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Existence of weak solutions to a convection–diffusion equation in amalgam spaces

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Abstract
We consider the local existence and uniqueness of a weak solution for a convection–diffusion equation in amalgam spaces. We establish the local existence and uniqueness of solution for the initial condition in amalgam spaces. Furthermore, we prove the validity of the Fujita–Weissler critical exponent for local existence and uniqueness of solution in the amalgam function class that is identified by Escobedo and Zuazua (J Funct Anal 100:119–161, 1991).

Keywords: Convection–diffusion equations, Amalgam spaces, Weak solution, Uniqueness

Mathematics Subject Classification: 35A01, 35K55, 35D30

Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be an unbounded domain with uniform C^2 boundary. We consider the Cauchy–Dirichlet problem for the convection–diffusion equation in amalgam spaces:

$$\begin{cases} \partial_t u - \Delta u = a \cdot \nabla(|u|^{p-1}u), & t > 0,\ x \in \Omega, \\ u(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $a \in \mathbb{R}^n \setminus \{0\}$ and $p \geq 1$, $u = u(t,x)$; $\mathbb{R}_+ \times \Omega \to \mathbb{R}$ is the unknown function and $u_0 = u_0(x); \Omega \to \mathbb{R}$ is given initial condition. Problem (1.1) has been considered by many authors (see, e.g., [2–12, 21, 22, 26, 32, 33]). Among others, for $\Omega = \mathbb{R}^n$, Escobedo and Zuazua [9] showed that, for any initial data $u_0 \in L^1(\mathbb{R}^n)$, there exists a unique global classical solution $u \in C([0,\infty); L^1(\mathbb{R}^n))$ of (1.1) in

$$u \in C((0,\infty);\ W^{2,q}(\mathbb{R}^n)) \cap C^1((0,\infty);\ L^q(\mathbb{R}^n)),$$

for every $q \in (1,\infty)$. They also studied the large-time behavior of solutions to (1.1) and obtained decay estimates for $L^1(\mathbb{R}^n)$ initial data. Haque, Ogawa and Sato [21] showed the existence and uniqueness of weak solutions in uniformly local Lebesgue spaces. One of the main reasons to study problem(1.1) in amalgam spaces is that they allow us to separate the global behavior from the local behavior. In applications, this makes amalgam spaces more applicable than to Lebesgue spaces and uniformly local Lebesgue spaces because the Lebesgue and uniformly local Lebesgue norm does not distinguish between local and global properties. Amalgam space has a long history and has been studied by many authors, [4–7, 16, 20, 25], etc. Amalgam spaces arise naturally in harmonic analysis. In 1926, Norbert Wiener, who was the first one to introduce the amalgam spaces, considers some special cases in [29–31]. Amalgams have been reinvented many times in the literature; the first systematic study appears by Holland in [23]; an excellent review article is [17]. H. Feichtinger [13–15] introduced a far-reaching generalization of amalgam spaces to general topological groups and general local/global function spaces.

Definition (Amalgam spaces). Let $1 \le r, \nu < \infty$. The amalgam spaces on Ω denoted by $L^{r,\nu}_\rho(\Omega)$ are defined by

$$L^{r,\nu}_\rho(\Omega) := \{f : \|f\|_{L^{r,\nu}_\rho} < \infty\},$$

where for $\rho > 0$

$$\|f\|_{L^{r,\nu}_\rho} = \left(\sum_{x_k \in \rho\mathbb{Z}^n} \|f\|^{\nu}_{L^r(B_\rho(x_k)\cap\Omega)} \right)^{\frac{1}{\nu}}, \tag{1.2}$$

where $\mathbb{Z}^n$ stands for the lattice points in $\mathbb{R}^n$. If $r = \nu$, then $L^{r,\nu}_\rho(\Omega) = L^r(\Omega)$. As well as if $\nu = \infty$, then $L^{r,\nu}_\rho(\Omega) = L^r_{\mathrm{uloc},\rho}(\Omega)$. The space $L^{r,\nu}_\rho(\Omega)$ is a Banach space with the norm defined in (1.2).

The Sobolev spaces $W^{k,r,\nu}_\rho(\Omega)$ for $1 \le r, \nu < \infty$, $\rho > 0$ and $k = 1, 2, \ldots$ are analogously introduced. We defined by

$$W^{k,r,\nu}_\rho(\Omega) := \left\{ f : \|f\|_{W^{k,r,\nu}_\rho} < \infty \right\},$$

where for $\rho > 0$,

$$\|f\|_{W^{k,r,\nu}_\rho} = \|f\|_{L^{r,\nu}_\rho} + \sum_{|\alpha|\le k} \|\partial^\alpha_x f\|_{L^{r,\nu}_\rho}.$$

We denote $W^{1,2,2}_\rho(\Omega)$ as $H^1_\rho(\Omega)$ for simplicity and $H^1_{0,\rho}(\Omega)$ be the closure of the $C^\infty_0(\Omega)$ in $H^1_\rho(\Omega)$.

To this end, we introduce the notion of weak solutions to (1.1) in amalgam spaces $L^{r,\nu}_\rho(\Omega)$ as follows.

Definition (Weak $L^{r,\nu}(\Omega)$-solutions) Let $1 \le r, \nu < \infty$ and $\rho > 0$. For an initial data $u_0 \in L^{r,\nu}_\rho(\Omega)$ and $T > 0$, we say that u is a weak $L^{r,\nu}(\Omega)$-solution of (1.1) in $(0, T) \times \Omega$, if

(1) $u \in C([0, T) : L^{r,\nu}_\rho(\Omega)) \cap L^2(0, T : H^1_{0,\rho}(\Omega) \cap L^{r,\nu}_\rho(\Omega))$,
(2) $u(t) \rightharpoonup u_0$ in $*$-weakly in $L^{r,\nu}_\rho(\Omega)$,
(3) u satisfies

$$\int_0^T \int_\Omega \left\{ -u\partial_t\phi + \nabla u \cdot \nabla\phi + a|u|^{p-1}u \cdot \nabla\phi \right\} \mathrm{d}x\mathrm{d}t = 0$$

for all $\phi \in C^\infty_0((0, T) \times \Omega)$.

We now state our main results concerning the existence and uniqueness of this problem.

Theorem 1.1 (Existence of a weak solution) *Let* $p > 1, 1 \leq r, \nu < \infty$ *and* $\nu \geq r$ *with*

$$\begin{cases} r \geq n(p-1) & \text{if} \quad p > 1 + \frac{1}{n}, \\ r > 1 & \text{if} \quad p = 1 + \frac{1}{n}, \\ r \geq 1 & \text{if} \quad 1 < p < 1 + \frac{1}{n}. \end{cases} \tag{1.3}$$

There exists a positive constant γ_0*, depending only on n, p and r, such that, if for any initial condition* $u_0 \in L^{r,\nu}_\rho(\Omega)$ *satisfies*

$$\rho^{\frac{1}{p-1} - \frac{n}{r}} \|u_0\|_{L^{r,\nu}_\rho} \leq \gamma_0 \tag{1.4}$$

for some $\rho > 0$*, then there exists a unique weak* $L^{r,\nu}(\Omega)$*- solution u of* (1.1) *in* $(0, \mu\rho^2) \times \Omega$ *such that*

$$\sup_{0<t<\mu\rho^2} \|u(t)\|_{L^{r,\nu}_\rho} \leq C \|u_0\|_{L^{r,\nu}_\rho},$$

where C and μ *are independent of u. Besides the solution has a uniform estimate*

$$\|u\|_{L^\infty((0,\mu\rho^2)\times\Omega)} \leq C \left(\int_0^{\mu\rho^2} \|u(t)\|^r_{L^{r,\nu}_\rho} dt \right)^{\frac{1}{r}}$$

and hence $u \in L^\infty\big((0, \mu\rho^2) \times \Omega\big)$ *for some* $\mu > 0$.

