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Exact solutions for MHD flow of couple stress fluid with heat transfer

Najeeb Alam Khan *, Hassam Khan, Syed Anwar Ali

Department of Mathematical Sciences, University of Karachi, Karachi 75270, Pakistan

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Abstract This paper aims at presenting exact solutions for MHD flow of couple stress fluid with heat transfer. The governing partial differential equations (PDEs) for an incompressible MHD flow of couple stress fluid are reduced to ordinary differential equations by employing wave parameter. The methodology is implemented for linearizing the flow equations without extra transformation and restrictive assumptions. Comparison is made with the result obtained previously.

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1. Introduction

Many non-linear science problems can appropriately and exactly be described by mathematical model of non-linear equations. Obtaining an exact solution of a non-linear PDEs [1–4] plays a dynamic role in the non-linear problems. If exact solution is obtainable, facilitate the confirmation of numerical solvers and stability theory. Numerous effective methods such as inverse scattering method, the tanh method, Exp-function method, the group analysis method have been extensively used to obtain exact solutions [5].

Since classical continuum theory enables to explain the rheological flow behavior of a Newtonian lubricant blended with various additives. A few of micro-continuum theories have been proposed [5–8]. The couple stress fluid model is one of the several models that anticipated to portray response characteristics of non-Newtonian fluids. They are a particular non-Newtonian fluid possessing "couple stress" effects. Couple stress fluid theory originated by Vijay Kumar Stokes in his treatise "Theories of Fluids with Microstructure" [9], is one among the polar fluid theories that takes into account couple stresses in addition to the classical Cauchy stress. In fact, the rotation vector is equal to the one-half the curl of the velocity vector as in the case in Newtonian fluids. Second order gradient of the velocity vector, rather than the kinematically independent rotation vector of asymmetric hydrodynamics is introduced into stress constitutive equations and consequently the theory yields only one vector equation to describe the velocity field. Moreover, microstructure of couple stress fluid is mechanically momentous. If the order of the magnitude of microstructure is equal as the characteristic geometric dimension of the problem considered, then the effect of microstructure on a liquid can be felt [10]. The spin field due to micro-rotation of these freely suspended particles in a vis-

* Corresponding author.

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cous medium result in an anti-symmetric stress, which is known as couple-stress, and thus forming couple-stress fluid. The study of couple stress fluid is very useful in understanding various physical problems because in the biomechanical area, this couple stress fluid model has been applied to study the mechanism of peristalsis [11,12]. One of the applications of couple-stress fluid is its use to the study of the mechanism of lubrication of synovial joints [13], which has grown to be the object of scientific research. The shoulder, knee, hip and ankle joints are the loaded-bearing synovial joints of human body. These joints Normal synovial fluid is clear or yellowish and is a viscous, non-Newtonian fluid. The couple stress fluids are proficient of describing different types of lubricants, suspension fluids, blood, etc. Colloidal fluids, liquid containing long chain molecules as polymeric suspension animal and human blood, polymer-thickened oils, lubricants containing small amount of polymer additive, electro-rheological fluids and synthetic fluids are examples of these fluids. Moreover, some of the non-Newtonian flow characteristics of blood can be explained by supposing the blood to be a fluid with couple stress. It is well recognized that at low shear stress rates during its flow through narrow vessels, being the suspension of cells, blood behave like a non-Newtonian fluid [14].

Ramanaiah discussed Squeeze films between finite plates lubricated by fluids with couple stresses [15]. Mokhiamar et al. [16] studied journal bearing lubricated by fluid with couple stresses. Zakaria [17] analyzed the unsteady free convection of couple stress fluid through a porous medium. Sreenadh et al. [18] examined the influence of MHD on the couple stress fluid in the porous medium. Khan and Riaz [19] studied the three dimensional flow of couple stress fluid over a rotating disk. Rani et al. [20] investigated the couple stress fluid over an infinite vertical cylinder. Effects of Hall and ion-slip on couple stress fluid between the parallel disks are presented by Srinivasacharya et al. [21]. El-Dabe and El-Mohandis [22] discussed the effect of couple stress on pulsatile hydromagnetic Poiseuille flow. Farooq et al. [23] examined the laminar flow of couple stress fluids for Vogel's model. Khan et al. [24] obtained the approximate solution of couple stress fluid with expanding or contracting porous channel.

