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ORIGINAL ARTICLE

On Humbert matrix functions

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Abstract In this paper, we consider a Humbert matrix function in the following form:

$$J_{A,B}(z) = \left(\frac{z}{3}\right)^{A+B} \Gamma^{-1}(A+I)\Gamma^{-1}(B+I)\,{}_0F_2\left(-,-;A+I,B+I;-\frac{z^3}{27}\right), \quad |z| < \infty,$$

where

$${}_0F_2\left(-,-;A+I,B+I;-\frac{z^3}{27}\right) = \Gamma(A+I)\Gamma(B+I)\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!}\left(\frac{z}{3}\right)^{3k},$$

and for this function we present order and type, integral representations and differential recurrence relations. Also, the Humbert matrix differential equation is studied.

1. Introduction

Humbert functions of scalar coefficients and variables is appeared in [1,2]. Special matrix functions appear in the literature related to statistics [3,4]. Recently, Laguerre, Hermite and Gegenbauer matrix polynomials have appeared in connection with the study of matrix differential equations [5–7]. The primary goal of this paper is to consider a new system of matrix functions, namely the Humbert matrix function. The paper is organized as follows. Section 2 is define and study of a new matrix functions, say, the Humbert matrix function, the radius of regularity and order and type on this function are established. In Section 3 Integral expressions of Humbert matrix functions are given. In Section 4 deducing some recurrence relations of Section 5, we prove that the Humbert matrix function satisfy a matrix differential equation.

A matrix P in $\mathbb{C}^{N\times N}$ is a positive stable matrix if $\mathrm{Re}(\lambda) > 0$ for all $\lambda \in \sigma(P)$ where $\sigma(P)$ is the set of all eigenvalues of P and its two-norm denoted by

$$\|P\| = \sup_{x\neq 0} \frac{\|Px\|_2}{\|x\|_2},$$

where for a vector y in $\mathbb{C}^N$, $\|y\|_2 = (y^T y)^{\frac{1}{2}}$ is the Euclidean norm of y.

Let $\alpha(P)$ and $\gamma(P)$ be the real numbers which were defined in [5] by

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$$\alpha(P) = \max\{\mathrm{Re}(z) : z \in \sigma(P)\}, \quad \gamma(P) = \min\{\mathrm{Re}(z) : z \in \sigma(P)\}. \tag{1.1}$$

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z which are defined in an open set Ω of the complex plane and P is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(P) \subset \Omega$, then from the properties of the matrix functional calculus (see [5]), it follows that

$$f(P)g(P) = g(P)f(P). \tag{1.2}$$

Hence, if Q in $\mathbb{C}^{N\times N}$ is a matrix for which $\sigma(Q) \subset \Omega$ and if $PQ = QP$, then

$$f(P)g(Q) = g(Q)f(P). \tag{1.3}$$

The reciprocal Gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then the image of $\Gamma^{-1}(z)$ acting on P denoted by $\Gamma^{-1}(P)$ is a well-defined matrix. Furthermore, if

$$P + nI \quad \text{is invertible for all integer } n \geqslant 0, \tag{1.4}$$

Then from [5], the Pochhammer symbol or shifted factorial defined by

$$(P)_n = P(P+I)\cdots(P+(n-1)I) = \Gamma(P+nI)\Gamma^{-1}(P); \quad n \geqslant 1; \; (P)_0 = I. \tag{1.5}$$

Jódar and Cortés have proved in [8] that

$$\Gamma(P) = \lim_{n\to\infty} (n-1)![(P)_n]^{-1} n^P. \tag{1.6}$$

Let P and Q be two positive stable matrices in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$. The gamma matrix function $\Gamma(P)$ and the beta matrix function $B(P,Q)$ have been defined in [9], as follows

$$\Gamma(P) = \int_0^\infty e^{-t} t^{P-I}\, dt; \quad t^{P-I} = \exp((P-I)\ln t), \tag{1.7}$$

and

$$B(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I}\, dt. \tag{1.8}$$

Let P and Q be commuting matrices in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ such that the matrices $P+nI$, $Q+nI$ and $P+Q+nI$ are invertible for every integer $n \geqslant 0$. Then according to [9], we have

$$B(P,Q) = \Gamma(P)\Gamma(Q)[\Gamma(P+Q)]^{-1}. \tag{1.9}$$

2. Humbert matrix function

In this section we deal with the Humbert matrix function $J_{A,B}(z)$ that is defined by

$$\begin{aligned} J_{A,B}(z) &= \left(\frac{z}{3}\right)^{A+B} \Gamma^{-1}(A+I)\Gamma^{-1}(B+I)\,{}_0F_2\left(-,-;A+I,B+I;-\frac{z^3}{27}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!} \left(\frac{z}{3}\right)^{A+B+3kI} \\ &= \sum_{k=0}^{\infty} U_{A+B+3kI} z^{A+B+3kI}, \end{aligned} \tag{2.1}$$

where $A+I$ and $B+I$ are matrices in $\mathbb{C}^{N\times N}$ such that $A+(k+1)I$ and $B+(k+1)I$ are invertible for every integer $k \geqslant -1$. The first few terms of the series are given by the formula

$$J_{A,B}(z) = \frac{z^{A+B}\Gamma^{-1}(A+I)\Gamma^{-1}(B+I)}{3^{A+B}}\left[1 - \frac{z^{3I}[(A+I)]^{-1}[(B+I)]^{-1}}{3^3 1!} + \frac{z^{6I}[(A+I)]^{-1}[(A+2I)]^{-1}[(B+I)]^{-1}[(B+2I)]^{-1}}{3^6 2!} - \cdots\right]$$