Local well-posedness problem for Fujita-type nonlinear heat equation was discussed by many authors: For $1 < p < \infty$,

$$\begin{cases} \partial_t u - \Delta u = u^p, & t > 0, x \in \Omega, \\ \quad u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{1.5}$$

In particular, Weissler [28] obtained the sharp well-posedness result in Lebesgue spaces: If

$$\begin{cases} r \geq \frac{n}{2}(p-1) & \text{if} \quad p > 1 + \frac{2}{n}, \\ r > 1 & \text{if} \quad p = 1 + \frac{2}{n}, \\ r \geq 1 & \text{if} \quad 1 < p < 1 + \frac{2}{n}, \end{cases}$$

then solution exists and well-posed in Lebesgue spaces $L^r(\Omega)$. The exponent appears naturally from the invariant scaling equipped with the equation itself;

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \tag{1.6}$$

where u_λ also solves equation (1.5). The threshold scaling space appears when the exponent of the coefficient $\lambda^{\frac{2}{p-1}}$ of the scaled function (1.6) coincides the L^1 invariant scaling. The corresponding result to the convection–diffusion equations (1.1) also holds for the critical exponent $p = 1 + \frac{1}{n}$ (cf. [9]). Our main finding is that even in amalgam spaces decouple the connection between local and global properties that is inherent in Lebesgue spaces, the well-posedness threshold coincides with the usual Lebesgue spaces case. Furthermore, amalgam spaces are a space between usual Lebesgue spaces and uniformly local Lebesgue spaces. We compare our result with the result of [21] and obtain stronger conclusion even though our initial data class smaller than that of [21].

This paper is organized as follows. In "Preliminaries" section, we will state some properties of amalgam spaces. In "A priori estimates" section, we will prove our key estimates: a priori estimates, difference estimates and L^∞ estimates for a weak solution in amalgam spaces. In "Proof of Theorem" section, we will prove our main Theorem 1.1 using the estimates that proved in "A priori estimates" section.

Preliminaries

In this section, we present important properties for functions belonging to amalgam spaces that will be used later.

Proposition 2.1 (Properties of amalgam spaces)

(i) *If* $r_1 \geq r_2$ *and* $\nu_1 \leq \nu_2$ *then for any* $\rho > 0$, *we have* $L^{r_1,\nu_1}_{\rho}(\Omega) \subset L^{r_2,\nu_2}_{\rho}(\Omega)$.

(ii) *Let* $1 \leq r < \infty$. *If* $f \in L^{r,\nu}_{\rho}(\Omega)$ *for some* $\rho > 0$, *then for any* $\rho' > 0$, $f \in L^{r,\nu}_{\rho'}(\Omega)$ *and*

$$\|f\|_{L^{r,\nu}_{\rho'}} \leq C\|f\|_{L^{r,\nu}_{\rho}} \tag{2.1}$$

for some constant C depending only on n, ρ and ρ' if $\rho' > \rho$.

For the proof, see ([25]).

Proposition 2.2 *The class of compact-supported smooth functions*; $C_0^\infty(\Omega)$ *is dense in* $L^{r,\nu}_{\rho}(\Omega)$, $1 \leq r, \nu < \infty$.

For the proof, see ([25]).

Proposition 2.3 (Gagliardo–Nirenberg's inequality) *Let* $\Omega \subset \mathbb{R}^n$, $1 \leq r \leq \infty$, $1 \leq p, q \leq \infty$, *and* $\theta \in [0, 1]$ *satisfying*

$$\frac{1}{q} = (1-\theta)\frac{1}{p} + \theta\left(\frac{1}{r} - \frac{1}{n}\right).$$

Then, there exists a constant $C_{GN} > 0$, *depending only on* p, q, r *and* n *such that for any* $f \in L^p(\Omega) \cap W_0^{1,r}(\Omega)$,

$$\|f\|_{L^q} \leq C_{GN}\|f\|_{L^p}^{1-\theta}\|\nabla f\|_{L^r}^{\theta}. \tag{2.2}$$

For the proof, see ([18]).

Proposition 2.4 *Let* $n \geq 1$, $\Omega \subset \mathbb{R}^n$, $x_0 \in \Omega$, $\rho > 0$ *and* $1 \leq p, q, r < \infty$ *with*

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{2\theta}{r}\left(\frac{1}{2} - \frac{1}{n}\right).$$

Then, there exists a constant $C > 0$ such that for any function f satisfying $f \in L^p(B_\rho(x_0) \cap \Omega)$ with $|f|^{\frac{r}{2}} \in H_0^1(B_\rho(x_0) \cap \Omega)$,

$$\left(\int_{B_\rho(x_0)\cap\Omega} |f|^q \, dy\right)^{\frac{1}{q}} \le C \left(\int_{B_\rho(x_0)\cap\Omega} |f|^p \, dy\right)^{\frac{1-\theta}{p}} \left(\int_{B_\rho(x_0)\cap\Omega} |\nabla |f|^{\frac{r}{2}}|^2 \, dy\right)^{\frac{\theta}{r}}. \tag{2.3}$$

For the proof, see [21]

A priori estimates

In this section, we give some a priori estimates for a weak solution to (1.1). All the estimates hold for the weak solutions to (1.1) if we assume that the solutions exist. In the remainder of this paper, we denote $B_\rho(x) \cap \Omega$ for $x \in \Omega$, $\rho > 0$ by simply $B_\rho(x)$ unless otherwise specified.

Proposition 3.1 (A priori estimate) *Let r satisfy (1.3) and $r > 1$. Let $u_0 \in L_\rho^{r,\nu}(\Omega)$ and u be a $L^{r,\nu}(\Omega)$- solution of (1.1) in $(0, T) \times \Omega$, where $T > 0$. There exists a positive constant γ_1 such that, if*

$$\rho^{\frac{1}{p-1} - \frac{1}{n}} \sup_{0 \le s \le T} \|u(s)\|_{L_\rho^{r,\nu}} \le \gamma_1 \tag{3.1}$$

for some $\rho > 0$, then there exists a constant $\mu > 0$ depending only on p, r, n and γ_1 such that

$$\sup_{0<s<t} \|u(s)\|_{L_\rho^{r,\nu}} \le C \|u_0\|_{L_\rho^{r,\nu}}$$

for $0 < t < \min\{\mu\rho^2, T\}$, where C is a positive constant depending only on n, p and r.

Proof of Proposition 3.1 Let $x \in \Omega$ and ζ be a smooth function in $C_0^\infty(\Omega)$ such that

$$\begin{cases} 0 \le \zeta \le 1 \text{ and } |\nabla \zeta| \le 2\rho^{-1} \text{ in } \Omega, \\ \zeta = 1 \text{ on } B_\rho(x), \quad \zeta = 0 \text{ in } \Omega \setminus B_{2\rho}(x). \end{cases}$$

For any $0 < \tau < t \le T$, multiplying (1.1) by $(\operatorname{sgn} u)|u|^{r-1}\zeta^k$ and integrating it in $(0, \tau) \times \Omega$, we have that,

$$\begin{aligned} &\frac{1}{r}\int_{B_{2\rho}(x)} \left|u(\tau,y)\right|^r \zeta(y)^k \mathrm{d}y - \frac{1}{r}\int_{B_{2\rho}(x)} \left|u(0,y)\right|^r \zeta(y)^k \mathrm{d}y \\ &\quad + \int_0^\tau \int_\Omega \nabla u(s,y) \cdot \nabla((\operatorname{sgn} u(s,y))\left|u(s,y)\right|^{r-1}\zeta(y)^k)\mathrm{d}y\mathrm{d}s \\ &\quad = \int_0^\tau \int_\Omega a \cdot \nabla(\left|u(s,y)\right|^{p-1}u(s,y))(\operatorname{sgn} u(s,y))\left|u(s,y)\right|^{r-1}\zeta(y)^k \mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.2}$$

As the relations

$$\begin{aligned}\nabla u\cdot\nabla((\operatorname{sgn}u)|u|^{r-1}\zeta^{k}) \geq&(r-1)|u|^{r-2}|\nabla u|^2\zeta^k-\left|k(\operatorname{sgn}u)|u|^{r-1}\zeta^{k-1}\nabla u\cdot\nabla\zeta\right|\\ \geq&\{(r-1)-k\varepsilon\}|u|^{r-2}|\nabla u|^2\zeta^k-\frac{k}{4\varepsilon}|u|^r\zeta^{k-2}|\nabla\zeta|^2\end{aligned} \tag{3.3}$$

and

$$\left|\nabla(|u|^{\frac{r}{2}}\zeta^{\frac{k}{2}})\right|^2\leq\frac{r^2}{2}|u|^{r-2}|\nabla u|^2\zeta^k+\frac{k^2}{2}|u|^r\zeta^{k-2}|\nabla\zeta|^2 \tag{3.4}$$

are hold. Hence, by inequalities (3.3) and (3.4), we have that,

$$\begin{aligned}&\nabla u\cdot\nabla((\operatorname{sgn}u)|u|^{r-1}\zeta^{k})\\ &\quad\geq\{(r-1)-\varepsilon k\}\left\{\frac{2}{r^2}\left|\nabla(|u|^{\frac{r}{2}}\zeta^{\frac{k}{2}})\right|^2-\frac{k^2}{r^2}|u|^r\zeta^{k-2}|\nabla\zeta|^2\right\}\\ &\quad-\frac{k}{4\varepsilon}|u|^r\zeta^{k-2}|\nabla\zeta|^2\\ &\quad=C_1\left|\nabla(|u|^{\frac{r}{2}}\zeta^{\frac{k}{2}})\right|^2-C_2|u|^r\zeta^{k-2}|\nabla\zeta|^2.\end{aligned} \tag{3.5}$$

