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ORIGINAL ARTICLE

Slip effects on a generalized Burgers' fluid flow between two side walls with fractional derivative

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Abstract This paper presents a research for the 3D flow of a generalized Burgers' fluid between two side walls generated by an exponential accelerating plate and a constant pressure gradient, where the no-slip assumption between the exponential accelerating plate and the Burgers' fluid is no longer valid. The governing equations of the generalized Burgers' fluid flow are established by using the fractional calculus approach. Exact analytic solutions for the 3D flow are established by employing the Laplace transform and the finite Fourier sine transform. Furthermore, some 3D and 2D figures for the fluid velocity and shear stress are plotted to analyze and discuss the effects of various parameters.

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1. Introduction

Studies of non-Newtonian fluids have obtained more and more attention in the last few years, the main reason may be that non-Newtonian fluids (such as humans blood, polymer suspension, slurries, and oil) are widely exist in life, production and nature. It is well known that the fluid which satisfies the law of Newton inner friction that the stress tensor is proportional to fluid velocity gradient, is called Newtonian fluid. However non-Newtonian fluids do not obey the law of Newton inner friction and show more complex rheological behaviors than Newtonian fluids. Among of all non-Newtonian fluids, viscoelastic fluid which has both viscous and elastic characteristics, is an important class. Various viscoelastic fluid models have been put forward to research the flow behaviors of the fluid. Among these models, the Maxwell fluid model [1,2], the second grade fluid model [3,4], the Oldroyd-B fluid model [5,6] and the Burgers' fluid model [7–9] are four typical viscoelastic fluid models. The constitutive equations of viscoelastic fluid with fractional derivative have been successfully used to describe viscoelastic characteristics, which are derived by replacing

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the integer order time derivative of the classical constitutive equations with the fractional derivative [10–13]. Last few years, many attempts about the generalized Burgers' fluid flow with fractional derivative have been done. Some flows of generalized Burgers' fluid with fractional derivative in the porous media space were considered by Xue [14,15], Khan and Hayat [16] and Hayat et al. [17]. Khan [18–20] obtained the exact analytical solutions for the accelerated, rotating and oscillating flows of a fractional generalized Burgers' fluid. Hyder Ali Muttaqi Shah [21] discussed the Poiseuille and Couette flows of a fractional Burgers' fluid between two parallel plates. Liu et al. [22] investigated the radiation effects on the heat transfer of a fractional generalized Burgers' fluid and obtained the exact analytical solutions for velocity and temperature fields.

In the past few years, unsteady flow problems of viscoelastic fluids between two side walls have obtained considerable attention. The exact analytical solutions for the Maxwell fluid flow between two side walls due to a constant velocity plate was investigated by Hayat et al. [23]. Fetecau et al. [24] and Khan and Wang [25] investigated the flows of a generalized second-grade fluid induced by a accelerated plate between two side walls. Some exact analytical solutions for generalized Oldroyd-B fluid flows between two side walls due to a accelerated plate were established by Fetecau [26,27] and Hyder Ali Muttaqi Shah [28]. Zheng et al. [29] for the first time presented the 3D figures for the flow of a generalized Oldroyd-B fluid with fractional derivative generated by a constant pressure gradient. Moreover, boundary slip is found to play an important role in various engineering technological applications. Zheng et al. [30] firstly studied slip effects on magnetohydrodynamic flow of a generalized Oldroyd-B fluid and obtained the exact analytical solutions.

Based on the above mentioned works and in order to better describe the flow of generalized Burgers' fluid, we investigate slip effects on the 3D flow between two side walls generated by an exponential accelerating plate and a constant pressure gradient. The governing equations of the generalized Burgers' fluid are established by making use of the fractional calculus approach. The exact analytical solutions for the 3D flow are calculated by employing the Laplace transform and the finite Fourier sine transform. Furthermore, some figures are plotted to analyze and discuss the effects of various parameters.

2. Basic governing equations

The constitutive relationship for an incompressible generalized Burgers' fluid with fractional derivative is [16–22]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad (1 + \lambda_1^\alpha \tilde{D}_t^\alpha + \lambda_2^\alpha \tilde{D}_t^{2\alpha})\mathbf{S} = \mu\lambda_3^\beta \tilde{D}_t^\beta \mathbf{A} \tag{1}$$

In the above equations, $-p\mathbf{I}$ indicates the indeterminate spherical stress, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{S}$ the extra stress tensor, $\mathbf{A}$ represents the first Rivlin–Ericksen tensor, μ indicates dynamic viscosity, λ_1 the relaxation time, λ_2 is the new material parameter, $\lambda_3 (< \lambda_1)$ the retardation time, α and β $(0 \leqslant \alpha \leqslant \beta \leqslant 1)$ are the fractional calculus parameters, $\tilde{D}_t^p$ denotes the upper convected time fractional derivative and is defined by [16–22]

$$\tilde{D}_t^p \mathbf{S} = D_t^p \mathbf{S} + (\mathbf{V} \cdot \nabla)\mathbf{S} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}} \tag{2}$$

$$\tilde{D}_t^{2p} \mathbf{S} = \tilde{D}_t^p (\tilde{D}_t^p \mathbf{S}) \tag{3}$$

where $\mathbf{V}$ denotes the fluid velocity, $\mathbf{L}$ denotes the velocity gradient, D_t^p denotes the fractional calculus operator of order p with respect to t and is defined by [31]

$$D_t^p f(t) = \frac{1}{\Gamma(1-p)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^p} d\tau, \quad 0 \leqslant p \leqslant 1, \tag{4}$$

where $\Gamma(\cdot)$ is the Gamma function.

The motion equation in the absence of body force is the following form [18]

$$\rho \frac{d\mathbf{V}}{dt} = \nabla \cdot \mathbf{T} \tag{5}$$

where ρ denotes the fluid density, ∇ is the gradient operator.