The equation of motion of non-Newtanian [25,26] fluids is undoubtedly non-linear and become higher order so for couple stress fluids. Their exact solutions are extraordinary or nonexistent. In special cases solutions have been obtained by Islam et al. [27,28].

The basic aim of this paper was to find the exact solution of two dimensional MHD couple stress further, heat transfer analysis is also taken into account. Traveling wave phenomenon was implemented for obtaining the exact solution of MHD aligned flow of second grade fluid by Khan et al. [29] They recovered the polynomial solution for the Ref. [30]. Moreover, Khan et al. [31] presented the traveling wave solutions of micropolar fluid. They showed that the result obtained by Shahzad et al. [25] can be found from their investigation as a special case. We offered the solution methodology for obtaining the exact solution of couple stress fluid.

2. Formulation of the problem

The equations of motion of an incompressible MHD couple stress fluids with heat transfer are governed by the system

$$\nabla \cdot V = 0 \tag{1}$$

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V = -\frac{\nabla p}{\rho} - \frac{\mu}{\rho}(\nabla \times \nabla \times V) - \frac{\eta}{\rho}(\nabla \times \nabla \times \nabla \times \nabla \times V) - \frac{\Omega}{\rho}(\nabla \times H) \times H \tag{2}$$

$$\nabla \cdot H = 0 \tag{3}$$

$$\frac{\partial H}{\partial t} - \nabla \times (V \times H) + \frac{1}{\mu^* \sigma}\nabla \times (\nabla \times H) = 0 \tag{4}$$

$$\rho C_p\left(\frac{\partial T}{\partial t} + (V \cdot \nabla)T\right) = k\nabla^2 T + \phi \tag{5}$$

where V is the velocity, p is the pressure function, H is the magnetic field, ρ is the density, μ is the constant viscosity, μ^* is the magnetic permeability, σ is electrical conductivity, T is the temperature, k is the thermal conductivity, C_p is the specific heat, η is material constant for couple stress fluid, ϕ is the dissipation function, $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity and $\eta^* = \frac{\eta}{\rho}$ is the couple stress parameter. Here, we shall consider a MHD couple stress fluid with heat transfer in a plane, taking

$V = (u(x, y, t), v(x, y, t), 0)$, $H = (H_1(x, y, t), H_2(x, y, t), 0)$, $p = p(x, y, t)$, and $T = T(x, y, t)$ so that our flow Eqs. (1)–(5) take the form

$$u_x + v_y = 0 \tag{6}$$

$$u_t + uu_x + vu_y = -\frac{p_x}{\rho} + \nu(u_{xx} + u_{yy}) - \eta^*(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) - \frac{\Omega}{\rho}H_2(H_{2x} - H_{1y}) \tag{7}$$

$$v_t + uv_x + vv_y = -\frac{p_y}{\rho} + \nu(v_{xx} + v_{yy}) - \eta^*(v_{xxxx} + 2v_{xxyy} + v_{yyyy}) - \frac{\Omega}{\rho}H_1(H_{2x} - H_{1y}) \tag{8}$$

$$H_{2xt} - H_{1yt} = \frac{1}{\mu^* \sigma}[H_{2xxx} + H_{2xyy} - H_{1xyy} - H_{1yyy}] + vH_{1xx} + v_{xx}H_1 + vH_{1yy} + v_{yy}H_1 - uH_{2xx} - u_{xx}H_2 - uH_{2yy} - u_{yy}H_2 \tag{9}$$

$$H_{2x} + H_{1y} = 0 \tag{10}$$

$$\rho C_p(T_t + uT_x + vT_y) = k(T_{xx} + T_{yy}) + \mu\left[u_y^2 + 2u_y v_x + v_x^2 + 4u_x^2\right] + \eta\left[(u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2\right] \tag{11}$$