Moreover, by Young's inequality, we have that,

$$\begin{aligned}&\int_0^\tau\int_{B_{2\rho}(x)}\frac{p}{p+r-1}a\cdot\nabla\left(\left|u(s,y)\right|^{p+r-1}\right)\zeta(y)^k\mathrm{d}y\mathrm{d}s\\ &=\frac{kp}{p+r-1}\int_0^\tau\int_{B_{2\rho}(x)}a\cdot\left|u(s,y)\right|^{p+r-1}\zeta(y)^{k-1}\nabla\zeta(y)\mathrm{d}y\mathrm{d}s\\ &\leq C_2\int_0^\tau\int_{B_{2\rho}(x)}\left|u(s,y)\right|^r\zeta(y)^{k-2}\left|\nabla\zeta(y)\right|^2\mathrm{d}y\mathrm{d}s\\ &\quad+C_3\int_0^\tau\int_{B_{2\rho}(x)}|u(s,y)|^{2p+r-2}\zeta(y)^k\mathrm{d}y\mathrm{d}s.\end{aligned} \tag{3.6}$$

Hence, by inequalities (3.2), (3.5) and (3.6), we have that,

$$\begin{aligned}&\frac{1}{r}\int_{B_{2\rho}(x)}\left|u(\tau,y)\right|^r\zeta(y)^k\mathrm{d}y-\frac{1}{r}\int_{B_{2\rho}(x)}\left|u(0,y)\right|^r\zeta(y)^k\mathrm{d}y\\ &\quad+C_1\int_0^\tau\int_{B_{2\rho}(x)}\left|\nabla(\left|u(s,y)\right|^{\frac{r}{2}}\zeta(y)^{\frac{k}{2}})\right|^2\mathrm{d}y\mathrm{d}s\\ &\leq C_2\int_0^\tau\int_{B_{2\rho}(x)}\left|u(s,y)\right|^r\zeta(y)^{k-2}\left|\nabla\zeta(y)\right|^2\mathrm{d}y\mathrm{d}s\\ &\quad+C_3\int_0^\tau\int_{B_{2\rho}(x)}\left|u(s,y)\right|^{2p+r-2}\zeta(y)^k\mathrm{d}y\mathrm{d}s.\end{aligned} \tag{3.7}$$

We now estimate the last term of the right hand side of (3.7) using Gagliardo–Nirenberg's inequality (Proposition 2.4). In particular, choosing $\tilde{q} = \frac{4}{r}(p-1)+2$ and $\frac{\tilde{q}\theta}{2} = 1$, and setting $g(s,y) := |u(s,y)|\zeta(y)^{\frac{k}{2p+r-2}}$, we have using Hölder's inequality for $r \geq n(p-1)$ that

$$\begin{aligned}
&\int_0^\tau \int_{B_{2\rho}(x)} |u(s,y)|^{2p+r-2}\zeta(y)^k \mathrm{d}y\mathrm{d}s \\
&\quad \leq C \sup_{0<s<\tau} \left(\int_{B_{2\rho}(x)} |g(s,y)|^{n(p-1)}\mathrm{d}y \right)^{\frac{2}{n}} \int_0^t \int_{B_{2\rho}(x)} |\nabla |g(s,y)|^{\frac{r}{2}}|^2 \mathrm{d}y\mathrm{d}s \\
&\quad \leq C \sup_{0<s<\tau} \left(\rho^{\frac{r}{p-1}-n} \int_{B_{2\rho}(x)} |u(s,y)|^r \mathrm{d}y \right)^{\frac{2(p-1)}{r}} \\
&\qquad \times \int_0^\tau \left(\int_{B_{2\rho}(x)} \left| \nabla (|u(s,y)|^{\frac{r}{2}}) \right|^2 \mathrm{d}y + \rho^{-2} \int_{B_{2\rho}(x)} |u(s,y)|^r \mathrm{d}y \right) \mathrm{d}s.
\end{aligned} \tag{3.8}$$

Hence, by Proposition 2.1, we conclude from (3.7) and (3.8) that

$$\begin{aligned}
&\frac{1}{r} \int_{B_{2\rho}(x)} |u(\tau,y)|^r \zeta(y)^k \mathrm{d}y - \frac{1}{r} \int_{B_{2\rho}(x)} |u(0,y)|^r \zeta(y)^k \mathrm{d}y \\
&+ C_1 \int_0^\tau \int_{B_{2\rho}(x)} \left| \nabla (|u(s,y)|^{\frac{r}{2}} \zeta(y)^{\frac{k}{2}}) \right|^2 \mathrm{d}y\mathrm{d}s \\
&\leq C_2 \int_0^\tau \int_{B_{2\rho}(x)} |u(s,y)|^r \zeta(y)^{k-2} |\nabla \zeta(y)|^2 \mathrm{d}y\mathrm{d}s \\
&+ C_3 \sup_{0<s<\tau} \left(\rho^{\frac{r}{p-1}-n} \int_{B_{2\rho}(x)} |u(s,y)|^r \mathrm{d}y \right)^{\frac{2(p-1)}{r}} \\
&\times \int_0^\tau \left(\int_{B_{2\rho}(x)} \left| \nabla (|u(s,y)|^{\frac{r}{2}}) \right|^2 \mathrm{d}y + \rho^{-2} \int_{B_{2\rho}(x)} |u(s,y)|^r \mathrm{d}y \right) \mathrm{d}s \\
&\leq C_2 \rho^{-2} \int_0^\tau \int_{B_\rho(x)} |u(s,y)|^r \mathrm{d}y\mathrm{d}s \\
&+ C_3 \sup_{0<s<\tau} \left(\rho^{\frac{r}{p-1}-n} \int_{B_\rho(x)} |u(s,y)|^r \mathrm{d}y \right)^{\frac{2(p-1)}{r}} \\
&\times \int_0^\tau \left(\int_{B_\rho(x)} \left| \nabla (|u(s,y)|^{\frac{r}{2}}) \right|^2 \mathrm{d}y + \rho^{-2} \int_{B_\rho(x)} |u(s,y)|^r \mathrm{d}y \right) \mathrm{d}s.
\end{aligned} \tag{3.9}$$

By taking, the supremum for $\tau \in (0,t)$ in the right hand side of (3.9) and using (3.1), we have that,

$$
\begin{aligned}
&\int_{B_\rho(x)} |u(\tau,y)|^r \mathrm{d}y + C_1 \int_0^\tau \int_{B_\rho(x)} \left|\nabla(|u(s,y)|^{\frac{r}{2}})\right|^2 \mathrm{d}y\mathrm{d}s \\
&\leq C_2 t\rho^{-2} \sup_{0<s<t} \int_{B_\rho(x)} |u(s,y)|^r \mathrm{d}y + \sup_{x\in\mathbb{R}^n} \int_{B_\rho(x)} |u(0,y)|^r \mathrm{d}y \\
&+ C_3 \sup_{0<s<t} \left(\rho^{\frac{r}{p-1}-n} \int_{B_\rho(x)} |u(s,y)|^r \mathrm{d}y\right)^{\frac{2(p-1)}{r}} \\
&\times \int_0^t \left(\int_{B_\rho(x)} \left|\nabla(|u(s,y)|^{\frac{r}{2}})\right|^2 \mathrm{d}y + \rho^{-2}\int_{B_\rho(x)} |u(s,y)|^r \mathrm{d}y\right)\mathrm{d}s \\
&\leq C_2 t\rho^{-2} \sup_{0<s<t} \int_{B_\rho(x)} |u(s,y)|^r \mathrm{d}y + \sup_{x\in\mathbb{R}^n} \int_{B_\rho(x)} |u(0,y)|^r \mathrm{d}y \\
&+ C_3\gamma_1^{2(p-1)} \int_0^t \left(\int_{B_\rho(x)} \left|\nabla(|u(s,y)|^{\frac{r}{2}})\right|^2 \mathrm{d}y + \rho^{-2}\int_{B_\rho(x)} |u(s,y)|^r \mathrm{d}y\right)\mathrm{d}s.
\end{aligned}
\tag{3.10}
$$

