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ORIGINAL ARTICLE

On zip and weak zip rings of skew generalized power series

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Abstract In this paper we show under certain conditions that the skew generalized power series $R[[S,w]]$ is a right zip (weak zip) ring if and only if R is a right zip (weak zip) ring.

1. Introduction

Throughout this paper R denotes an associative ring with identity. Recall from Faith [3] that R is a right zip ring if the right annihilator $r_R(X)$ of a subset $X \subseteq R$ is zero, then $r_R(X_0) = 0$ for a finite subset X_0 of X, equivalently for a left ideal L of R if $r_R(L) = 0$, then there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. Although the concept of zip rings was initiated by Zelmanowitz [17] it was not called so at that time. However, He showed that any ring satisfying the descending chain condition on right annihilators is a right zip ring but the converse is not true.

Extensions of zip rings were studied by several authors. In [1] Beach and Blair showed that if R is a commutative zip ring, then $R[x]$ is a zip ring. The pioneering paper [12] introduced the notion of an Armendariz ring: a ring R is called Armendariz if whenever polynomials $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{j=0}^{m} b_j x^j \in R[x]$ satisfy $fg = 0$, then $a_i b_j = 0$ for each $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant m$. In Hong et al. [7, Theorem 1] showed that if R is an Armendariz ring, then R is a right zip ring if and only if $R[x]$ is a right zip ring.

Rege and Chhawchharia in [12] motivated the other researchers to adapt the Armendariz condition for different extensions. Cortes in [2] defined and extended the condition for skew polynomial rings ($R[x,\sigma]$), skew Laurant polynomial rings ($R[x,x^{-1},\sigma]$), skew power series rings ($R[[x,\sigma]]$), and skew Laurant power series rings ($R[[x,x^{-1},\sigma]]$). These extensions share the right zip property with the base rings satisfying the corresponding Armendariz condition. In Zhongkui [18] extended the notion of an Armendariz ring to the generalized power series ring $A = [[R^{S,\leqslant}]]$, where $(S,\leqslant)$ is a commutative strictly ordered monoid as follows: whenever $f, g \in [[R^{S,\leqslant}]]$ such that $fg = 0$, then $f(s)g(t) = 0$ for all $s \in supp(f)$ and $t \in supp(g)$.

In Marks et al. [10] unified all versions of Armendariz rings and called it (S,w)-Armendariz ring as follows. For a ring R, $(S,\leqslant)$ a strictly ordered monoid, and $w: S \to (End\ R, +)$ a monoid homomorphism, whenever $fg = 0$ for f, g in the skew

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generalized power series ring $R[[S,w]]$, then $f(s)w_s(g(t)) = 0$ for all $s \in supp(f)$ and $t \in supp(g)$.

Motivated by the above Ouyang in [8] introduced the notion of right weak zip rings (i.e., rings provided that if the right weak annihilator of a subset X of R, $Nr_R(X) \subseteq nil(R)$, then there exists a finite subset $X_0 \subseteq X$ such that $Nr_R(X_0) \subseteq nil(R)$), where $nil(R)$ is the set of all nilpotent elements of R and $Nr_R(X) = \{a \in R | xa \in nil(R)$ for each $x \in X\}$. The author in [8] studied the transfer of the right (left) weak zip property between the base ring R and Ore extension $R[x,\sigma,\delta]$, where σ is an endomorphism and δ is a σ-derivation. A ring R is called σ-compatible if for each $x,y \in R, xy = 0 \iff x\sigma y = 0$. In this case it is clear that σ is a monomorphism. A ring R is called NI if $nil(R)$ forms an ideal, i.e., if the set of all nilpotent elements forms an ideal. Ribenboim studied extensively rings of generalized power series (see [13,14]). In [11] Mazurek and Ziembowski generalized Ribenboim construction and introduced a twisted version of the generalized power series rings as follows.

Let $(S,+,\leqslant)$ be a strictly ordered monoid, R a ring, w: $S \to End(R)$ a monoid homomorphism and let $w_s = w(s)$ denotes the image of $s \in S$ under w for any $s \in S$. Consider the set Λ of all maps $f:S \to R$ such that $supp(f) = \{s \in S | f(s) \neq 0\}$ is Artinian and narrow subset of S, i.e., every strictly decreasing sequence of elements of $supp(f)$ is finite and every subset of pairwise order-incomparable elements of $supp(f)$ is finite with pointwise addition and product operation called convolution defined by

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)w_u(g(v)) \quad \text{for each } f,g \in \Lambda$$

where $X_s(f,g) = \{(u,v) \in S \times S | u + v = s, f(u)$, and $g(v) \neq 0\}$ is finite.

Hence, $\Lambda = R[[S,w]]$ becomes a ring called skew generalized power series with coefficients in R and exponents in S, for more details on the structure of $\Lambda = R[[S,w]]$ (see[11]).