We assume the motion of the fluid is unidirectional, the velocity field of the fluid may be

$$\mathbf{V} = [u(y,z,t), 0, 0] \tag{6}$$

where $u(y,z,t)$ denotes the fluid velocity in the x-axis direction. Substituting Eq. (6) into (1), (5) and considering initial condition $\mathbf{S}(y,z,0) = 0$, we obtain the following equations

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{xy}}{\partial y} + \frac{\partial S_{xz}}{\partial z} \tag{7}$$

$$(1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha}) S_{xy} = \mu(1 + \lambda_3^\beta D_t^\beta) \frac{\partial u}{\partial y} \tag{8}$$

$$(1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha}) S_{xz} = \mu(1 + \lambda_3^\beta D_t^\beta) \frac{\partial u}{\partial z} \tag{9}$$

where $\partial p / \partial x$ is the pressure gradient along x-axis. Eliminating S_{xy} and S_{xz} between Eqs. (7)–(9), we get the governing equation of the flow problem

$$\left(1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha}\right) \frac{\partial u(y,z,t)}{\partial t} = -\frac{1}{\rho}\left(1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha}\right) \frac{\partial p}{\partial x} + v(1 + \lambda_3^\beta D_t^\beta)\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(y,z,t) \tag{10}$$

where $v = \mu/\rho$ denotes kinematic viscosity.

3. Flow between two side walls

In the following sections, we investigate an incompressible generalized Burgers' fluid with fractional derivative between two side walls occupying the full space above the plate which is perpendicular to the side walls. The Burgers' fluid begins to move due to the exponential accelerating plate with a motion of the speed e^{-t} and a constant pressure gradient $A = -1/\rho \cdot \partial p / \partial x$ in the x-axis direction. The slip between the exponential accelerating plate and the Burgers' fluid is considered. The governing equation of the fluid is Eq. (10) and the corresponding initial and boundary conditions are

$$u(y,z,0) = u_t(y,z,0) = u_{tt}(y,z,0) = 0, \quad (y > 0, 0 \leqslant z \leqslant h) \tag{11}$$

$$u(0,z,t) = e^{-t} + \theta \frac{\partial u(0,z,t)}{\partial y}, \quad (t \geqslant 0, 0 \leqslant z \leqslant h) \tag{12}$$

$$u(y,z,t) = u_y(y,z,t) = 0, \quad (y, t \geqslant 0; z = 0, h) \tag{13}$$

$$u_y(y,z,t) \to 0, \quad (y \to \infty, t \geqslant 0; 0 < z < h) \tag{14}$$

where θ is the slip coefficient, h denotes the distance between the side walls, and Eq. (14) is the natural boundary condition of the flow problem.

3.1. Solution of the velocity field

In this section, we will use the finite Fourier sine transform [21,29] and the Laplace transform with sequential fractional derivative [31] to obtain the solution of the velocity field. Now, multiplying both sides of Eq. (10) by $\sin\left(\frac{n\pi z}{h}\right)$, and integrating the result with respect to z from 0 to h [23–29], we can get the following equation

$$\left(1+\lambda_1^\alpha D_t^\alpha+\lambda_2^\alpha D_t^{2\alpha}\right)\frac{\partial u_n}{\partial t}=A\frac{h}{n\pi}(1-(-1)^n)\left(1+\frac{\lambda_1^\alpha t^{-\alpha}}{\Gamma(1-\alpha)}+\frac{\lambda_2^\alpha t^{-2\alpha}}{\Gamma(1-2\alpha)}\right)+v(1+\lambda_3^\beta D_t^\beta)\frac{\partial^2 u_n}{\partial y^2}-v\left(\frac{n\pi}{h}\right)^2(1+\lambda_3^\beta D_t^\beta)u_n \quad (15)$$

Applying the Laplace transform to the above equation, we can get the following differential equation of $u_{sn}(y,n,s)$

$$\frac{\partial^2 u_{sn}}{\partial y^2}-\left[\xi^2+\frac{s(1+\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha})}{v(1+\lambda_3^\beta s^\beta)}\right]u_{sn}+\frac{A}{v\xi}(1-(-1)^n)\frac{(1+\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha})}{s(1+\lambda_3^\beta s^\beta)}=0 \quad (16)$$

where $\xi=\frac{n\pi}{h}$, the Laplace transform formula [31] is

$$u_s(y,z,s)=\int_0^\infty u(y,z,t)e^{-st}dt, s>0. \quad (17)$$

By using the knowledge of ordinary differential equations, we easily obtain the solution of Eq. (16)

$$u_{sn}=\frac{\left[\frac{(1-(-1)^n)}{\xi(s+1)}-\frac{A(1-(-1)^n)}{\xi s^2(v\xi^2B+1)}\right]}{1+\theta(\xi^2+1/vB)^{1/2}}\exp\left[-(\xi^2+1/vB)^{1/2}y\right]+\frac{A(1-(-1)^n)}{\xi s^2(v\xi^2B+1)} \quad (18)$$

where $B=\frac{(1+\lambda_3^\beta s^\beta)}{s(1+\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha})}$.