for $0 < \tau < t \leq T$. Taking a sufficiently small γ_1 and $t\rho^{-2}$ if necessary, and by taking the-supremum for $\tau \in (0,t)$, we deduce from (3.10) that

$$
\sup_{0<\tau<t} \int_{B_\rho(x)} |u(\tau,y)|^r \mathrm{d}y \leq \quad Ct\rho^{-2} \sup_{0<s<t} \int_{B_\rho(x)} |u(s,y)|^r \mathrm{d}y\mathrm{d}s + \int_{B_\rho(x)} |u(0,y)|^r \mathrm{d}y
$$

for $0 < \tau < t \leq T$. This implies that,

$$
\left(1 - Ct\rho^{-2}\right)^{\frac{\nu}{r}} \sup_{0<s<t} \left(\int_{B_\rho(x)} |u(s,y)|^r \mathrm{d}y\right)^{\frac{\nu}{r}} \leq \left(\int_{B_\rho(x)} |u(0,y)|^r \mathrm{d}y\right)^{\frac{\nu}{r}}.
$$

Taking summation on both sides on the lattice point $x_k \in \mathbb{Z}^n$, we have that,

$$
\begin{aligned}
&\left(1 - Ct\rho^{-2}\right)^{\frac{\nu}{r}} \sup_{0<s<t} \sum_{x_k\in\rho\mathbb{Z}^n} \left(\int_{B_\rho(x_k)} |u(s,y)|^r \mathrm{d}y\right)^{\frac{\nu}{r}} \\
&\leq \sum_{x_k\in\rho\mathbb{Z}^n} \left(\int_{B_\rho(x_k)} |u(0,y)|^r \mathrm{d}y\right)^{\frac{\nu}{r}}.
\end{aligned}
\tag{3.11}
$$

Hence, from (3.11), we have that,

$$
\sup_{0<s<t} \|u(s)\|_{L^{r,\nu}_\rho} \leq C\|u(0)\|_{L^{r,\nu}_\rho},
$$

for $0 < t < \min\{\mu\rho^2, T\}$. □

Proposition 3.2 (Difference estimate) *Let r satisfy* (1.3), $r > 1$ *and* $T > 0$. *Let* u_0 *and* $v_0 \in L^{r,\nu}_\rho(\Omega)$ *be two initial data and suppose that u and v be a corresponding $L^{r,\nu}(\Omega)$-solution of* (1.1) *in* $(0,T)\times\Omega$, *respectively. There exists a positive constant* γ_2 *such that, if*

$$
\begin{aligned}
&\rho^{\frac{1}{p-1}-\frac{n}{r}} \sup_{0\leq s\leq T} \|u(s)\|_{L^{r,\nu}_\rho} \leq \gamma_2, \\
&\rho^{\frac{1}{p-1}-\frac{n}{r}} \sup_{0\leq s\leq T} \|v(s)\|_{L^{r,\nu}_\rho} \leq \gamma_2,
\end{aligned}
\tag{3.12}
$$

for some $\rho > 0$, *then there exists a constant* $\mu > 0$ *depending only on p, r, n and* γ_2 *such that*

$$\sup_{0<s<t} \|u(s) - v(s)\|_{L^{r,v}_\rho} \le C\|u_0 - v_0\|_{L^{r,v}_\rho}$$

for $0 < t < \min\{\mu\rho^2, T\}$, *where C and* μ *are positive constants depending only on n, p and r.*

Proof of Proposition 3.2

Let $x \in \Omega$ *and* ζ *be a smooth function in* $C_0^\infty(\Omega)$ *defined in (1.1) Suppose that u and v are two strong solutions of (1.1) in* $(0,T)\times\Omega$ *and let* $w = u - v$. *Then multiply* $|w|^{r-1}(\operatorname{sgn} w)\zeta^k$ *for* $k \in \mathbb{N}$ *to the difference of equation*

$$\partial_t w - \Delta w = a \cdot \nabla(|u|^{p-1}u - |v|^{p-1}v)$$

and integrate it over Ω we obtain that

$$\begin{aligned} &\frac{1}{r}\frac{d}{dt}\int_\Omega |w(s)|^r\zeta^k \mathrm{d}y + \int_\Omega \nabla w(s)\cdot\nabla(|w(s)|^{r-1}(\operatorname{sgn} w(s))\zeta^k\mathrm{d}y \\ &\quad = -\int_\Omega (|u|^{p-1}u - |v|^{p-1}v)\big(a\cdot\nabla(\operatorname{sgn} w(s)|w(s)|^{r-1})\big)\zeta^k\mathrm{d}y \\ &\quad\quad - \int_\Omega (|u|^{p-1}u - |v|^{p-1}v)(\operatorname{sgn} w(s))|w(s)|^{r-1}a\cdot\nabla\zeta^k\mathrm{d}y. \end{aligned} \tag{3.13}$$

Observing that

$$\nabla w\cdot\nabla(|w|^{r-1}(\operatorname{sgn} w)\zeta^k) \ge C_1\Big|\nabla(|w|^{\frac{r}{2}}\zeta^{\frac{k}{2}})\Big|^2 - C_2|w|^r\zeta^{k-2}|\nabla\zeta|^2. \tag{3.14}$$

By mean value's theorem, we know that

$$\begin{aligned} \big||u|^{p-1}u - |v|^{p-1}v\big| =& \Big|\int_0^1 \frac{d}{d\theta}\big(|v+\theta(u-v)|^{p-1}(v+\theta(u-v))\big)d\theta\Big| \\ \le& p|u-v|\int_0^1\Big(|v+\theta(u-v)|^{p-1}\Big)d\theta \\ \le& p|w|\big(\max(|u|,|v|)\big)^{p-1}. \end{aligned} \tag{3.15}$$

Therefore, by (3.14) and (3.15), we obtain from (3.13) that

$$\begin{aligned} &\frac{1}{r}\frac{d}{dt}\int_{B_{2\rho}(x)} |w(s)|^r\zeta^k\mathrm{d}y + C\int_{B_{2\rho}(x)}\Big|\nabla(|w(s)|^{\frac{r}{2}}\zeta^{\frac{k}{2}})\Big|^2\mathrm{d}y \\ &\quad - C\int_{B_{2\rho}(x)} |w(s)|^r\zeta^{k-2}|\nabla\zeta|^2\mathrm{d}y \\ &\quad \le C\int_{B_{2\rho}(x)}\big(\max(|u(s)|,|v(s)|)\big)^{p-1}\big|\nabla|w(s)|^r\big|\zeta^k\mathrm{d}y \\ &\quad + C\int_{B_{2\rho}(x)}\big(\max(|u(s)|,|v(s)|)\big)^{p-1}|w(s)|^r|\nabla\zeta^k|\mathrm{d}y. \end{aligned} \tag{3.16}$$

Now we estimate the first and last term of the right hand side of (3.16) using the Young and the Hölder inequalities. The first term of the right hand side of (3.16) follows:

Let $U(s) = \max(|u(s)|, |v(s)|)$, then

$$\begin{aligned}
&\int_{B_{2\rho}(x)} \big(\max(|u(s)|, |v(s)|)\big)^{p-1} \big|\nabla |w(s)|^r\big| \zeta^k \mathrm{d}y \\
&= C\int_{B_{2\rho}(x)} U(s)^{p-1} |w(s)|^{\frac{r}{2}} \big|\nabla |w(s)|^{\frac{r}{2}}\big| \zeta^k \mathrm{d}y \\
&\leq C\int_{B_{2\rho}(x)} U(s)^{2p-2} |w(s)|^r \zeta^k \mathrm{d}y + C\int_{B_{2\rho}(x)} \big|\nabla |w(s)|^{\frac{r}{2}}\big|^2 \zeta^k \mathrm{d}y.
\end{aligned} \tag{3.17}$$