Let $\pi(f)$ denotes the set of all minimal elements of $supp(f)$. If $(S,\leqslant)$ is totally ordered, then $\pi(f)$ consists of only one element which is still denoted by $\pi(f)$. Let $T = C(f)$ be the content of f i.e., $C(f) = \{f(s) | s \in supp(f)\}$. Since, $R \simeq c_R$ we can identify, the content of f with

$$c_{C(f)} = \{c_{f(u_i)} | u_i \in supp(f)\} \subseteq \Lambda.$$

For any nonempty subset $X \subseteq R$, let $X[[S,w]] = \{f \in \Lambda | f(s) \in X \cup \{0\}$ *for each* $s \in supp(f)\}$.

The motivation of this paper is to continue the studying of the transfer of some algebraic properties between the base ring R and the generalized power series ring $[[R^{S,\leqslant}]]$ (see [5,15]) also to extend the results of Cortes [2], Oynang [8] and Salem [16] to the skew generalized power series over zip and weak zip rings.

2. Skew generalized power series over zip rings

Hirano [6], Cortes [2] and Ouyang [8] studied the relation between the right annihilators of R and those of $R[x]$ and $R[x,\sigma,\delta]$ respectively. In [10] Marks et al presented a characterization theorem for (S,w)-Armendariz rings in terms of one-sided annihilator and for the sake of completeness of this note we give a version of [10, Theorem 3.4].

Let R be a ring, $(S,\leqslant)$ a strictly ordered monoid and $w:S \to End(R)$ a monoid homomorphism. R is called S-compatible if w_s is compatible for every $s \in S$. In fact w_s is a monomorphism for each $s \in S$ (see [4]).

Lemma 2.1. *Suppose that R is a ring, S a strictly ordered monoid, and let $\Lambda = R[[S,w]]$. If R is S-compatible and $U \subset R$, then*

$$r_\Lambda(U) = r_R(U)[[S,w]] \quad (l_\Lambda(U) = l_R(U)[[S,w]]).$$

Proof 1. Let $f \in r_\Lambda(U)$. Then $0 = c_u f$ for each $u \in U$. So, $0 = (c_u f)(s) = uw_0(f(s)) = uf(s)$ for each $s \in supp(f)$. Consequently, $f(s) \in r_R(u)$ for each $s \in supp(f)$. Hence, $f \in r_R(U)[[S,w]]$ and it follows that $r_\Lambda(U) \subseteq r_R(U)[[S,w]]$.

Conversely, let $f \in r_R(U)[[S,w]]$. Then $0 = Uf(s)$ for each $s \in supp(f)$. So, for each $u \in U$, $0 = uf(s) = uw_0(f(s)) = (c_u f)(s)$. Hence, $f \in r_\Lambda(u)$ and it follows that $r_R(U)[[S,w]] \subseteq r_\Lambda(U)$.

Consequently, $r_\Lambda(U) = r_R(U)[[S,w]]$. □

Using Lemma 2.1 we have the map ϕ: $r_R(2^R) \to r_\Lambda(2^\Lambda)$ defined by $\phi(I) = I[[S,w]]$ for every $I \in r_R(2^R)$ and the map ψ: $l_R(2^R) \to l_\lambda(2^\Lambda)$ defined by $\psi(J) = J[[S,w]]$ for every $J \in l_R(2^R)$ without any condition on R, where $r_R(2^R) = \{r_R(U) | U \subseteq R\}(l_R(2^R) = \{l_R(U) | U \subseteq R\})$. Obviously ϕ (ψ) is injective. In the following lemma we show that ϕ (ψ) is a bijective map if and only if R is an (S,w)-Armendariz ring.

Lemma 2.2. *Suppose that R is a ring, S a strictly ordered monoid, and let $\Lambda = R[[S,w]]$. If R is S-compatible, then the following are equivalent:*

1. *R is an (S,w)-Armendariz ring.*
2. *$\phi:r_R(2^R) \to r_\Lambda(2^\Lambda)$ defined via $\phi(I) = I[[S,w]]$ ($\psi:l_\Lambda(2^R) \to l_\Lambda(2^\Lambda)$ defined via $\psi(J) = J[[S,w]]$) is a bijective map.*

Proof 2. $1 \Rightarrow 2$

Let $Y \subseteq \Lambda$ and $T = \cup_{f\in Y} C(f) = \cup_{f\in Y}\{f(s) | s \in supp(f)\}$

From Lemma 2.1 it is sufficient to show that $r_\Lambda(f) = r_R C(f)[[S,w]]$ for each $f \in Y$. So, let $g \in r_\Lambda(f)$, it follows that $fg = 0$. Since, R is an (S,w)-Armendariz ring and S-compatible, then $0 = f(u)w_u(g(v)) = f(u)g(v)$ for each $u \in supp\ (f)$ and $v \in supp(g)$.

So, for a fixed $u \in supp(f)$ and each $v \in supp(g)$, $0 = f(u)g(v)$ and it follows that $g \in r_R f(u)[[S,w]]$. Consequently, $g \in r_R C(f)[[S,w]]$, hence $r_\Lambda(f) \subseteq r_R C(f)[[S,w]]$.

Conversely, let $g \in r_R C(f)[[S,w]]$, hence $f(u)g(v) = 0$ for each $u \in supp(f)$ and $v \in supp(g)$. Since R is S-compatible, then $0 = f(u)w_u(g(v))$ for each $u \in supp(f)$ and $v \in supp(g)$. So, $(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)w_u g(v) = 0$. Therefore, $g \in r_\Lambda f$ and it follows that $r_R C(f)[[S,w]] \subseteq r_\Lambda(f)$.