For the above Eq. (18), it is very hard to obtain the analytical solution by calculating the residues and contour integrals of Eq. (18). Therefore we will use the discrete inverse Laplace transform method [16–30]. We rewrite Eq. (18) as

$$\begin{aligned}u_{sn}=&(1-(-1)^n)\cdot\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{i+r+k+m+j+l+p}y^k}{\theta^{r+1}k!m!j!l!p!}\\&\times\frac{\lambda_1^{p\alpha}\lambda_3^{(m-j-c)\beta}}{\lambda_2^{(l+m+p-c)\alpha}}\cdot\frac{\Gamma(j-m+c)\Gamma(l+m+p-c)}{\Gamma(-c)\Gamma(c-m)}\\&\times\frac{v^{m-c}\xi^{2m-1}}{s^{(i+1+m-c)+(j-m+c)\beta+(p+2l+2m-k+r+1)\alpha}}\\&-A(1-(-1)^n)\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{i+r+k+m+j+l+p}y^k}{\theta^{r+1}k!m!j!l!p!}\\&\cdot\frac{\lambda_1^{p\alpha}\lambda_3^{(i+m-j-c)\beta}}{\lambda_2^{(i+l+m+p-c)\alpha}}\frac{\Gamma(m-c)\Gamma(j-i-m+c)\Gamma(l+i+m+p-c)}{\Gamma(-c)\Gamma(c-i-m)\Gamma(i+m-c)}\\&\cdot\frac{v^{i+m-c}\xi^{2i+2m-1}}{s^{(i+2+m-c)+(j-i-m+c)\beta+(p+2l+2i+2m-k+r+1)\alpha}}\\&+A(1-(-1)^n)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{i+j+l+p}}{j!l!p!}\frac{\lambda_1^{p\alpha}\lambda_3^{(i-j)\beta}}{\lambda_2^{(l+i+p)\alpha}}v^i\xi^{2i-1}\\&\cdot\frac{\Gamma(j-i)\Gamma(l+i+p)}{\Gamma(-i)\Gamma(i)}\frac{1}{s^{(i+2)+(j-i)\beta+(p+2i+2l)\alpha}}\end{aligned} \quad (19)$$

where $c=(k-r-1)/2$

By using the inverse Laplace transform [16–30] to Eq. (19), we can get the following equation

$$\begin{aligned}u_n=&(1-(-1)^n)\cdot\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{i+r+k+m+j+l+p}y^k}{\theta^{r+1}k!m!j!l!p!}\frac{\lambda_1^{p\alpha}\lambda_3^{(m-j-c)\beta}}{\lambda_2^{(l+m+p-c)\alpha}}\cdot\\&\frac{\Gamma(j-m+c)\Gamma(l+m+p-c)}{\Gamma(-c)\Gamma(c-m)}\frac{v^{m-c}\xi^{2m-1}t^{(i+m-c)+(j-m+c)\beta+(p+2l+2m-2c)\alpha}}{\Gamma((i+1+m-c)+(j-m+c)\beta+(p+2l+2m-2c)\alpha)}\\&-A(1-(-1)^n)\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{i+r+k+m+j+l+p}y^k}{\theta^{r+1}k!m!j!l!p!}\cdot\\&\frac{\lambda_1^{p\alpha}\lambda_3^{(i+m-j-c)\beta}}{\lambda_2^{(i+l+m+p-c)\alpha}}\frac{\Gamma(m-c)\Gamma(j-i-m+c)\Gamma(l+i+m+p-c)}{\Gamma(-c)\Gamma(c-i-m)\Gamma(i+m-c)}\\&v^{i+m-c}\xi^{2i+2m-1}\cdot\frac{t^{(i+1+m-c)+(j-i-m+c)\beta+(p+2l+2i+2m-2c)\alpha}}{\Gamma((i+2+m-c)+(j-i-m+c)\beta+(p+2l+2i+2m-2c)\alpha)}\\&+A(1-(-1)^n)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{i+j+l+p}\Gamma(j-i)\Gamma(l+i+p)}{j!l!p!\Gamma(-i)\Gamma(i)}\cdot\\&v^i\xi^{2i-1}\frac{\lambda_1^{p\alpha}\lambda_3^{(i-j)\beta}}{\lambda_2^{(l+i+p)\alpha}}\frac{t^{(i+1)+(j-i)\beta+(p+2i+2l)\alpha}}{\Gamma((i+2)+(j-i)\beta+(p+2i+2l)\alpha)}\end{aligned} \quad (20)$$

Finally, by employing the inverse finite Fourier sine transform to Eq. (20), we can obtain the exact analytical solution