Now we estimate the first term of the right hand side of (3.17) using the Hölder and the Sobolev inequalities and obtain that

$$\begin{aligned}
&\int_0^\tau \int_{B_{2\rho}(x)} |U(s,y)|^{2p-2} |w(s,y)|^r \zeta(y)^k \mathrm{d}y\mathrm{d}s \\
&\leq C\int_0^\tau \left(\int_{B_{2\rho}(x)} |U(s,y)|^{n(p-1)} \mathrm{d}y\right)^{\frac{2}{n}} \left(\int_{B_{2\rho}(x)} \left||w(s,y)|^{\frac{r}{2}}\right|^{\frac{2n}{n-2}} \zeta(y)^{\frac{kn}{n-2}} \mathrm{d}y\right)^{\frac{n-2}{n}} \mathrm{d}s \\
&\leq C \sup_{0<s<\tau} \left(\rho^{\frac{r}{p-1}-n} \int_{B_{2\rho}(x)} |U(s,y)|^r \mathrm{d}y\right)^{\frac{2(p-1)}{r}} \\
&\times \int_0^\tau \left(\int_{B_{2\rho}(x)} \left|\nabla(|w(s,y)|^{\frac{r}{2}})\right|^2 \mathrm{d}y + \rho^{-2} \int_{B_{2\rho}(x)} |w(s,y)|^r \mathrm{d}y\right) \mathrm{d}s.
\end{aligned} \tag{3.18}$$

Therefore, by (3.17), (3.18), we obtain from (3.16) that

$$\begin{aligned}
&\frac{1}{r}\int_{B_{2\rho}(x)} |w(\tau,y)|^r \zeta(y)^k \mathrm{d}y - \frac{1}{r}\int_{B_{2\rho}(x)} |w(0,y)|^r \zeta(y)^k \mathrm{d}y \\
&\qquad + C_1 \int_0^\tau \int_{B_{2\rho}(x)} \left|\nabla(|w(s,y)|^{\frac{r}{2}} \zeta(y)^{\frac{k}{2}})\right|^2 \mathrm{d}y\mathrm{d}s \\
&\leq C_2 \int_0^\tau \int_{B_{2\rho}(x)} |w(s,y)|^r \zeta(y)^{k-2} |\nabla\zeta(y)|^2 \mathrm{d}y\mathrm{d}s \\
&\qquad + C_3 \sup_{0<s<\tau} \left(\rho^{\frac{r}{p-1}-n} \int_{B_{2\rho}(x)} \big|\max(|u|, |v|)\big|^r \mathrm{d}y\right)^{\frac{2(p-1)}{r}} \\
&\qquad \times \int_0^\tau \left(\int_{B_{2\rho}(x)} \left|\nabla(|w(s,y)|^{\frac{r}{2}})\right|^2 \mathrm{d}y + \rho^{-2}\int_{B_{2\rho}(x)} |w(s,y)|^r \mathrm{d}y\right) \mathrm{d}s.
\end{aligned} \tag{3.19}$$

By the Gagliardo–Nirenberg inequality, we obtain from (3.19) that

$$
\begin{aligned}
&\frac{1}{r}\int_{B_{2\rho}(x)}|w(\tau,y)|^r\zeta(y)^k\mathrm{d}y-\frac{1}{r}\int_{B_{2\rho}(x)}|w(0,y)|^r\zeta(y)^k\mathrm{d}y\\
&\quad+C_1\int_0^\tau\int_{B_{2\rho}(x)}\left|\nabla(|w(s,y)|^{\frac{r}{2}}\zeta(y)^{\frac{k}{2}})\right|^2\mathrm{d}y\mathrm{d}s\\
&\quad\le C_2\int_0^\tau\int_{B_\rho(x)}|w(s,y)|^r\mathrm{d}y\mathrm{d}s\\
&\quad+C_3\left(\rho^{\frac{r}{p-1}-n}\sup_{0<s<\tau}\int_{B_\rho(x)}\big||u(s,y)|+|v(s,y)|\big|^r\mathrm{d}y\right)^{\frac{2(p-1)}{r}}\\
&\quad\times\int_0^\tau\left(\int_{B_\rho(x)}\left|\nabla(|w(s,y)|^{\frac{r}{2}})\right|^2\mathrm{d}y+\rho^{-2}\int_{B_\rho(x)}|w(s,y)|^r\mathrm{d}y\right)\mathrm{d}s\\
&\quad\le C_2\rho^{-2}\int_0^\tau\int_{B_\rho(x)}|w(s,y)|^r\mathrm{d}y\mathrm{d}s\\
&\quad+C_3\rho^{\frac{r}{p-1}-n}\sup_{0<s<\tau}\left(\int_{B_\rho(x)}|u(s,y)|^r\mathrm{d}y+\int_{B_\rho(x)}|v(s,y)|^r\mathrm{d}y\right)^{\frac{2(p-1)}{r}}\\
&\quad\times\int_0^\tau\left(\int_{B_\rho(x)}\left|\nabla(|w(s,y)|^{\frac{r}{2}})\right|^2\mathrm{d}y+\rho^{-2}\int_{B_\rho(x)}|w(s,y)|^r\mathrm{d}y\right)\mathrm{d}s.
\end{aligned}\tag{3.20}
$$

for all $0<\tau<t\le T$.

By taking the supremum for $\tau\in(0,t)$ in the right hand side of (3.20) and using (3.12), we obtain that

$$
\begin{aligned}
&\int_{B_\rho(x)}|w(\tau,y)|^r\mathrm{d}y+C_1\int_0^\tau\int_{B_\rho(x)}\left|\nabla(|w(s,y)|^{\frac{r}{2}})\right|^2\mathrm{d}y\mathrm{d}s\\
&\quad\le C_2t\rho^{-2}\sup_{0<s<t}\int_{B_\rho(x)}|w(s,y)|^r\mathrm{d}y+\int_{B_\rho(x)}|w(0,y)|^r\mathrm{d}y\\
&\quad+C_3\gamma_2^{2(p-1)}\left(\int_0^t\int_{B_\rho(x)}\left|\nabla(|w(s,y)|^{\frac{r}{2}})\right|^2\mathrm{d}y\mathrm{d}s+t\rho^{-2}\sup_{0<s<t}\int_{B_\rho(x)}|w(s,y)|^r\mathrm{d}y\right).
\end{aligned}\tag{3.21}
$$

for $0<\tau<t\le T$. Taking a sufficiently small γ_2 and $t\rho^{-2}$ if necessary, and by taking the supremum for $\tau\in(0,t)$, we deduce from (3.21) that

$$
\sup_{0<\tau<t}\int_{B_\rho(x)}|w(\tau,y)|^r\mathrm{d}y\le\quad Ct\rho^{-2}\sup_{0<s<t}\int_{B_\rho(x)}|w(s,y)|^r\mathrm{d}y\mathrm{d}s+\int_{B_\rho(x)}|w(0,y)|^r\mathrm{d}y
$$

for $0<\tau<t\le T$. This implies that

$$
\left(1-Ct\rho^{-2}\right)^{\frac{\nu}{r}}\sup_{0<s<t}\left(\int_{B_\rho(x)}|w(s,y)|^r\mathrm{d}y\right)^{\frac{\nu}{r}}\le\left(\int_{B_\rho(x)}|w(0,y)|^r\mathrm{d}y\right)^{\frac{\nu}{r}}.
$$

Taking summation on both sides on the lattice point $x_k\in\mathbb{Z}^n$, we have

$$
\left(1-Ct\rho^{-2}\right)^{\frac{\nu}{r}}\sup_{0<s<t}\sum_{x_k\in\rho\mathbb{Z}^n}\left(\int_{B_\rho(x_k)}|w(s,y)|^r\mathrm{d}y\right)^{\frac{\nu}{r}}\le\sum_{x_k\in\rho\mathbb{Z}^n}\left(\int_{B_\rho(x_k)}|w(0,y)|^r\mathrm{d}y\right)^{\frac{\nu}{r}}.\tag{3.22}
$$

Hence, from (3.22), we obtain that

$$\sup_{0<s<t} \|w(s)\|_{L^{r,\nu}_\rho} \le C\|w(0)\|_{L^{r,\nu}_\rho},$$

for $0 < t < \min\{\mu\rho^2, T\}$. □

To obtain the critical existence of the weak solutions, the L^∞ a priori estimate for the weak solutions is essential. For related results, see ([1, 24]).