Hence,
$r_\Lambda(Y) = \cap_{f\in Y} r_\Lambda f = \cap_{f\in Y} r_R C(f)[[S,w]] = r_R(T)[[S,w]]$.

$2 \Rightarrow 1$

Let $f, g \in \Lambda$ be such that $fg = 0$ then using Lemma 2.1 $g \in r_\Lambda(f) = T[[S,w]]$ for some right ideal T of R. Hence, $g(v) \in T$ for each $v \in supp(g)$. So, $0 = fc_{g(v)}$. Thus, $0 = (fc_{g(v)})(u) = f(u)w_u(g(v)))$ for each $u \in supp(f)$. Therefore, R is a (S,w)-Armendariz ring. □

Now, we are ready to prove the main result of this section.

Theorem 2.3. *Suppose that R is an (S,w) Armendariz ring, S a strictly ordered monoid, and let $\Lambda = R[[S,w]]$. If R is S-compatible, then R is a right (left) zip ring if and only if Λ is a right (left) zip ring.*

Proof 3. Suppose Λ is right zip and $X \subseteq R$ satisfies $r_R(X) = 0$. Let $Y = \{c_x | x \in X\}$, then $r_\Lambda(Y) = 0$ by Lemma 2.1. Since Λ is right zip, $r_\Lambda(c_{x_1}, \ldots, c_{x_n}) = 0$ for some $x_1, \ldots, x_n \in X$. Now Lemma 2.1 shows that $r_R(x_1, \ldots, x_n) = 0$. Hence R is right zip.

Conversely, suppose R is right zip and $Y \subseteq \Lambda$ satisfies $r_\Lambda(Y) = 0$. Let $T = C(Y)$ be the content of Y, then $r_R(T) = 0$ by [10, Theorem 3.4]. Since R is right zip, $r_R(t_1, \ldots, t_n) = 0$ for some $t_1, \ldots, t_n \in T$. For any $i \in \{1, \ldots, n\}$ there exists $f_i \in Y$ with $t_i \in f_i(s)$. Set $Y_0 = \{f_1, \ldots, f_n\}$. Since $\{t_1, \ldots, t_n\} \subseteq C(Y_0)$, $r_R(C(Y_0)) = 0$ and thus $r_\Lambda = 0$ by [10, Theorem 3.4]. Hence Λ is right zip. □

3. Skew generalized power series over weak zip rings

The following results introduce some properties of S-compatible rings.

Lemma 3.1. *Suppose that R is a ring, S a strictly ordered monoid, and let $\Lambda = R[[S,w]]$. If R is S-compatible, then we have the following:*

(i) *If $ab = 0$, then $w_s(a)b = 0$ and $aw_t(w_s(b)) = aw_{t+s}(b) = 0$ for every $s, t \in S$.*
ii) *If $w_s(a)b = 0$ for some $s \in S$, then $ab = 0$.*

Proof 4

(i) Suppose that $ab = 0$, then for each $s \in S$, $0 = w_s(ab) = w_s(a)w_s(b)$. Since R is S-compatible, then $w_s(a)b = aw_s(b) = 0$. Again since, R is a S-compatible, then for each $t \in S$, $0 = aw_t(w_s(b)) = aw_{t+s}(b)$.
(ii) Suppose that $w_s(a)b = 0$. Since, R is S-compatible in fact it is a monomorphism, then $0 = w_s(a)b = w_s(a)w_s(b) = w_s(ab)$. Hence, $ab = 0$. □

Let $f^k_{w_s} = w_s + w_{2s} + \cdots + w_{ks-1}$ denotes the map which is the sum of endomorphisms where k is a positive integer. Then we can deduce the following.

Lemma 3.2. *Suppose that R is a ring, S a strictly ordered monoid, and let $\Lambda = R[[S,w]]$. If R is S-compatible, then $ab = 0$ implies that $0 = af^k_{w_s}(b) = aw_s(b) + aw_{2s}(b) + \cdots + aw_{ks-1}(b)$*

Proof 5. Since, R is S-compatible, then for each $s \in S$ $aw_s(b) = 0$. Thus by Lemma 3.1 $aw_{2s}(b) = 0$, and it follows that $0 = aw_s(b) + aw_{2s}(b) + \cdots + aw_{ks-1}(b)$. □

Lemma 3.3. *Suppose that R is a ring, S a strictly ordered monoid, and let $\Lambda = R[[S,w]]$. If R is S-compatible and $aw_s(b)$ is nilpotent, then ab is nilpotent.*

Proof 6. Since, $aw_s(b)$ is nilpotent, then there exists an integer k such that $(aw_s(b))^k = aw_s(b)aw_s(b)\cdots aw_s(b) = 0$ (k − times). Since, R is S-compatible, then

$$0 = aw_s(b)aw_s(b)\cdots aw_s(b)ab = aw_s(b)aw_s(b)\cdots aw_s(bab)$$
$$= aw_s(b)aw_s(b)\cdots abab$$

Continuing on this process we can deduce that $0 = (ab)^k$ and the lemma is proved. □