3. Methodology implementation

The method under consideration can be summarized as follows: for the present system of coupled PDEs

$$R_0(u_x, u_y) = 0 \tag{12}$$

$$R_1(u, v, p_x, u_t, u_x, u_y, u_{xx}, \ldots, v_t, v_x, v_y, v_{xx}, \ldots, H_1, H_{1y}, H_{2x}) = 0 \tag{13}$$

$$R_2(u, v, p_y, u_t, u_x, u_y, u_{xx}, \ldots, v_t, v_x, v_y, v_{xx}, \ldots, H_1, H_{1y}, H_{2x}) = 0 \tag{14}$$

$$R_3(u, v, u_{xy}, u_{xx}, \ldots, v_{xx}, v_{yy}, \ldots, H_1, H_2, H_{1x}, H_{2x} \ldots) = 0 \tag{15}$$

$$R_4(H_{1x}, H_{2y}) = 0 \tag{16}$$

We look for the following traveling wave solutions:

$$u = u(\xi), \quad v = v(\xi), \quad p = p(\xi), \quad H_1 = H_1(\xi), \quad H_2 = H_2(\xi) \tag{17}$$

where $\xi = mx + ny + Ct$, then system (12)–(16) are reduced to system of ordinary differential equations

$$R_0(u', v') = 0 \tag{18}$$

$$R_1\left(u, v, p', u', u'', u''', \ldots, v', v'', v''', \ldots, H_1, H_1', H_2'\right) = 0 \tag{19}$$

$$R_2\left(u, v, p', u', u'', u''', \ldots, v', v'', v''', \ldots, H_1, H_1', H_2'\right) = 0 \tag{20}$$

$$R_3\left(u, v, u'', v'', H_1, H_2, H_1', \ldots, H_2, H_2' \ldots\right) = 0 \tag{21}$$

$$R_4\left(H_1', H_2'\right) = 0 \tag{22}$$

where m, n and C are the constants and the prime denotes the derivation with respect to ξ. In order to obtain the solution of the system of ordinary differential equation, we eliminate pressure p from Eqs. (19) and (20) making the use of equation of the continuity (18). The obtained equation may integrate simply for variables u, v. Let us employ this methodology for an incompressible MHD couple stress fluid.

4. Solutions

On utilizing the system of Eqs. (6)–(10), we obtain

$$mu' + nv' = 0 \tag{23}$$

$$(C + mu + nv)u' = -\frac{np'}{\rho} + \nu(m^2 + n^2)u'' - \eta^*(m^2 + n^2)^2 u^{iv} \tag{24}$$

$$(C + mu + nv)v' = -\frac{np'}{\rho} + \nu(m^2 + n^2)v'' - \eta^*(m^2 + n^2)^2 v^{iv} \tag{25}$$

$$C\left(mH_2'' - nH_1''\right) = (m^2 + n^2)\left(\frac{1}{\mu^*\sigma}\left(mH_2''' - nH_1'''\right) + vH_1'' + v''H_1 + vH_1'' - uH_2'' - u''H_2\right) \tag{26}$$

$$mH_1' + nH_2' = 0 \tag{27}$$

$$\rho C_p(C + mu + nv)T' = kT''(m^2 + n^2) + \mu\left(n^2u'^2 + 4m^2u'^2 + 2m\,nu'v'\right) + \eta\left[(m^2 + n^2)^2\left(u''^2 + v''^2\right) + n^2v''^2\right] \tag{28}$$

On integrating Eqs. (23) and (27), we obtain

$$mu + nv = a_0 \tag{29}$$

$$mH_1 + nH_2 = a_1 \tag{30}$$

where a_0, a_1, are arbitrary constants of integration. Assuming that $a_1 = 0$. On eliminating pressure p, from Eqs. (24) and (25) invoking. Eqs. (29) and (30), we get

$$\eta^* u^{iv} - \frac{\nu}{(m^2 + n^2)}u'' + \frac{(C + a_0)}{(m^2 + n^2)^2}u' = 0 \tag{31}$$

If we take $\eta^* \to 0$ reduce to classical Newtonian flow problem. Here, two cases shall be discussed independently.

I. $C + a_0 = 0$.
II. $C + a_0 \neq 0$.

Case I

In this case Eq. (31) takes the form

$$u^{iv} - Au'' = 0 \tag{32}$$

where $A = \frac{\nu}{\eta^*(m^2+n^2)}$.

Which is on solving gives

$$u = a_2 + a_3\xi + a_4e^{\sqrt{A}\xi} + a_5e^{-\sqrt{A}\xi} \tag{33}$$

Using Eq. (33) in Eq. (29), we find

$$v = -\frac{m\left(a_2 + a_3\xi + a_4e^{\sqrt{A}\xi} + a_5e^{-\sqrt{A}\xi}\right)}{n} + a_0 \tag{34}$$

Substituting Eqs. (30)–(34) in Eq. (26) with a_1, we have

$$H_1''' = 0 \tag{35}$$

Which on integration gives

$$H_1 = a_6\xi^2 + a_7\xi + a_8 \tag{36}$$

Utilizing Eq. (35) in Eq. (30), we get

$$H_2 = -\frac{m}{n}(a_6\xi^2 + a_7\xi + a_8) \tag{37}$$

The heat equation for this case is

$$T'' + \chi u'^2 + \delta u''^2 = 0 \tag{38}$$

where $\chi = \frac{\mu(m^2+n^2)}{kn^2}$, $\delta = \frac{\eta(m^2+n^2)}{k}$.