Proposition 3.3 (L^∞-a priori estimate) *Let u be a $L^{r,\nu}(\Omega)$-solution of (1.1) in $(0,T)\times\Omega$, where $0 < T < \infty$ and $r > 1$. For some positive constant γ_3, if*

$$\rho^{\frac{1}{p-1}-\frac{n}{r}} \sup_{0\le s\le T} \|u(s)\|_{L^{r,\nu}_\rho} \le \gamma_3 \tag{3.23}$$

for some $\rho > 0$, then there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty\left((t_1,t)\times B_{R_1}(x)\right)} \le CD^{\frac{n+2}{2r}}\left(\int_{t_2}^t \int_{B_{R_2}(x)} |u|^r \mathrm{d}y\mathrm{d}s\right)^{\frac{1}{r}}, \tag{3.24}$$

$$\int_{t_1}^t \int_{B_{R_1}(x)} |\nabla u|^2 \mathrm{d}y\mathrm{d}s \le CD\int_{t_2}^t \int_{B_{R_2}(x)} |u|^2\mathrm{d}y\mathrm{d}s, \tag{3.25}$$

for all $x\in\Omega$, $0 < R_1 < R_2$ and $0 < t_2 < t_1 \le T$, where

$$D = C_1(R_2-R_1)^{-2} + (t_1-t_2)^{-1}.$$

Proof of Proposition 3.3 Let $x\in\Omega$, $0 < R_1 < R_2$, $0 < t_2 < t_1 < t \le T$. For $j = 0, 1, 2, ...,$ set

$$r_j := R_1 + (R_2-R_1)2^{-j}, \quad \tau_j := t_1 - (t_1-t_2)2^{-2j}, \quad Q_j = (\tau_j, t)\times B_{r_j}(x).$$

Let ζ_j be a piecewise smooth function in Q_j satisfying

$$\begin{cases} 0 \le \zeta_j(t,x) \le 1 \quad \text{in } \Omega, \\ \quad \zeta_j(t,x) \equiv 1 \quad \text{on } Q_{j+1}, \\ \zeta_j = 0 \quad \text{near} \quad [\tau_j, t]\times\partial B_{r_j}(x) \cup \{\tau_j\}\times B_{r_j}(x), \\ |\nabla\zeta_j| \le \dfrac{2^{j+1}}{R_2-R_1}, \quad \text{in } Q_j \\ 0 \le \partial_t\zeta_j \le \dfrac{2^{2(j+1)}}{t_2-t_1} \quad \text{in } Q_j. \end{cases} \tag{3.26}$$

Multiplying (1.1) by $|u(t,y)|^{\beta-2}u(t,y)\zeta^k(t,y)$ and integrating it in Ω, we obtain that

$$
\begin{aligned}
&\frac{1}{\beta}\frac{d}{dt}\left(\int_{B_{r_j}(x)}|u(t,y)|^{\beta}\zeta_j(t,y)^k\mathrm{d}y\right)+\frac{2(2\beta-k-2)}{\beta^2}\int_{B_{r_j}(x)}\left|\nabla|u(t,y)|^{\frac{\beta}{2}}\right|^2\zeta_j(t,y)^k\mathrm{d}y\\
&\quad\le\frac{pk|a|^2}{2(p+\beta-1)}\int_{B_{r_j}(x)}|u(t,y)|^{2p+\beta-2}\zeta_j(t,y)^k\mathrm{d}y\\
&\quad+\frac{k}{2}\left(\frac{p}{p+\beta-1}+1\right)\int_{B_{r_j}(x)}|u(t,y)|^{\beta}\zeta_j(t,y)^{k-2}|\nabla\zeta_j(t,y)|^2\mathrm{d}y\\
&\quad+\frac{k}{\beta}\int_{B_{r_j}(x)}\zeta_j(t,y)^{k-1}|u(t,y)|^{\beta}\partial_t\zeta_j(t,y)\mathrm{d}y.
\end{aligned}
\tag{3.27}
$$

For the highest-order term, using the Hölder and the Sobolev inequalities, we obtain that

$$
\begin{aligned}
&\int_{B_{r_j}(x)}|u(t,y)|^{2(p-1)}|u(t,y)|^{\beta}\zeta_j{}^k\mathrm{d}y\\
&\quad\le\left(\int_{B_{r_j}(x)}|u(t,y)|^{n(p-1)}\mathrm{d}y\right)^{\frac{2}{n}}\left(\int_{B_{r_j}(x)}\left(|u(t,y)|^{\beta}\zeta_j{}^k\right)^{\frac{n}{n-2}}\mathrm{d}y\right)^{\frac{n-2}{n}}\\
&\quad\le C_s^2\left(\int_{B_{r_j}(x)}|u(t,y)|^{n(p-1)}\mathrm{d}y\right)^{\frac{2}{n}}\left(\int_{B_{r_j}(x)}\left|\nabla\left(|u(t,y)|\zeta_j(t,y)^{\frac{k}{\beta}}\right)^{\frac{\beta}{2}}\right|^2\mathrm{d}y\right).
\end{aligned}
\tag{3.28}
$$

Since

$$
\frac{1}{2}\left|\nabla(u\zeta_j^{\frac{k}{\beta}})^{\frac{\beta}{2}}\right|^2-\frac{k^2}{4}u^{\beta}\zeta_j{}^{k-2}|\nabla\zeta_j|^2\le\left|\nabla u^{\frac{\beta}{2}}\right|^2\zeta_j{}^k
$$

and using (3.28), integrating (3.27) over $t\in I_j$, we obtain that

$$
\begin{aligned}
&\sup_{t\in I_j}\int_{B_{r_j}(x)}|u(s,y)|^{\beta}\zeta_j(s,y)^k\mathrm{d}y+\frac{2\beta-k-2}{\beta}\int_{I_j}\int_{B_{r_j}(x)}\left|\nabla(|u(s,y)|\zeta_j(s,y)^{\frac{k}{\beta}})^{\frac{\beta}{2}}\right|^2\mathrm{d}y\mathrm{d}s\\
&\le\frac{pk|a|^2\beta}{2(p+\beta-1)}C_s^2\int_{I_j}\left\{\left(\int_{B_{r_j}(x)}|u(s,y)|^{n(p-1)}\mathrm{d}y\right)^{\frac{2}{n}}\int_{B_{r_j}(x)}\left|\nabla\left(|u(s,y)|\zeta_j(s,y)^{\frac{k}{\beta}}\right)^{\frac{\beta}{2}}\right|^2\mathrm{d}y\right\}\mathrm{d}s\\
&\quad+\frac{k}{2}\left(\frac{p\beta}{p+\beta-1}+\beta+\frac{k(2\beta-k-2)}{\beta}\right)\int_{I_j}\int_{B_{r_j}(x)}|u(s,y)|^{\beta}\zeta_j(s,y)^{k-2}|\nabla\zeta_j(s,y)|^2\mathrm{d}y\\
&\quad+k\int_{I_j}\int_{B_{r_j}(x)}|u(s,y)|^{\beta}\zeta_j^{k-1}(t,y)\partial_t\zeta_j(s,y)\mathrm{d}y.
\end{aligned}
\tag{3.29}
$$

Let $\gamma_3>0$ be taken as

$$
\frac{pk|a|^2\beta}{2(p+\beta-1)}C_s^2\gamma_3^{\frac{p-1}{2}}\le1-\frac{k+2}{n(p-1)}.
$$