Lemma 3.4. *Suppose that R is an NI ring, S a strictly ordered monoid, and let $\Lambda = R[[S,w]]$. If R is S-compatible, then $ab \in nil(R)$ implies that $aw_s(b) \in nil(R)$.*

Proof 7. Since, ab is nilpotent, then there exists an integer k such that $(ab)^k = 0$. We use the S-compatibility of R many times. Hence

$$0 = (ab)^k = abab\cdots abab \qquad k - \text{times} = aw_s(bab\cdots abab)$$
$$= aw_s(b)w_s(abab\cdots abab) = aw_s(b)(abab\cdots abab)$$
$$= aw_s(b)aw_s(bab\cdots abab) = aw_s(b)aw_s(b)ab\cdots ab$$

Continuing on this process it can be easily shown that $0 = (aw_sb)^k$ and the lemma is proved. □

Proposition 3.5. *Suppose that R is an NI ring and S a strictly totally ordered monoid. If R is S-compatible and $f \in \Lambda = R[[S,w]]$ is nilpotent, then $f(u)$ is nilpotent for each $u \in supp(f)$.*

Proof 8. Suppose that $f \in \Lambda$ is a nilpotent element, hence there exists $k \in \mathbb{N}$ such that $f^k = 0$, i.e., $supp(f^k) = \phi$. Since, S is a totally ordered monoid, let $\pi(f) = u_0$.

Therefore, $0 = f^k(ku_0) = f(u_0)w_{u_0}f(u_0)w_{2u_0}f(u_0)\cdots w_{(k-1)u_0}f(u_0) + \sum_{(t_1,\ldots,t_k)\in X_{ku_0}-\{(u_0,u_0,\ \ldots,u_0)\}} f(t_1)w_{t_1}f(t_2)\cdots w_{t_1+\cdots+t_{k-1}}f(t_k)$.

Since, $\pi(f) = u_0$, then for some $i \in \{1, \ldots, k\}$, $t_i > u_0$.

Hence, $ku_0 = u_0 + \cdots + u_0 < t_1 + \cdots + t_i + \cdots + t_k = u_0 + \cdots + u_0$ a contradiction. So, $t_i = u_0$ for each $i \in \{1, \ldots, k\}$.

Consequently, $0 = f^k(ku_0) = f(u_0)w_{u_0}f(u_0)\cdots w_{(k-1)u_0}f(u_0)$. Since, R is S-compatible, then by freely using Lemma 3.3 it follows that $0 = (f(u_0))^k$ and $f(u_0)$ is a nilpotent element of R.

Consider now, $f = (f - c_{f(u_0)}e_{u_0}) + c_{f(u_0)}e_{u_0} = (f - f_0') + f_0' = f_0 + f_0'$, where $supp(f - f_0') = supp(f) - \{u_0\}$ and $supp(f_0') = supp(c_{f(u_0)}e_{u_0}) = \{u_0\}$.

Hence, $0 = f^k = (f_0 + f_0')^k = (f_0 + f_0')(f_0 + f_0') \cdots (f_0 + f_0') = f_0^k + f_0 f_0'^{k-1} + \cdots + f_0'^{k-1} f_0 + f_0^2 f_0'^{k-2} + f_0 f_0'^{k-2} f_0 + \cdots + f_0'^k = f_0^k + \Delta + f_0'^k$, where $supp(f_0'^k) \subseteq supp(f_0') + \cdots + supp(f_0')$ k-times $\subseteq ksupp(f_0') = \{ku_0\}$.

Thus $f_0'^k(ku_0) = f(u_0) w_{u_0} f(u_0) w_{2u_0} f(u_0) \cdots w_{(k-1)u_0} f(u_0)$ and by freely using Lemma 3.3 $f_0'^k(ku_0) = (f(u_0))^k = 0$ and f_0' is nilpotent.

Now, it is clear that Δ is the sum of monomials each monomial is the product of ℓcopies of f_0 and $(k - \ell)$ copies of f_0', where *supp* each monomial $\subseteq$ the sum of ℓ copies of $supp(f_0)$ and $(k - \ell)$ copies of $supp(f_0')$.

Since, $f_0'(u_0) = f(u_0)$ is nilpotent and R is an NI ring, then nilpotent elements of R form an ideal. Therefore it can be easily shown that each monomial is a nilpotent element of Λ and it follows that f_0 is also nilpotent.

If $f = f_0'$, then $f \in \Lambda$ is a nilpotent element of Λ and $f_0'(u_0) = (c_{f(u_0)} e_{u_0})(u_0) = f(u_0)$ is a nilpotent element of R and there is nothing to prove.

So, suppose that $0 \neq f_0 = f - f_0'$ and $\pi(f_0) = \pi(f - f_0') = u_i$.

Since, $0 \neq (f - f_0')(u_i) = (f - c_{f(u_0)} e_{u_0})(u_i) = f(u_i)$ and $f_0(u_0) = (f - f_0')(u_0) = (f - c_{f(u_0)} e_{u_0})(u_0) = f(u_0) - f(u_0) e_{u_0}(u_0) = f(u_0) - f(u_0) = 0$ then $u_0 < u_i$.