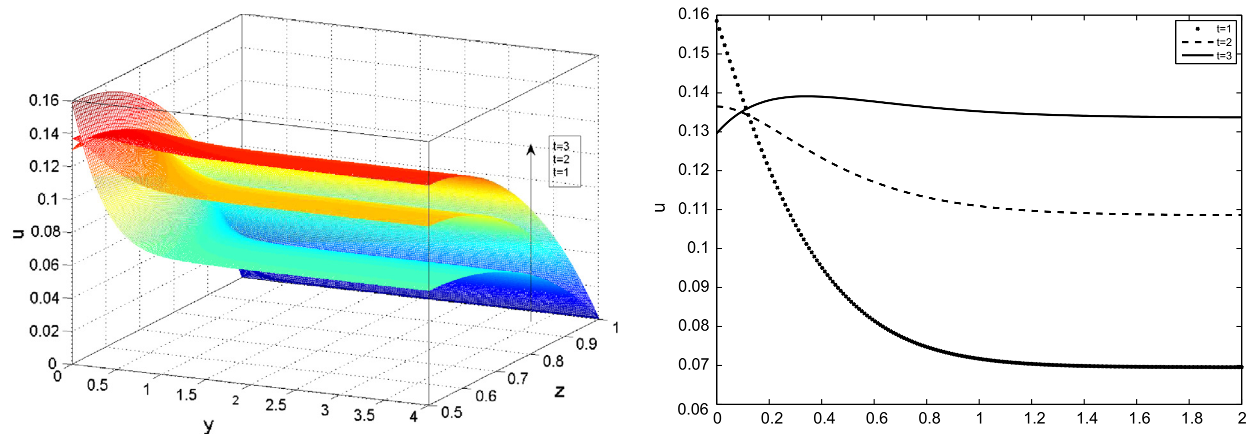
$$\begin{aligned}u=&\frac{2}{h}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{h}\right)u_n=\frac{2}{h}\sum_{n=1}^{\infty}(1-(-1)^n)\sin\left(\frac{n\pi z}{h}\right)\cdot\\&\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{i+r+k+m+j+l+p}y^k}{\theta^{r+1}k!m!j!l!p!}\frac{\lambda_1^{p\alpha}\lambda_3^{(m-j-c)\beta}v^{m-c}\xi^{2m-1}}{\lambda_2^{(l+m+p-c)\alpha}}\\&\cdot\frac{\Gamma(j-m+c)\Gamma(l+m+p-c)}{\Gamma(-c)\Gamma(c-m)}\\&\frac{t^{(i+m-c)+(j-m+c)\beta+(p+2l+2m-2c)\alpha}}{\Gamma((i+1+m-c)+(j-m+c)\beta+(p+2l+2m-2c)\alpha)}\\&-A(1-(-1)^n)\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{i+r+k+m+j+l+p}y^k}{\theta^{r+1}k!m!j!l!p!}\cdot\\&\frac{\lambda_1^{p\alpha}\lambda_3^{(i+m-j-c)\beta}v^{i+m-c}\xi^{2i+2m-1}}{\lambda_2^{(i+l+m+p-c)\alpha}}\frac{\Gamma(m-c)\Gamma(j-i-m+c)\Gamma(l+i+m+p-c)}{\Gamma(-c)\Gamma(c-i-m)\Gamma(i+m-c)}\cdot\\&\frac{t^{(i+1+m-c)+(j-i-m+c)\beta+(p+2l+2i+2m-2c)\alpha}}{\Gamma((i+2+m-c)+(j-i-m+c)\beta+(p+2l+2i+2m-2c)\alpha)}\\&+A(1-(-1)^n)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{i+j+l+p}\Gamma(j-i)\Gamma(l+i+p)}{j!l!p!\Gamma(-i)\Gamma(i)}\\&\cdot\frac{\lambda_1^{p\alpha}\lambda_3^{(i-j)\beta}v^i\xi^{2i-1}}{\lambda_2^{(l+i+p)\alpha}}\frac{t^{(i+1)+(j-i)\beta+(p+2i+2l)\alpha}}{\Gamma((i+2)+(j-i)\beta+(p+2i+2l)\alpha)}\end{aligned} \quad (21)$$

By using the property of the Fox H-function [32] which is

$$\sum_{n=0}^{\infty}\frac{(-z)^n\prod_{j=1}^{p}\Gamma(a_j+A_jn)}{n!\prod_{j=1}^{q}\Gamma(b_j+B_jn)}=H_{p,q+1}^{1,p}\left[z\left|{}_{(0,1),(1-b_1,B_1),\cdots,(1-b_q,B_q)}^{(1-a_1,A_1),\cdots,(1-a_p,A_p)}\right.\right] \quad (22)$$

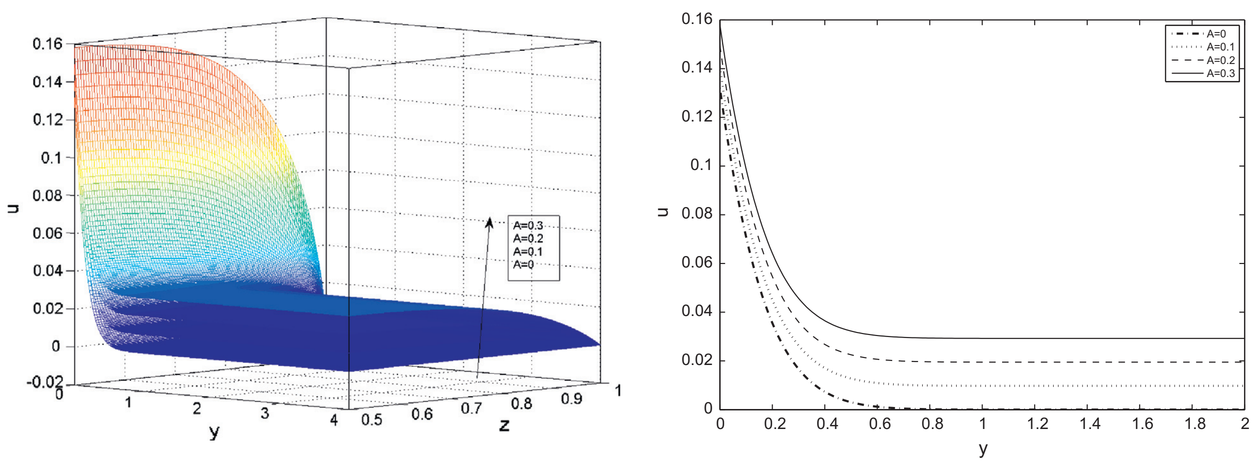
we can get a simpler form for Eq. (21)