Then, under the assumption (3.23), we estimate the first term of the right hand side of (3.29) and it cancels by the second term of the right hand side. Thus from (3.29) and using the estimate for the derivatives ζ_j in (3.26), that

$$\begin{aligned}&\sup_{t\in I_j}\int_{B_{r_j}(x)}u(s)^\beta\zeta_j(s)^k\mathrm{d}y+\int_{I_j}\int_{B_{r_j}(x)}\left|\nabla(u\zeta_j^{\frac{k}{\beta}})^{\frac{\beta}{2}}\right|^2\mathrm{d}y\mathrm{d}s\\&\le 2k\left[\left(\frac{p\beta}{p+\beta-1}+\beta+\frac{k(2\beta-k-2)}{\beta}\right)\frac{2^{2j}}{(R_2-R_1)^2}+\frac{2^{2j}}{t_1-t_2}\right]\\&\quad\int_{I_j}\int_{B_{r_j}(x)}|u(s,y)|^\beta\mathrm{d}y\mathrm{d}s\\&=C2^{2j}\left\{\frac{\beta}{(R_2-R_1)^2}+\frac{1}{t_1-t_2}\right\}\int_{I_j}\int_{B_{r_j}(x)}|u(s,y)|^\beta\mathrm{d}y\mathrm{d}s,\end{aligned}\tag{3.30}$$

for any $j=0,1,2,\ldots$ and $\beta>r$. Now applying the Gagliardo–Nirenberg inequality, Proposition 2.4, for any function $f\in C_0^1(B_{r_j}(x))$ and $\theta\in(0,1)$ with choosing $r=2+\frac{4}{n}=2(1+\frac{2}{n})$, $p=2, q=2$. we obtain for letting $\gamma=1+\frac{2}{n}$

$$\int_{B_{r_j}(x)}|f|^{2\gamma}\mathrm{d}y\le C^{2\gamma}\left(\int_{B_{r_j}(x)}|f|^2\mathrm{d}y\right)^{\frac{2}{n}}\int_{B_{r_j}(x)}|\nabla f|^2\mathrm{d}y.\tag{3.31}$$

Integrating (3.31) with respect to time $t\in I_j$, we have

$$\int_{I_j}\int_{B_{r_j}(x)}|u|^{\beta\gamma}\zeta_j^{k\gamma}\mathrm{d}y\mathrm{d}s\le C^{2\gamma}\left(\sup_{t\in I_j}\int_{B_{r_j}(x)}|u^\beta\zeta_j^k|\mathrm{d}y\right)^{\frac{2}{n}}\int_{I_j}\int_{B_{r_j}(x)}|\nabla(u\zeta_j^{\frac{k}{\beta}})^{\frac{\beta}{2}}|^2\mathrm{d}y\mathrm{d}s.$$

Hence, we obtain the reversed Hölder estimate:

$$\begin{aligned}&\left(\int\int_{Q_{j+1}}|u(t,y)|^{\beta\gamma}\mathrm{d}y\mathrm{d}s\right)^{\frac{1}{\gamma}}\\&\qquad\le\left(\int_{I_j}\int_{B_{r_j}(x)}|u(t,y)|^{\beta\gamma}\zeta_j(t,y)^{k\gamma}\mathrm{d}y\mathrm{d}s\right)^{\frac{1}{\gamma}}\\&\qquad\le C2^{2j}\left\{\frac{\beta}{(R_2-R_1)^2}+\frac{1}{t_1-t_2}\right\}\int_{I_j}\int_{B_{r_j}(x)}|u(t,y)|^\beta\mathrm{d}y\mathrm{d}s,\end{aligned}\tag{3.32}$$

where $Q_j=I_j\times B_{r_j}(x)=(\tau_j,t)\times B_{r_j}(x)$ and $\zeta_j=1$ on Q_{j+1}. Furthermore, by (3.30) with $\beta=2$ and $k=2$ we have (3.25). We use the estimate (3.32) iteratively with choosing $\beta=\beta_j=r\gamma^j$, where $\gamma=1+\frac{2}{n}$ and $j=1,2,\cdots$. Since it holds

$$\begin{aligned}&\left(\int\int_{Q_{j+1}}|u(t,y)|^{\beta_j\gamma}\mathrm{d}y\mathrm{d}s\right)^{\frac{1}{\gamma\beta_j}}\\&\qquad\le\left(C2^{2j}\right)^{\frac{1}{\beta_j}}\left[\frac{\beta_j}{(R_2-R_1)^2}+\frac{1}{t_1-t_2}\right]^{\frac{1}{\beta_j}}\left(\int\int_{Q_j}|u(t,y)|^{\beta_j}\mathrm{d}y\mathrm{d}s\right)^{\frac{1}{\beta_j}},\end{aligned}$$

we see that

$$\begin{aligned}M_{j+1}&\le\left(C2^{2j}\right)^{\frac{1}{\beta_j}}\left[\frac{r\gamma^j}{(R_2-R_1)^2}+\frac{1}{t_1-t_2}\right]^{\frac{1}{\beta_j}}M_j\\&=C^{\frac{j}{\beta_j}}(CD)^{\frac{1}{\beta_j}}M_j,\end{aligned}\tag{3.33}$$

where

$$D = C_1(R_2 - R_1)^{-2} + (t_1 - t_2)^{-1}.$$

The inequality (3.33) implies that

$$M_{j+1} \leq M_0 \prod_{k=0}^{j} C^{\frac{k}{\beta_k}} (CD)^{\frac{1}{\beta_k}}.$$

This follows that

$$\lim_{j\to\infty} M_j \leq C^{\sum_{j=0}^{\infty} \frac{j}{\beta_j}} (CD)^{\sum_{j=0}^{\infty} \frac{1}{\beta_j}} M_0. \tag{3.34}$$

Since $\gamma = 1 + \frac{2}{n}$,

$$\sum_{j=0}^{\infty} \frac{1}{\beta_j} \equiv \sum_{j=0}^{\infty} \frac{1}{r\gamma^j} = \frac{\gamma}{r(\gamma-1)} = \frac{n+2}{2r}$$

and $\sum_{j=0}^{\infty} \frac{j}{\beta_j} < \infty$. We obtain from (3.34)

$$\|u\|_{L^\infty(Q_\infty)} \leq CD^{\frac{n+2}{2r}} \|u\|_{L^r(Q_0)}.$$

Hence, we have that

$$\|u\|_{L^\infty\left((t_1,t)\times B_{R_1}(x)\right)} \leq CD^{\frac{n+2}{2r}} \left(\int_{t_2}^{t} \int_{B_{R_2}(x)} |u|^r \mathrm{d}y\mathrm{d}s \right)^{\frac{1}{r}}.$$

□

Proof of Theorem

Proof of Theorem 1.1 Let $u_0 \in L^{r,\nu}_\rho(\Omega)$. As $C_0^\infty(\Omega)$ is dense in $L^{r,\nu}_\rho(\Omega)$. Then, there exists a sequence $\{u_{k,0}\}$ in $C_0^\infty(\Omega)$ such that

$$u_{k,0} \longrightarrow u_0 \quad \text{in} \quad L^{r,\nu}_\rho(\Omega).$$

For each k, $u_{k,0}$ in $C_0^\infty(\Omega)$ as an initial data, we obtain a unique $L^r(\Omega)$-strong solution, $u_k(t) = u_k(t,x) \in C([0,T); L^r(\Omega))$ for the Cauchy problem (1.1) by [9]. As $L^r(\Omega) \subset L^{r,\nu}_\rho(\Omega)(\nu \geq r)$, it follows that for any $0 < T' < T$ such that $C\big([0,T'); L^r(\Omega)\big) \subset C\big([0,T'); L^{r,\nu}_\rho(\Omega)\big)$. Hence, we have that $u_k \in C\big([0,T'); L^{r,\nu}_\rho(\Omega)\big)$. Secondly $u_k \in L^2(0,T'; H^1_{0,\rho}(\Omega) \cap L^{r,\nu}_\rho(\Omega))$ by combining with Proposition 3.1 and Proposition 3.3. Then, the weak form of equation (1.1) is satisfied, and therefore, $u_k(t)$ is a $L^{r,\nu}$-weak solution to (1.1).

We then claim that $\{u_k(t)\}_k$ satisfies the assumption (3.1). Indeed, since $u_{k,0} \to u_0$ in $L^{r,\nu}_\rho(\Omega)$ as $k \to \infty$, we regard, by taking k_0 sufficiently large if necessary, that

$$\|u_{k,0}\|_{L^{r,\nu}_\rho} \le 2\|u_0\|_{L^{r,\nu}_\rho} \tag{4.1}$$

for all $k \ge k_0$. Let $u_k(t)$ be the corresponding strong solution in $L^r(\Omega)$ to $u_{k,0}$ and choose γ_0 such that

$$\gamma_0 < \min\left\{\frac{1}{2}, \frac{1}{2C}\right\}\gamma_4, \qquad \gamma_4 = \min\{\gamma_1, \gamma_2, \gamma_3\} \tag{4.2}$$

and γ_1, γ_2 and γ_3 are the constants appeared in Proposition 3.1, Proposition 3.2 and Proposition 3.3. By the assumption (1.4) on the data u_0;

$$\rho^{\frac{1}{p-1}-\frac{n}{r}}\|u_0\|_{L^{r,\nu}_\rho} \le \gamma_0$$

(4.1) and (4.2) , it follows that

$$\rho^{\frac{1}{p-1}-\frac{n}{r}}\|u_{k,0}\|_{L^{r,\nu}_\rho} \le 2\rho^{\frac{1}{p-1}-\frac{n}{r}}\|u_0\|_{L^{r,\nu}_\rho} \le \gamma_4$$

for all $k \ge k_0$. Since the strong solution $u_k \in C([0,T'); L^r(\Omega)) \subset C\big([0,T'); L^{r,\nu}_\rho(\Omega)\big)$, one can find a time $0 < \tilde{T}_k \le T'$ such that

$$\rho^{\frac{1}{p-1}-\frac{n}{r}} \sup_{0\le s\le \tilde{T}_k} \|u_k(s)\|_{L^{r,\nu}_\rho} \le \gamma_4.$$

According to Proposition 3.1 and (4.1), we see that

$$\sup_{0\le s\le \tilde{T}_k} \|u_k(s)\|_{L^{r,\nu}_\rho} \le C\|u_{k,0}\|_{L^{r,\nu}_\rho} \le 2C\|u_0\|_{L^{r,\nu}_\rho}$$

for all $k \ge k_0$.