Which provides the solution

$$T = E_0e^{2\sqrt{A}\xi} + E_1e^{-2\sqrt{A}\xi} + E_2a_4e^{\sqrt{A}\xi} + E_2a_5e^{-\sqrt{A}\xi} + (AE_3 + E_4)\xi^2 + a_{10}\xi + a_{11} \tag{39}$$

where

$$E_0 = -\frac{1}{4}a_4^2(\delta A + \chi), \quad E_1 = -\frac{1}{4}a_5^2(\delta A + \chi),$$

$$E_2 = -\frac{2a_3\chi}{\sqrt{A}}, \quad E_3 = -Aa_4a_5(A\delta - \chi), \quad E_4 = -\frac{1}{2}a_3^2\chi$$

And a_i, $i = 1, 2, \ldots, 11$ are constants of integration. Returning back to original variables, we get

$$u(x, y, t) = a_2 + a_3(mx + ny + Ct) + a_4e^{\sqrt{A}(mx+ny+Ct)} + a_5e^{-\sqrt{A}(mx+ny+Ct)} \tag{40}$$

$$v(x, y, t) = \frac{-m\left(a_2 + a_3(mx + ny + Ct) + a_4e^{\sqrt{A}(mx+ny+Ct)} + a_5e^{-\sqrt{A}(mx+ny+Ct)}\right)}{n} + a_0 \tag{41}$$

$$H_1(x,y,t) = a_6(mx+ny+Ct)^2 + a_7(mx+ny+Ct) + a_8 \quad (42)$$

$$H_2(x,y,t) = -\frac{m}{n}\left(a_6(mx+ny+Ct)^2 + a_7(mx+ny+Ct) + a_8\right) \quad (43)$$

$$T = E_0 e^{2\sqrt{A}(mx+ny+Ct)} + E_1 e^{-2\sqrt{A}(mx+ny+Ct)} + E_2 a_4 e^{\sqrt{A}(mx+ny+Ct)} + E_2 a_5 e^{-\sqrt{A}(mx+ny+Ct)} + (AE_3+E_4)(mx+ny+Ct)^2 + a_{10}(mx+ny+Ct) + a_{11} \quad (44)$$

Case II

For this case Eq. (31) takes the form

$$u^{iv} - Au'' + Bu' = 0 \quad (45)$$

where $A = \frac{\nu}{\eta^*(m^2+n^2)}$, $B = \frac{(C+a_0)}{\eta^*}$.

The auxiliary equation

$$m^3 - Am^2 + B = 0 \quad (46)$$

Eq. (45) admits the solution

$$u = a_{12} + a_{13}e^{\lambda_1\xi} + a_{14}e^{\lambda_2\xi} + a_{15}e^{\lambda_3\xi} \quad (47)$$

where λ_i, $i = 1, 2, 3$ are the roots of equation. Utilizing Eq. (47) in Eq. (30), we find

$$v = -\frac{m(a_{12} + a_{13}e^{\lambda_1\xi} + a_{14}e^{\lambda_2\xi} + a_{15}e^{\lambda_3\xi})}{n} + a_0 \quad (48)$$

Substituting Eqs. (47) and (48) and (30) in Eq. (26), we have

$$H_1''' + \Gamma H_1'' = 0 \quad (49)$$

where $\Gamma = \frac{\mu^*\sigma(C+a_0)}{(m^2+n^2)}$.

Which gives the solution

$$H_1 = a_{16}e^{\Gamma\xi} + a_{17}\xi + a_{18} \quad (50)$$

Utilizing Eq. (50) in Eq. (30), we get

$$H_2 = -\frac{m}{n}(a_{16}e^{\Gamma\xi} + a_{17}\xi + a_{18}) \quad (51)$$

Invoking Eqs. (47) and (29) in Eq. (28)

$$T'' - \varepsilon T' + \chi u'^2 + \delta u''^2 = 0 \quad (52)$$

where $\varepsilon = \frac{\rho C_p(C+a_0)}{m^2+n^2}$, $\chi = \frac{\mu(m^2+n^2)}{kn^2}$, $\delta = \frac{\eta(m^2+n^2)}{k}$.