Since, f_0 is nilpotent, then there exists a positive integer k' such that $(f_0)^{k'} = 0$,

using the same procedure above it can be easily shown that $0 = f_0^{k'}(k' u_i) = (f(u_i))^{k'}$. Continuing on this process $f = f_\mu + f_\mu'$, where $f_\mu' : S \to R$ is defined by $f_\mu'(u_m) = (c_{f(u_m)} e_{u_m})(u_m) = f(u_m)$ which is nilpotent for each $u_m \in supp(f)$, $m \leqslant \mu$ and f_μ is a nilpotent element of Λ. Let $\pi(f_\mu) = \pi(f - f_\mu') = \pi(f - c_{f(u_\mu)} e_{u_\mu}) = u_\theta,\ u_\mu < u_\theta$.

Using [14, 5.3] we can define a relation on Λ called section relation for f_μ' and $f_\nu' \in \Lambda$ as follows:

(i) $f_\mu' \preceq f_\nu'$ if $\mu < \nu$
(ii) $\pi(f - f_\mu') < \pi(f - f_\nu')$, where $\mu < \nu$
(iii) $u < \pi(f - f_\mu')$ for each $u \in supp(f_\mu')$
(iv) $f_\mu = f - f_\mu' \in \Lambda$ is nilpotent and $f_\mu'(u_m) \in R$ is nilpotent for each $u_m \in supp(f)$, $m \leqslant \mu$.

Let * denotes the section relation $\preceq$ with the above properties. Let α be an ordinal such that card $\alpha >$ *card*| *supp f*| and Γ the set of all ordinals $\lambda < \alpha$. We show that for each $\lambda \in \Gamma$ there exists $f_\lambda' \in \Lambda$ such that * is satisfied. In fact let $\lambda \in \Gamma$ and assume that we have already found the element $f_\mu' \in \Lambda$ for every $\mu < \lambda$ satisfying * for ordinals $\mu < \nu < \lambda$.

Now, we will determine an element f_λ', where * is satisfied for $\mu < \nu \leqslant \lambda$. Suppose that there exists an ordinal η such that $\lambda = \eta + 1$. If $f - f_\eta' = 0$, then $f = f_\eta'$ is nilpotent. Thus $f_\eta'(u_m) = (c_{f(u_m)} e_{u_m})(u_m) = f(u_m) \in R$ is nilpotent for each $u_m \in supp(f)$, $m \leqslant \eta$ and there is nothing to prove.

Hence, suppose that $f_\eta = f - f_\eta' \neq 0$, and let $\pi(f_\eta) = \pi(f - f_\eta') = u_\lambda$. Let $f_\lambda' : S \to R$ be defined by $f_\lambda' = f_\eta' + c_{f(u_\lambda)} e_{u_\lambda}$. So, $f_\lambda' \in \Lambda$ and we show that $f_\eta' \preceq f_\lambda'$ and this implies that $f_\mu' \preceq f_\lambda'$ for every $\mu < \lambda$.

In fact $0 \neq (f - f_\eta')(u_\lambda) = f(u_\lambda)$ and it follows that $supp(f_\lambda' - f_\eta') = supp(c_{f(u_\lambda)} e_{u_\lambda}) = \{u_\lambda\}$. If $u \in supp(f_\eta')$, then by * $u < \pi(f - f_\eta') = u_\lambda \in supp(f_\lambda' - f_\eta')$. Thus $f_\eta' \preceq f_\lambda'$, if $f_\eta' = f_\lambda'$, then $c_{f(u_\lambda)} e_{u_\lambda} = 0$ which is a contradiction. If $f_\mu' = f_\lambda'$, then $f_\mu' \preceq f_\eta' \preceq f_\lambda' = f_\mu'$ and $f_\lambda' = f_\eta'$ which is again a contradiction. Hence, $f_\mu' \neq f_\lambda'$ for each $\mu < \eta \leqslant \lambda$, and it can be easily shown that $f_\lambda = f - f_\lambda'$ is nilpotent and $f_\lambda'(u_m) = (c_{f(u_m)} e_{u_m})(u_m) = f(u_m) \in R$ is nilpotent for each $m \leqslant \lambda$.

If $f_\lambda = f - f_\lambda' = 0$, there is nothing to prove, otherwise there exists $u_\xi \in supp(f)$ such that $\pi(f - f_\lambda') = u_\xi$ where $f_\lambda' = f_\eta' + c_{f(u_\lambda)} e_{u_\lambda}$. Since, $(f - f_\lambda') = f - f_\eta' - c_{f(u_\lambda)} e_{u_\lambda}$ and $(f - f_\lambda')(u_\lambda) = (f - f_\eta' - c_{f(u_\lambda)} e_{u_\lambda})(u_\lambda) = f(u_\lambda) - f_\eta'(u_\lambda) - f(u_\lambda) = 0$ then $u_\lambda < u_\xi$. By the fact that $0 \neq (f - f_\lambda')\ (u_\xi) = (f - f_\eta' - c_{f(u_\lambda)} e_{u_\lambda})(u_\xi) = (f - f_\eta')(u_\xi)$, we have that, $u_\xi \in supp(f - f_\eta')$. Hence, $u_\lambda < u_\xi$ and $\pi(f - f_\eta') < \pi(f - f_\lambda')$ and this implies that $\pi(f - f_\mu') < \pi(f - f_\lambda')$ for each $\mu < \lambda$.

