002 y_n}, \tag{33}$$

with initial conditions $x_0 = 57$, $x_{-1} = 56$, $y_{-1} = 52$, $y_0 = 53$.

In this case the unique positive equilibrium point of the system (33) is given by $(\bar{x}, \bar{y}) = (56.942, 52.4369)$. Moreover, in Fig. 1 the plot of $x_n$ is shown in Fig. 1a, the plot of $y_n$ is shown in Fig. 1b, and an attractor of the system (33) is shown in Fig. 1c.

**Example 2.** Let $\alpha_1 = 0.5$, $\beta_1 = 12$, $a_1 = 13$, $b_1 = 0.2$, $\alpha_2 = 0.1$, $\beta_2 = 17$, $a_2 = 17.5$, $b_2 = 0.3$. Then, system (1) can be written as

$$x_{n+1} = \frac{0.5 + 12 y_{n-1}}{13 + 0.2 x_n}, \quad y_{n+1} = \frac{0.1 + 17 x_{n-1}}{17.5 + 0.3 y_n}, \tag{34}$$

with initial conditions $x_{-1} = 0.37$, $x_0 = 0.36$, $y_{-1} = 0.35$, $y_0 = 0.34$.

In this case the unique positive equilibrium point of the system (34) is given by $(\bar{x}, \bar{y}) = (0.380527, 0.372984)$. Moreover, in Fig. 2 the plot of $x_n$ is shown in Fig. 2a, the plot of $y_n$ is shown in Fig. 2b, and an attractor of the system (34) is shown in Fig. 2c.

## 7. Conclusion

Anti-competitive system of planar rational difference equations has been studied by several authors [15]. It is very interesting mathematical problem to study the qualitative behavior of anti-competitive systems in higher dimension. This work is related to qualitative behavior of an anti-competitive system of second-order rational difference equations. We have investigated the existence and uniqueness of positive steady-state of system (1). Under certain parametric conditions the boundedness and persistence of positive solutions is proved. Moreover, we have shown that unique positive equilibrium point of system (1) is locally as well as globally asymptotically stable under certain parametric conditions. Furthermore, rate of convergence of positive solutions of (1) which converge to its unique positive equilibrium point is demonstrated. Furthermore, periodicity nature of positive solutions of this

anti-competitive system is given. Illustrative numerical examples are provided to support our theoretical discussion.

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