Which admits the solution

$$T = b_0e^{2\lambda_1\xi} + b_1e^{2\lambda_2\xi} + b_2e^{2\lambda_3\xi} + b_3e^{(\lambda_1+\lambda_2)\xi} + b_4e^{(\lambda_2+\lambda_3)\xi} + b_5e^{(\lambda_1+\lambda_3)\xi} + b_6e^{\varepsilon\xi} + a_{20} \quad (53)$$

where a_j, $j = 1,2,\ldots,20$, are the constants of the integration.

$$b_0 = -\frac{a_{13}^2\lambda_1(\delta\lambda_1^2+\chi)}{2(-\varepsilon+2\lambda_1)}, \quad b_1 = -\frac{a_{14}^2\lambda_2(\delta\lambda_2^2+\chi)}{2(-\varepsilon+2\lambda_2)},$$
$$b_2 = -\frac{a_{15}^2\lambda_3(\delta\lambda_3^2+\chi)}{2(-\varepsilon+2\lambda_3)}, \quad b_3 = -\frac{2a_{13}a_{14}\lambda_1\lambda_2(\delta\lambda_1\lambda_2+\chi)}{(\lambda_1+\lambda_2)(-\varepsilon+\lambda_1+\lambda_2)},$$
$$b_4 = -\frac{2a_{14}a_{15}\lambda_2\lambda_3(\delta\lambda_2\lambda_3+\chi)}{(\lambda_2+\lambda_3)(-\varepsilon+\lambda_2+\lambda_3)},$$
$$b_5 = -\frac{2a_{13}a_{15}\lambda_1\lambda_3(\delta\lambda_1\lambda_3+\chi)}{(\lambda_1+\lambda_3)(-\varepsilon+\lambda_1+\lambda_3)}, \quad b_6 = \frac{a_{19}}{\varepsilon}$$

The velocity components, magnetic field and temperature in the original variables are

$$u(x,y,t) = a_{12} + a_{13}e^{\lambda_1(mx+ny+Ct)} + a_{14}e^{\lambda_2(mx+ny+Ct)} + a_{15}e^{\lambda_3(mx+ny+Ct)} \quad (54)$$

$$v(x,y,t) = -\frac{m\left(a_{12} + a_{13}e^{\lambda_1(mx+ny+Ct)} + a_{14}e^{\lambda_2(mx+ny+Ct)} + a_{15}e^{\lambda_3(mx+ny+Ct)}\right)}{n} + a_0 \quad (55)$$

$$H_1(x,y,t) = a_{16}e^{a(mx+ny+Ct)} + a_{17}(mx+ny+Ct) + a_{18} \quad (56)$$

$$H_2(x,y,t) = -\frac{m}{n}\left(a_{16}e^{a(mx+ny+Ct)} + a_{17}(mx+ny+Ct) + a_{18}\right) \quad (57)$$

$$T = b_0e^{2\lambda_1(mx+ny+Ct)} + b_1e^{2\lambda_2(mx+ny+Ct)} + b_2e^{2\lambda_3(mx+ny+Ct)} + b_3e^{(\lambda_1+\lambda_2)(mx+ny+Ct)} + b_4e^{(\lambda_2+\lambda_3)(mx+ny+Ct)} + b_5e^{(\lambda_1+\lambda_3)(mx+ny+Ct)} + b_6e^{\varepsilon(mx+ny+Ct)} + a_{20} \quad (58)$$

5. Conclusions

An attempt to obtain exact solutions of MHD flow of couple stress fluid with heat transfer has been made. The methodology in the present work is easy to linearize the equation of MHD flow for couple stress fluid. It should be noted that when $\eta \to 0$ and $C + a_0 < 0$, there is no couple stresses, exponential type of solutions has been recovered for viscous and non-Newtonian fluids [32–34]. The method has been used in a direct way without any restrictive assumptions and labourious calculation. For future research, we will solve the three dimensional Newtonian and couple stress flow equations by using proposed method. Moreover, this methodology can be employed on other non-Newtonian fluids to obtain an exact solutions, for instance pseudoplastic and rate type fluids.

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