Now, we show that $u < \pi(f - f_\lambda')$ for each $u \in supp(f_\lambda')$. In fact $supp(f_\lambda') = supp(f_\eta' + c_{f(u_\lambda)} e_{u_\lambda}) \subseteq supp(f_\eta') \cup supp(c_{f(u_\lambda)} e_{u_\lambda})$. If $u \in supp(f_\eta')$, then $u < \pi(f - f_\eta')$ and if $u \in supp(c_{f(u_\lambda)} e_{u_\lambda})$, then $u = u_\lambda = \pi(f - f_\eta') < \pi(f - f_\lambda')$.

Now, let λ be a limit ordinal for the family $\{f_\lambda' | \mu < \lambda\}$ of elements $f_\mu' \in \Lambda$ it was proved in [14, 5.3] that there exists an element $b = \preceq - \sup(f_\mu')_{\mu<\lambda} \in \Lambda$ such that

(i) $f_\mu' \preceq b$ for every $\mu < \lambda$

(ii) If $b' \in \Lambda$ and $f_\mu' \preceq b'$ for every $\mu < \lambda$, then $b \preceq b'$.

Let $f_\lambda' = b = \preceq - \sup(f_\mu')_{\mu<\lambda}$. Then by i) we know that $f_\mu' \preceq f_\lambda'$ for every $\mu < \lambda, f_\lambda = f - f_\lambda'$ is a nilpotent element of Λ and that $f_\lambda'(u_m) \in R$ is nilpotent for each $u_m \leqslant u_\lambda$. If $f_\mu' = f_\lambda'$, then $f_\mu' \preceq f_{\mu+1}' \preceq f_\lambda' = f_\mu'$ and $f_\mu' = f_{\mu+1}'$ which is a contradiction. Hence, $f_\mu' \neq f_\lambda'$ for every $\mu < \lambda$.

Since, $f - f_\lambda' = (f - f_\mu') - (f_\lambda' - f_\mu')$ for every $\mu < \lambda$, then by [14, 5.3], $u_\xi = \pi(f - f_\lambda') \geqslant \min\{\pi(f - f_\mu'), \pi(f_\lambda' - f_\mu')\}$. Note that,

$$\pi(f_\lambda' - f_\mu') = \pi(f_{\mu+1}' - f_\mu') = \pi(f - f_\mu') - (f - f_{\mu+1}') \geqslant \min\{u_\theta, u_{\theta+1}\}$$

Hence, $u_\xi \geqslant u_\theta$ for all $\mu < \lambda$ and if $u_\theta \leqslant u_{\theta+1} \leqslant u_\xi$, then $u_\theta < u_\xi$.

We now show that $u < \pi(f - f_\lambda')$ for each $u \in supp(f_\lambda')$. In fact, $supp(f_\lambda') = \cup_{\mu<\lambda} supp(f_\mu')$, then there exists an ordinal $\mu < \lambda$ such that $u \in supp(f_\mu')$, thus $u < u_\mu < u_\lambda$.

Hence, for μ, $\nu \in \Gamma$, $\mu < \nu$, then $u_\mu < u_\nu$ and we have that $|\{u_\lambda\}$ such that $\lambda \in \Gamma| = |\Gamma| > |S|$ which is a contradiction. So, $f = f_\lambda'$ and the proposition is proved. □

Proposition 3.6. *Suppose that R is a right Noetherian NI ring, S a strictly totally ordered monoid, and let $\Lambda = R[[S,w]]$. If R is S-compatible and $f \in \Lambda$ such that $f(u)$ is nilpotent for each $u \in supp(f)$, then f is nilpotent.*

Proof 9. Let $f \in \Lambda$ be such that $f(u)$ is nilpotent for each $u \in supp(f)$ and I the ideal generated by $\{f(u)| u \in supp(f)\}$. Since, R is an NI right Noetherian ring, then by [19, Lemma 3.1] I is a finitely generated nilpotent ideal. Thus, there exists a positive integer n such that $I^n = (0)$.

So, for each

$(u_1, \ldots, u_n) \in X_u(f, \ldots, f)$
$\mathrm{f}(u_1)w_{u_1}(f(u_2)) \cdots w_{u_1+u_2+\cdots+u_{n-1}}(f(u_n)) = 0.$

Thus, $f^n(u) = \sum_{(u_1,\ldots,u_n)\in X_n(f,\ldots,f)} f(u_1)w_{u_1}(f(u_2)) \cdots w_{u_1+u_2+\cdots+u_{n-1}}(f(u_n)) = 0$, for each $u \in S$ and it follows that f is nilpotent. □

We combine Propositions 3.5 and 3.6 to get the following.