$$\begin{aligned}u=&\frac{2}{h}\sum_{n=1}^{\infty}(1-(-1)^n)\sin\left(\frac{n\pi z}{h}\right)\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}v^{m-c}\xi^{2m-1}\cdot\\&\frac{(-1)^{i+r+k+m+j+l}y^k}{\theta^{r+1}k!m!j!l!}\frac{\lambda_3^{(m-j-c)\beta}}{\lambda_2^{(l+m-c)\alpha}}t^{a+(j-m+c)\beta+(2l+2m-2c)\alpha}\\&\times H_{2,4}^{1,2}\left[\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\left|{}_{(0,1),(c,0),(m+c-1,0),(-a-(j-m+c)\beta-(2l+2m-2c)\alpha,\alpha)}^{(1-j+m-c,0),(1-l-m+c,1)}\right.\right]\\&-\frac{2A}{h}\sum_{n=1}^{\infty}(1-(-1)^n)\sin\left(\frac{n\pi z}{h}\right)\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{\lambda_3^{(a-j)\beta}}{\lambda_2^{(a+l)\alpha}}\cdot\\&\frac{(-1)^{i+r+k+m+j+l}y^k}{\theta^{r+1}k!m!j!l!}v^a\xi^{2i+2m-1}t^{1+a+(j-a)\beta+(2l+2a)\alpha}\\&\times H_{3,5}^{1,3}\left[\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\left|{}_{(0,1),(c+1,0),(1+a,0),(1-a,0),(-1-a-(j-a)\beta-(2l+2a)\alpha,\alpha)}^{(1-m+c,0),(1-j+a,0),(1-l-a,1)}\right.\right]\\&+\frac{2A}{h}\sum_{n=1}^{\infty}(1-(-1)^n)\sin\left(\frac{n\pi z}{h}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+l}}{j!l!}\cdot\frac{\lambda_3^{(i-j)\beta}}{\lambda_2^{(l+i)\alpha}}\\&v^i\xi^{2i-1}t^{i+1+(j-i)\beta+(2i+2l)\alpha}\times H_{2,4}^{1,2}\left[\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\left|{}_{(0,1),(1+i,0),(1-i,0),(-i-1+(i-j)\beta+(2i+2l)\alpha,\alpha)}^{(1-j+i,0),(1-l-i,1)}\right.\right]\end{aligned} \quad (23)$$



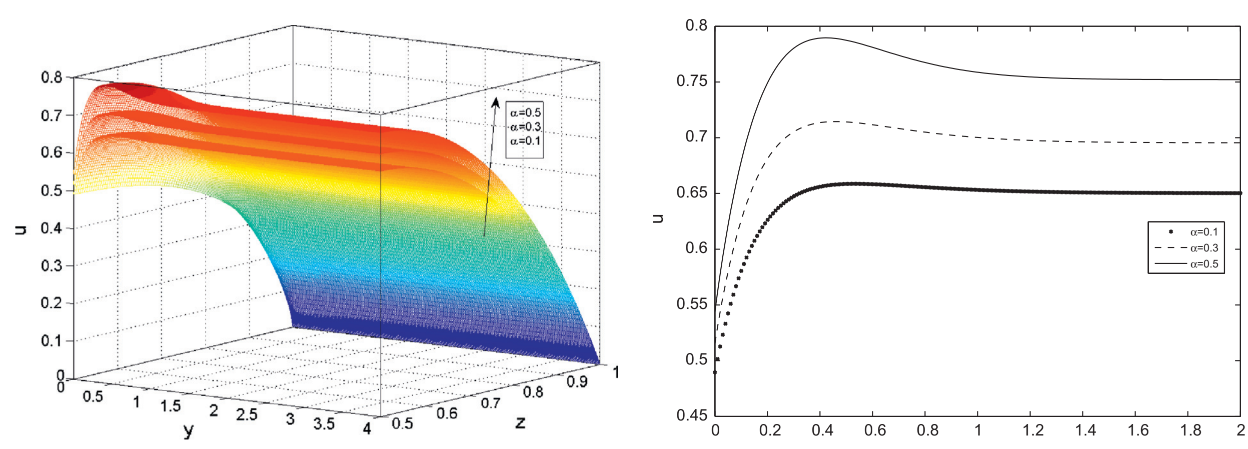
(a) 3D surface (b) 2D figure when $z = 0.5$

Figure 1 Profiles of velocity u for different values t when $A = 0.1,\ \alpha = 0.3,\ \beta = 0.8,\ \theta = 1,\ \lambda_2 = 1,\ \lambda_1 = 2,\ \lambda_3 = 3,\ v = 0.1.$



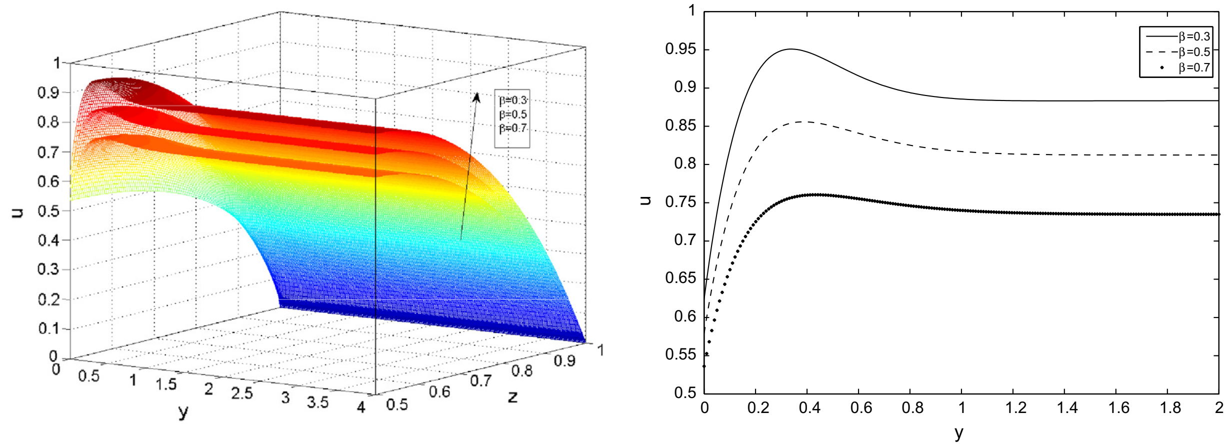
(a) 3D surface (b) 2D figure when $z = 0.5$

Figure 2 Profiles of velocity u for different values A when $t = 0.1,\ \alpha = 0.3,\ \beta = 0.8,\ \theta = 1,\ \lambda_2 = 1,\ \lambda_1 = 2,\ \lambda_3 = 3,\ v = 0.1.$