Therefore, for each fixed solution $u_k(t)$, we obtain that

$$\rho^{\frac{1}{p-1}-\frac{n}{r}} \sup_{0\le s\le \tilde{T}_k} \|u_k(s)\|_{L^{r,\nu}_\rho(\Omega)} \le 2C\rho^{\frac{1}{p-1}-\frac{n}{r}}\|u_0\|_{L^{r,\nu}_\rho} \le \gamma_4$$

for all $k \ge k_0$.

Applying Proposition 3.2, for any m and $\ell \in \mathbb{N}$ with $m > \ell \ge 1$ it follows that

$$\sup_{0<s<\mu\rho^2} \|u_m(s) - u_\ell(s)\|_{L^{r,\nu}_\rho} \le C\|u_{m,0} - u_{\ell,0}\|_{L^{r,\nu}_\rho}. \tag{4.3}$$

This estimate (4.3) shows that $\{u_k(t)\}_{k=1}^\infty$ is a Cauchy sequence in $C\big([0,\mu\rho^2); L^{r,\nu}_\rho(\Omega)\big)$ since $\{u_{k,0}\}_k$ is the Cauchy sequence in $L^{r,\nu}_\rho(\Omega)$. Noticing the fact that $L^\infty\big([0,T'); L^{r,\nu}_\rho(\Omega)\big)$ is complete and $u_k \in C\big([0,\mu\rho^2); L^{r,\nu}_\rho(\Omega)\big)$, there exists a limit function

$$u \in BUC\big([0,\mu\rho^2); L^{r,\nu}_\rho(\Omega)\big)$$

such that

$$u_k \to u \in C([0,\mu\rho^2); L^{r,\nu}_\rho(\Omega)) \qquad \text{as } k \to \infty. \tag{4.4}$$

Besides u_k satisfies the equation in the weak sense, Proposition 3.3 yields that $\{u_k\}_k$ is uniformly bounded under the condition (3.23). Hence by taking subsequence if necessary, we see that

$$u \in BUC\big([0,\mu\rho^2); L^{r,\nu}_\rho(\Omega)\big) \cap L^\infty\big((0,\mu\rho^2)\times\Omega\big)$$

and

$$u_k \to u \text{ weak* in } L^\infty\big((0,\mu\rho^2)\times\Omega\big) \qquad \text{as } k\to\infty.$$

Since u_k is a $L^{r,\nu}$-weak solution, it satisfies equation (1.1) in the weak form. Namely, for each $\phi \in C_0^\infty((0,T)\times\Omega)$ with $T \le \mu\rho^2$,

$$\int_0^T\int_\Omega \big\{ -u_k\partial_t\phi + \nabla u_k\cdot\nabla\phi + a|u_k|^{p-1}u_k\cdot\nabla\phi\big\}\mathrm{d}x\mathrm{d}t = 0.$$

By (4.4) and using Proposition 2.1 finitely many times depending on the support of the test function ϕ,

$$\left|\int_0^T\int_\Omega u_k\partial_t\phi\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega u\partial_t\phi\mathrm{d}x\mathrm{d}t\right| \le C(\phi)\int_0^T \|u_k(t)-u(t)\|_{L^{r,\nu}_\rho}\|\partial_t\phi\|_{L^{r',\nu'}_\rho}dt \to 0,$$

and we obtain that

$$\int_0^T\int_\Omega u_k\partial_t\phi\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega u\partial_t\phi\mathrm{d}x\mathrm{d}t \tag{4.5}$$

as $k\to\infty$. Analogously using (3.25) we have

$$\begin{aligned}\int_0^T\int_\Omega \nabla u_k\cdot\nabla\phi\mathrm{d}x\mathrm{d}t &= -\int_0^T\int_\Omega u_k\Delta\phi\mathrm{d}x\mathrm{d}t\\ &\to -\int_0^T\int_\Omega u\Delta\phi\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \nabla u\cdot\nabla\phi\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.6}$$

Furthermore, by applying (3.15) and Proposition 3.3, we see that

$$\begin{aligned}&\left|\int_0^T\int_\Omega |u_k(t)|^{p-1}u_k(t)a\cdot\nabla\phi(t)\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega |u(t)|^{p-1}u(t)a\cdot\nabla\phi(t)\mathrm{d}x\mathrm{d}t\right|\\ &\le |a|\int_0^T\int_\Omega |u_k(t)-u(t)|\big(\max(|u_k(t)|,|u(t)|)\big)^{p-1}|\nabla\phi(t)|\mathrm{d}x\mathrm{d}t\\ &\le CT\max\big(\|u_k(t)\|_{L^\infty(K)},\|u(t)\|_{L^\infty(K)}\big)^{p-1}\sup_{0<t<T}\|\nabla\phi(t)\|_{L^{r',\nu'}_{\tilde\rho}}\sup_{0<t<T}\|u_k(t)-u(t)\|_{L^{r,\nu}_{\tilde\rho}}\\ &\le CT\max\big(\|u_k(t)\|_{L^\infty(K)},\|u(t)\|_{L^\infty(K)}\big)^{p-1}\sup_{0<t<T}\|u_k(t)-u(t)\|_{L^{r,\nu}_{\tilde\rho}},\end{aligned}$$

where $K = \mathrm{supp}\phi$ and $\tilde\rho > 0$ is taken such that $K\subset B_{\tilde\rho}(x)$ for some $x\in\Omega$. Hence

$$\int_0^T\int_\Omega a|u_k|^{p-1}u_k\cdot\nabla\phi\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega a|u|^{p-1}u\cdot\nabla\phi\mathrm{d}x\mathrm{d}t \tag{4.7}$$

as $k\to\infty$. Passing $k\to\infty$, we obtain from (4.5)-(4.7) that

$$\int_0^T \int_\Omega \{ - u\partial_t \phi + \nabla u \cdot \nabla \phi + a|u|^{p-1} u \cdot \nabla \phi \} \mathrm{d}x\mathrm{d}t = 0.$$

This proves the existence of an $L^{r,\nu}(\Omega)$-weak solution for $u_0 \in L^{r,\nu}_\rho(\Omega)$.

To see the uniqueness of weak solution, let u and v be two $L^{r,\nu}(\Omega)$-weak solutions of (1.1) with the same initial data $u_0 \in L^{r,\nu}_\rho(\Omega)$ satisfying the condition (1.4). Then, it holds in a similar observation that both u and v satisfy the condition (3.12). Then, Proposition 3.2 now implies $u = v$ in $C([0, T'); L^{r,\nu}_\rho(\Omega))$. Finally, the solution u is approximated from compact-supported smooth function u_k uniformly in t, and it belongs to the class $C([0, T'); L^{r,\nu}_\rho(\Omega))$.

This completes the proof of Theorem 1.1. □

Conclusion

In this paper, we consider existence and uniqueness problem for a convection–diffusion equation in amalgam spaces. We proved the local existence and uniqueness of solution for a convection–diffusion equation with initial condition in amalgam spaces. Moreover, we identified the Fujita–Weissler critical exponent for the local existence and uniqueness found by Escobedo and Zuazua [9] is also valid for the amalgam function class.

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Author contributions
I am the individual author of the manuscript. The author read and approved the final manuscript.

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The author declares that he has no competing interests.

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