Theorem 3.7. *Suppose that R is a right Noetherian NI ring, S a strictly totally ordered monoid, and let $A = R[[S,w]]$. If R is S-compatible, then $f \in A$ is a nilpotent element if and only if $f(u) \in R$ is nilpotent for each $u \in supp(f)$.*

Proof 10. Is clear. □

Lemma 3.8. *Suppose that R is a right Noetherian NI ring, S a strictly totally ordered monoid, and let $A = R[[S,w]]$. If R is S-compatible and $X \subseteq R$, then $Nr_A(X) = Nr_R(X)[[S,w]]$ $(Nl_A(X) = Nl_R(X)[[S,w]])$*

Proof 11. Suppose that $f \in Nr_R(X)[[S,w]]$. Thus $xf(u) \in nil(R)$ for each $x \in X$ and $u \in supp(f)$. Hence, $xf(u) = xw_0(f(u)) = (c_xf)(u) \in nil(R)$ and using Proposition 3.6 $c_xf \in nil(A)$ for each $x \in X$. Therefore, $f \in Nr_A(X)$ and $Nr_R(X)[[S,w]] \subseteq Nr_A(X)$.

Conversely, suppose that $f \in Nr_A(X)$. Then $c_xf \in nil(A)$ for each $x \in X$. So, for each $u \in supp(f)$ and using Proposition 3.5 $(c_xf)(u) = xw_0f(u) = xf(u) \in nil(R)$. Hence, for each $x \in X$, $f \in Nr_R(X)[[S,w]]$ and we can deduce that $Nr_A(X) \subseteq Nr_R(X)A$. Hence, $Nr_A(X) = Nr_R(X)[[S,w]]$ □

Lemma 3.8 supplies us with the following maps $\phi : Nr_R(2^R) \to Nr_A(2^A)$ given by $\phi(I) = I[[S,w]]$ and $\psi : Nr_l(2^R) \to Nl_A(2^A)$ given by $\psi(J) = J[[S,w]]$. It is clear that both ϕ and ψ are injective maps. In the next theorem we will show that those maps are bijective.

Theorem 3.9. *Suppose that R is an NI ring, S a strictly totally ordered monoid, and let $A = R[[S,w]]$. If R is S-compatible, then*

$\phi : Nr_R(2^R) \to Nr_A(2^A)$ defined by $\phi(I) = I[[S,w]]$

$(\psi : Nr_l(2^R) \to Nl_A(2^A)$ defined by $\psi(J) = J[[S,w]])$

is bijective.

Proof 12. It is sufficient to show that ϕ (ψ) is a surjective map.

Suppose that $V \subseteq A$ and $f \in Nr_A(V)$. Then $gf \in nil(A)$ for each $g \in V$. Using Proposition 3.5 $(gf)(w) \in nil(R)$ for each $w \in supp(gf) \subseteq supp(g) + supp(f)$. Since, S is a totally ordered monoid, let $\pi(g) = v_0$ and $\pi(f) = u_0$. Then

$$(gf)(v_0 + u_0) = g(v_0)w_{v_0}f(u_0) + \sum_{(v_i,u_i)\in X_{v_0+u_0}(g,f)-\{(v_0,u_0)\}} g(v_i)w_{v_i}f(u_i)$$

Since, $\pi(g) = v_0$ and $\pi(f) = u_0$, then for some i, $v_i > v_0$ and $u_i > u_0$. Therefore $v_0 + u_0 > v_i + u_0 = v_0 + u_0$ and it follows that $v_0 = v_i$ and $u_0 = u_i$ for each i. Therefore, $(gf)(v_0 + u_0) = g(v_0)w_{u_0}f(u_0)$ is nilpotent and using Lemma 3.3 it follows that $g(v_0)f(u_0)$ is nilpotent. Hence $f(u_0)g(v_0)$ is nilpotent.

Now, suppose that $g(v)f(u)$, hence $f(u)g(v)$, is nilpotent for each $u \in supp(f)$ and $v \in supp(g)$ such that $u + v < w \in supp(gf)$. Using the transfinite induction we show that $f(u)g(v)$ and $g(v)f(u)$ are nilpotent for each $u + v = w$. Since, $X_w(g,f) = \{(v,u) | \ u + v = w$ where $v \in supp(g)$ and $u \in supp(f)\}$ is a finite subset. Then let

$X_w(g,f) = \{(v_i, u_i) | \quad i = 1, \ldots, n\}$

By assumption, S is a totally ordered monoid, then S is a cancellative monoid. Let $u_1 < u_2 < \cdots < u_n$ if $u_1 = u_2$ and $u_1 + v_1 = u_2 + v_2$, then $v_1 = v_2$. As $<$ is strictly order if $u_1 < u_2$ and $u_1 + v_1 = u_2 + v_2$ it must $v_1 > v_2$ and it follows that $v_1 > v_2 > \cdots > v_n$.