(a) 3D surface (b) 2D figure when $z = 0.5$

Figure 3 Profiles of velocity u for different values α when $t = 1,\ A = 1,\ \beta = 0.8,\ \theta = 0.1,\ \lambda_2 = 1,\ \lambda_1 = 2,\ \lambda_3 = 3,\ v = 0.1.$



(a) 3D surface (b) 2D figure when $z = 0.5$

Figure 4 Profiles of velocity u for different values β when $t = 1,\ A = 1,\ \alpha = 0.3,\ \theta = 0.1,\ \lambda_2 = 1,\ \lambda_1 = 2,\ \lambda_3 = 3,\ v = 0.1$.

where $a = i + m - c$

3.2. Solutions of the shear stresses

By employing the Laplace transform to Eqs. (8) and (9), we can get the following equations

$$\bar{\tau}_1 = \frac{\mu(1+\lambda_3^\beta s^\beta)}{(1+\lambda_1^\alpha s^\alpha + \lambda_2^\alpha s^{2\alpha})} \cdot \frac{\partial u_s(y,z,s)}{\partial y} \tag{24}$$

$$\bar{\tau}_2 = \frac{\mu(1+\lambda_3^\beta s^\beta)}{(1+\lambda_1^\alpha s^\alpha + \lambda_2^\alpha s^{2\alpha})} \cdot \frac{\partial u_s(y,z,s)}{\partial z} \tag{25}$$

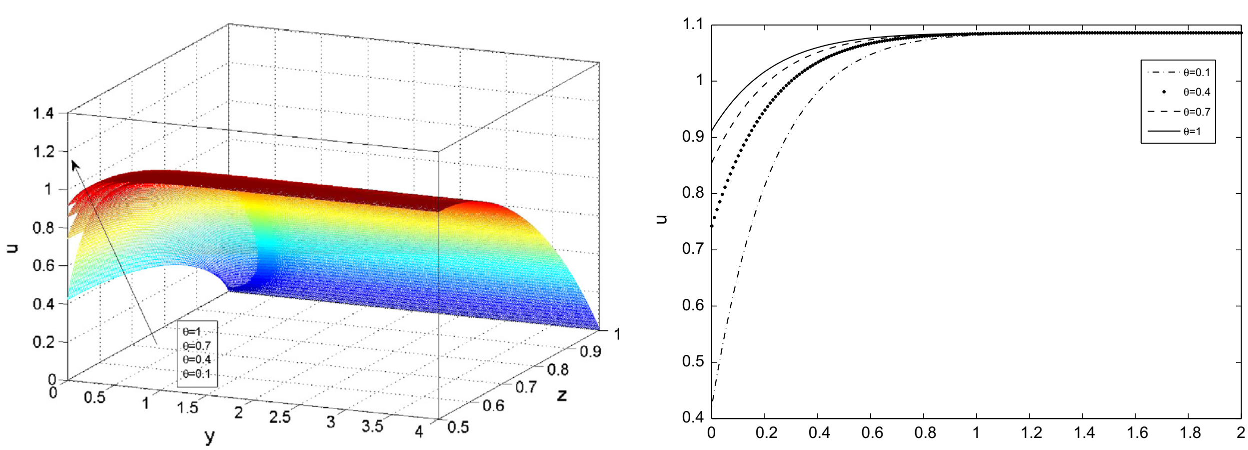
In the above equations, $u_s(y,z,s)$ can be easily obtained by using the inverse finite Fourier sine transform to Eq. (18). Substituting $u_s(y,z,s)$ into (24), we get

$$\bar{\tau}_1 = -\frac{2\mu s B}{h}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{h}\right)\frac{\left[\frac{(1-(-1)^n)}{\xi(s+1)} - \frac{A(1-(-1)^n)}{\xi s^2(v\xi^2 B+1)}\right]}{1+\theta(\xi^2+1/vB)^{1/2}} \cdot (\xi^2+1/vB)^{1/2}\exp\left[-(\xi^2+1/vB)^{1/2}y\right] \tag{26}$$