Now, from the above ordering on v_i and u_i it follows that

$$(gf)(w) = g(v_1)w_{v_1}(f(u_1)) + g(v_2)w_{v_2}(f(u_2)) + \cdots + g(v_n)(w_{v_n}(f(u_n))) \in nil(R)$$

Hence

$$g(v_1)w_{v_1}(f(u_1)) = (gf)(w) - g(v_2)w_{v_2}(f(u_2)) - \cdots - g(v_n)(w_{v_n}(f(u_n))) \in nil(R)$$

and for $i \geqslant 2$ it follows that $u_1 + v_i < v_i + u_i$, then by induction hypothesis we have $g(v_i)f(u_1)$ and $f(u_1)g(v_i)$ are nilpotent elements, then multiply from the left side by $f(u_1)$ it follows that

$$f(u_1)g(v_1)w_{v_1}(f(u_1)) = f(u_1)gf(w) - f(u_1)g(v_2)w_{v_2}(f(u_2)) - f(u_1)g(v_n)w_{v_n}(f(u_n))$$

Since, R is an NI, then $nil(R)$ is an ideal and by induction $f(u_1)g(v_1)w_{v_1}(f(u_)))$ is a nilpotent element again as R is S-compatible it follows that $f(u_1)g(v_1)f(u_1)$ is nilpotent. Hence, $f(u_1)g(v_1)$ and $g(v_1)f(u_1)$ are nilpotent. Therefore, multiplying ** from the left by $f(u_2) \cdots f(u_n)$ respectively yields $f(u_i)g(v_i)$ and $g(v_i)f(u_i)$ are nilpotent for each $u_i \in supp(f)$ and $v_i \in supp(g)$. Consequently, $f \in Nr_R(C(g))[[S,w]]$ for each $g \in V$ and it follows that $f \in Nr_R(C(V))[[S,w]]$. Hence, $Nr_A(V) \subseteq Nr_R(C(V))[[S,w]]$ and ϕ is a surjective map. □

Theorem 3.10. *Suppose that R is a right Noetherian NI ring, S a strictly totally ordered monoid, and let $A = R[[S,w]]$. If R is S-compatible, then R is a right (left) weak zip ring if and only if A is a right (left) weak zip ring.*

Proof 13. Suppose that A is a right weak zip ring and $X \subseteq R$ such that $Nr_R(X) \subseteq nil(R)$. Let $Y = \{c_x \in A | \ x \in X\}$ and $0 \neq f \in Nr_A(Y)$. Then $c_xf \in nil(A)$ for each $c_x \in Y$ and $x \in X$. Using Proposition 3.5 $(c_xf)(u) = xw_0(f(u_0)) = xf(u) \in nil(R)$ for each $u \in supp(f)$.

Hence, $f(u) \in Nr_R(X) \subseteq nil(R)$ for each $u \in supp(f)$. Then using Proposition 3.6 $f \in nil(A)$. Therefore, $Nr_A(Y) \subseteq nil(A)$. Since A is a right weak zip ring, then it follows that there exists finite subset $Y_0 \subseteq Y$ such that $Nr_A Y_0 \subseteq nil$ (A), where $Y_0 = \{c_{x_i} | i = 1, \ldots, n\}$ and $X_0 = \{x_i | \ i = 1, \ldots, n\}$. Let $f \in Nr_A(Y_0)$, then $c_{x_i}f \in nil(A)$ for each $c_{x_i} \in Y_0$ and using Lemma 3.5 it follows that $(c_{x_i}f)(u) = x_iw_0(f(u)) = x_if(u) \in nil(R)$ for each $u \in supp(f)$ and $x_i \in X_0 \subseteq X$ So, $T = \cup_{f \in Nr_A Y}\{f(u) | u \in supp(f)\} \subseteq nil(R)$ and R is right weak zip ring.

Conversely, assume that R is a right weak zip ring and $Y \subseteq \Lambda$ such that $Nr_\Lambda(Y) \subseteq nil(\Lambda)$. Let $T = C(Y)$ be the content of Y and $a \in Nr_R(T)$, then $f(u)a \in nil\ R$ for each $u \in supp(f)$.

Since, R is an NI ring then $f(u)w_u(a) = (fc_a)(u) \in nil(R)$ for each $u \in supp(f)$. Then using Proposition 3.6 $fc_a \in nil(\Lambda)$. Hence $c_a \in Nr_\Lambda(Y) \subseteq nil(\Lambda)$. Therefore, using Lemma 3.5 $a \in nil(R)$. Thus, $Nr_R(T) \subseteq nil(R)$.

Since, R is a right weak zip ring there exists a finite subset $T_0 \subseteq T$ such that $Nr_R(T_0) \subseteq nil(R)$. Hence for each $t \in T_0$, there exist $f_t \in Y$ such that $t \in \{f_t(u) | u \in supp(f_t)\}$. Let Y_0 be a minimal subset of Y which contains each f_t such that $t \in T_0$ and it clear that Y_0 is finite subset. Let $T_1 = \cup_{f_t \in Y_0}\{f_t(u)|u \in supp(f_t)\}$. Hence $T_0 \subseteq T_1$ and $Nr_R(T_1) \subseteq Nr_R(T_0) \subseteq nil(R)$.

Now, suppose that $g \in Nr_\Lambda(Y_0)$, then $fg \in nil(\Lambda)$ for each $f \in Y_0$. Using Proposition 3.5 $(fg)(w) \in nil(R)$ for each $w \in supp(fg)$. Tracing the same procedure used in Theorem 3.9 we can show that $f(u)g(v)$ is nilpotent for each $u \in supp(f)$ and $v \in supp(g)$. Consequently $g(v) \in Nr_R(T_1) \subseteq nil(R)$ for each $v \in supp(g)$, then using Proposition 3.6 $g \in nil(\Lambda)$.

Hence $Nr_\Lambda(Y_0) \subseteq nil\Lambda$and Λ is a right weak zip ring. □

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