For the ease of calculation, Eq. (26) can be written as the following form

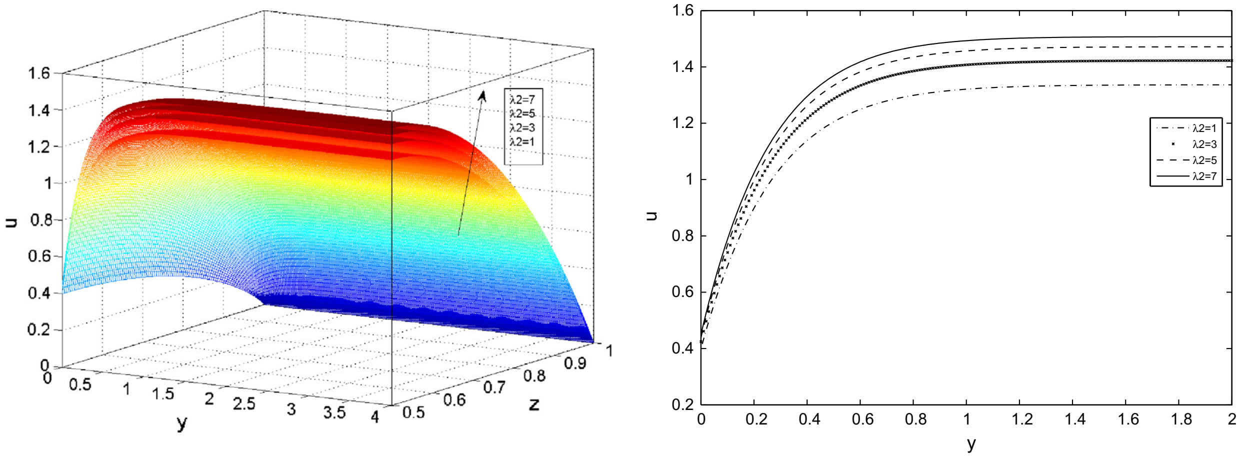
$$\begin{aligned}\bar{\tau}_1 = &-\frac{2\mu}{h}\sum_{n=1}^{\infty}(1-(-1)^n)\sin\left(\frac{n\pi z}{h}\right)\cdot\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\\ &\frac{(-1)^{i+r+k+m+j+l+p}y^k}{\theta^{r+1}k!m!j!l!p!}\cdot\frac{\Gamma(m-c-1/2)\Gamma(j-m+c-1/2)\Gamma(l+m+p-c+1/2)}{\Gamma(-c-1/2)\Gamma(c-1/2-m)\Gamma(m-c+1/2)}\\ &v^{m-c-1/2}\xi^{2m-1}\cdot\frac{\lambda_1^{p\alpha}\lambda_3^{(m-c-j+1/2)\beta}}{\lambda_2^{(l+m+p-c+1/2)\alpha}}\cdot\frac{1}{s^{(i+m-c+1/2)+(j-m+c-1/2)\beta+(p+2l+2m-2c+1)\alpha}}\\ &+\frac{2A\mu}{h}\sum_{n=1}^{\infty}(1-(-1)^n)\sin\left(\frac{n\pi z}{h}\right)\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\xi^{2m-1}\cdot\\ &v^{m-c-1/2}\frac{(-1)^{r+k+m+j+l+p}y^k}{\theta^{r+1}k!m!j!l!p!}\frac{\Gamma(j-m+c-1/2)\Gamma(l+m+p-c+1/2)}{\Gamma(-c+1/2)\Gamma(c-m-1/2)}\frac{\lambda_1^{p\alpha}\lambda_3^{(m-c-j+1/2)\beta}}{\lambda_2^{(l+m+p-c+1/2)\alpha}}\cdot\\ &\frac{1}{s^{(m-c+3/2)+(j-m+c-1/2)\beta+(p+2l+2m-2c+1)\alpha}}\end{aligned} \tag{27}$$

By employing the inverse Laplace transform to Eq. (27), we can obtain the exact analytical solution for $\tau_1(y,z,t)$



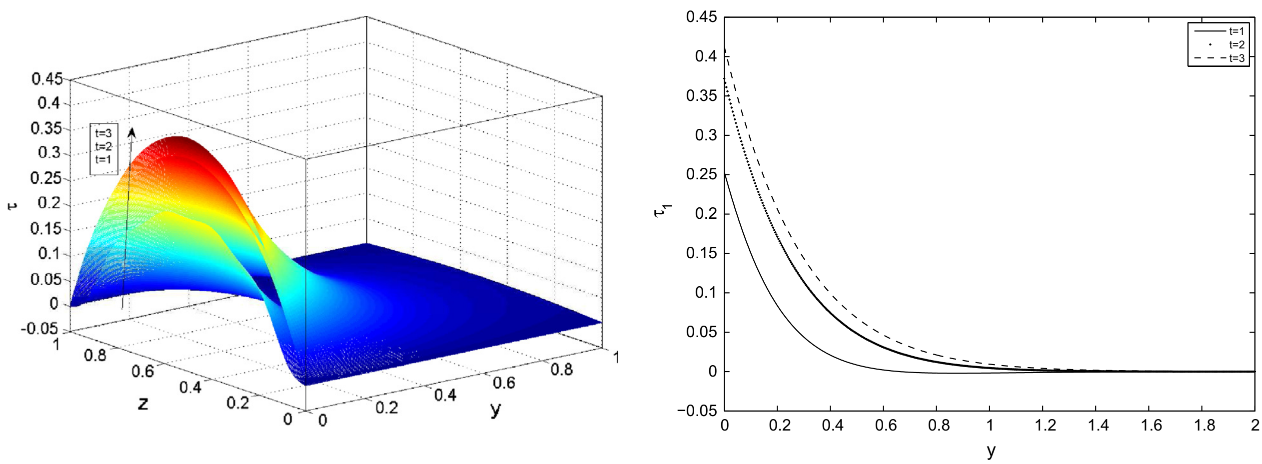
(a) 3D surface (b) 2D figure when $z = 0.5$

Figure 5 Profiles of velocity u for different values θ when $t = 2,\ A = 1,\ \alpha = 0.3,\ \beta = 0.8,\ \lambda_2 = 1,\ \lambda_1 = 2,\ \lambda_3 = 3,\ v = 0.1$.



(a) 3D surface (b) 2D figure when $z = 0.5$

Figure 6 Profiles of velocity u for different values λ_2 when $t = 3,\ A = 1,\ \alpha = 0.3,\ \beta = 0.8,\ \theta = 0.1,\ \lambda_1 = 2,\ \lambda_3 = 3,\ v = 0.1$.



(a) 3D surface (b) 2D figure when $z = 0.5$

Figure 7 Profiles of shear stress τ_1 for different values t when $A = 1,\ \alpha = 0.3,\ \beta = 0.8,\ \theta = 1,\ \lambda_2 = 1,\ \lambda_1 = 2,\ \lambda_3 = 3,\ v = 0.1$.

$$\begin{aligned}
\tau_1 = &-\frac{2\mu}{h}\sum_{n=1}^{\infty}(1-(-1)^n)\sin\left(\frac{n\pi z}{h}\right)\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}v^{m-c-1/2}\cdot\\
&\xi^{2m-1}\frac{(-1)^{i+r+k+m+j+l+p}y^k}{\theta^{r+1}k!m!j!l!p!}\frac{\Gamma(m-c-1/2)\Gamma(j-m+c-1/2)\Gamma(l+m+p-c+1/2)}{\Gamma(-c-1/2)\Gamma(c-1/2-m)\Gamma(m-c+1/2)}\cdot\\
&\frac{\lambda_1^{p\alpha}\lambda_3^{(m-c-j+1/2)\beta}}{\lambda_2^{(l+m+p-c+1/2)\alpha}}\frac{t^{(i+m-c-1/2)+(j-m+c-1/2)\beta+(p+2l+2m-2c+1)\alpha}}{\Gamma((i+m-c+1/2)+(j-m+c-1/2)\beta+(p+2l+2m-2c+1)\alpha)}\\
&+\frac{2A\mu}{h}\sum_{n=1}^{\infty}(1-(-1)^n)\sin\left(\frac{n\pi z}{h}\right)\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}v^{m-c-1/2}\xi^{2m-1}\cdot\\
&\frac{(-1)^{r+k+m+j+l+p}y^k}{\theta^{r+1}k!m!j!l!p!}\frac{\lambda_1^{p\alpha}\lambda_3^{(m-c-j+1/2)\beta}}{\lambda_2^{(l+m+p-c+1/2)\alpha}}\frac{\Gamma(j-m+c-1/2)\Gamma(l+m+p-c+1/2)}{\Gamma(-c+1/2)\Gamma(c-m-1/2)}\cdot\\
&\frac{t^{(m-c+1/2)+(j-m+c-1/2)\beta+(p+2l+2m-2c+1)\alpha}}{\Gamma((m-c+3/2)+(j-m+c-1/2)\beta+(p+2l+2m-2c+1)\alpha)}
\end{aligned}\tag{28}$$

By using the Fox H-function, we can get a simpler form of Eq. (28):

$$\begin{aligned}
\tau_1 = &-\frac{2\mu}{h}\sum_{n=1}^{\infty}(1-(-1)^n)\sin\left(\frac{n\pi z}{h}\right)\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{\lambda_3^{(b-j)\beta}}{\lambda_2^{(l+b)\alpha}}\cdot\\
&v^{b-1}\xi^{2m-1}\frac{(-1)^{i+r+k+m+j+l}y^k}{\theta^{r+1}k!m!j!l!}t^{(i+b-1)+(j-b)\beta+(2l+2b)\alpha}\\
&\times H_{3,5}^{1,3}\left[\frac{\lambda_1^{\alpha}t^{\alpha}}{\lambda_2^{\alpha}}\middle|\begin{matrix}(2-b,0),(1-j+b,0),(1-l-b,1)\\(0,1),(c+3/2,0),(b+1,0),(1-b,0),(1-i-b-(j-b)\beta-(2l+2b)\alpha,\alpha)\end{matrix}\right]\\
&+\frac{2A\mu}{h}\sum_{n=1}^{\infty}(1-(-1)^n)\sin\left(\frac{n\pi z}{h}\right)\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{r+k+m+j+l}y^k}{\theta^{r+1}k!m!j!l!}\cdot\\
&v^{b-1}\xi^{2m-1}\frac{\lambda_3^{(b-j)\beta}}{\lambda_2^{(l+b)\alpha}}t^{b+(j-b)\beta+(2l+2b)\alpha}\\
&\times H_{2,4}^{1,2}\left[\frac{\lambda_1^{\alpha}}{\lambda_2^{\alpha}}t^{\alpha}\middle|\begin{matrix}(1-j+b,0),(1-l-b,1)\\(0,1),(c+1/2,0),(1+b,0),(-b-(j-b)\beta-(2l+2b)\alpha,\alpha)\end{matrix}\right]
\end{aligned}\tag{29}$$

where $b = m - c + 1/2$. As for the shear stress $\tau_2(y, z, t)$, it can be easily obtained from Eqs. (18) and (25) by performing the same calculation steps with those of $\tau_1(y, z, t)$.

4. Analysis and discussion

In the Section 3, we have studied the problem of the unidirectional flow of a generalized Burgers' fluid with fractional derivative between two side walls induced by an exponential accelerating plate and a constant pressure gradient, where the slip boundary condition between the exponential accelerating plate and the Burgers' fluid is considered. The exact analytical solutions for the 3D flow are obtained by using the Laplace transform, the finite Fourier sine transform and their inverse transforms. Moreover, some 3D and 2D figures are plotted to analyze and discuss the effects of various parameters.

Fig. 1 shows the time parameter t effects on fluid velocity. It is clearly seen that near the plate, the increasing t results in the decrease of the velocity surfaces, while, away from the plate, the increasing t results an opposite trend for the fluids. The reason for this is that the plate moves in the speed of exponential decay function e^{-t}, the increasing t results in the decrease of e^{-t}, in far away from the plate area, because of the constant pressure gradient, the increasing t results in the increase of the fluid velocity. Fig. 2 shows the variations of parameter A. The effect of increasing A results in the increase of the velocity surfaces. Figs. 3 and 4 show the fractional derivative parameters α and β effect on the fluid velocity. The fluid velocity speeds up with the increasing the values of α. However, the increasing β has the opposite effect to that of α. Fig. 5 displays the effect of slip coefficient θ. The fluid flows faster with the increasing of the slip coefficient θ. $\theta = 0$ indicates that the slip between the exponential accelerating plate and the Burgers' fluid is neglected. Fig. 6 shows the material constant λ_2 of the Burgers' fluid effects on the velocity field. This figure indicates that the fluid velocity is increasing in pace with the increasing of the material constant λ_2. Fig. 7 displays the profiles of the shear stress τ_1 for different values of t. This figure shows that the strongest shear stress takes place in the boundary between the Burgers' fluid and the exponential accelerating plate, along with away from the plate, shear stress τ_1 decreases rapidly.

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