For A and B are equal the zero matrix $\mathbf{0}$ it follows

$$J_{\mathbf{0},\mathbf{0}}(z) = \left[1 - \frac{z^{3I}}{3^3} + \frac{z^{6I}}{3^6\cdot 2^3} - \frac{z^{9I}}{3^9\cdot 2^3\cdot 3^3} + \cdots\right].$$

Now we prove that the matrix power series (2.1) converges uniformly in any bounded domain of the complex variable z, by the following inequality

$$\begin{aligned} \|J_{A,B}(z)\| &\leq \sum_{k=0}^{\infty} \left\|\frac{\Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!}\left(\frac{z}{3}\right)^{A+B+3kI}\right\| \\ &\leq \left\|\left(\frac{z}{3}\right)^{A+B}\right\| \left\|\sum_{k=0}^{\infty} \frac{\Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)\left(\frac{z}{3}\right)^{3k}}{k!}\right\| \\ &\leq \left\|\left(\frac{z}{3}\right)^{A+B}\Gamma^{-1}(A+I)\Gamma^{-1}(B+I)\right\| \sum_{k=0}^{\infty}\left\|\frac{[(A+I)_k]^{-1}[(B+I)_k]^{-1}\left(\frac{z}{3}\right)^{3k}}{k!}\right\|. \end{aligned}$$

So,

$$\begin{aligned} \|J_{A,B}(z)\| &\leqslant \left\|\left(\frac{z}{3}\right)^{A+B}\Gamma^{-1}(A+I)\Gamma^{-1}(B+I)\right\| \exp\left(\left\|(A+I)^{-1}(B+I)^{-1}\frac{z^3}{27}\right\|\right) \\ &\leqslant \left\|\left(\frac{z}{3}\right)^{A+B}\Gamma^{-1}(A+I)\Gamma^{-1}(B+I)\right\| \exp\left(\frac{1}{27}|z|^3\right). \end{aligned} \tag{2.2}$$

By considering all the terms of the series for $J_{A,B}(z)$ except the first, it is found that

$$J_{A,B}(z) = \left(\frac{z}{3}\right)^{A+B}\Gamma^{-1}(A+I)\Gamma^{-1}(B+I)(1+\Theta), \tag{2.3}$$

where

$$\begin{aligned} \|\Theta\| &\leqslant \exp\left(\left\|(A+I)^{-1}(B+I)^{-1}\frac{z^3}{27}\right\|\right) - 1 \\ &\leqslant \|(A+I)^{-1}(B+I)^{-1}\|\left(\exp\left(\frac{1}{27}|z|^3\right) - 1\right). \end{aligned}$$

Thus, the series on the right in (2.1) converges uniformly in any bounded domain of the complex variables z. We define the radius of regularity of the function $J_{A,B}(z)$ given in the form

$$\begin{aligned} \frac{1}{R} &= \limsup_{k\to\infty}(\|U_{A+B+3kI}\|)^{\frac{1}{A+B+3kI}} = \limsup_{k\to\infty}\left(\left\|\frac{1}{3^{A+B+3kI}k!(A+kI)!(B+kI)!}\right\|\right)^{\frac{1}{A+B+3kI}} \\ &= \limsup_{k\to\infty}\left(\left\|\frac{1}{3^{A+B+3kI}\sqrt{2\pi k}\left(\frac{k}{e}\right)^k\sqrt{2\pi(A+kI)}\left(\frac{A+kI}{e}\right)^{A+kI}\sqrt{2\pi(B+kI)}\left(\frac{B+kI}{e}\right)^{B+kI}}\right\|\right)^{\frac{1}{A+B+3kI}} \\ &\leqslant \limsup_{k\to\infty}\left(\frac{1}{k^{k+\frac{1}{2}}(A+kI)^{A+\left(k+\frac{1}{2}\right)I}(B+kI)^{B+\left(k+\frac{1}{2}\right)I}}\right)^{\frac{1}{A+B+3kI}} = 0. \end{aligned}$$

Therefore, the order and type of the Humbert matrix function is formulated as follows

$$\rho = \limsup_{k\to\infty} \frac{(A+B+3kI)\ln(A+B+3kI)}{\ln\left(\frac{1}{\|U_{A+B+3kI}\|}\right)}$$

$$= \limsup_{k\to\infty} \frac{(A+B+3kI)\ln(A+B+3kI)}{\ln(3^{A+B+3kI}k!(A+kI)!(B+kI)!)}$$

$$= \limsup_{k\to\infty} \frac{(A+B+3kI)\ln(A+B+3kI)}{\ln\left(\sqrt{2\pi k}\left(\frac{k}{e}\right)^k \sqrt{2\pi(A+kI)}\left(\frac{A+kI}{e}\right)^{A+kI}\sqrt{2\pi(B+kI)}\left(\frac{B+kI}{e}\right)^{B+kI}3^{A+B+3kI}\right)}$$

$$\leqslant \limsup_{k\to\infty} \frac{1}{\frac{k\ln k}{(A+B+3kI)\ln(A+B+3kI)} + \frac{(A+kI)\ln(A+kI)}{(A+B+3kI)\ln(A+B+3kI)} + \frac{(B+kI)\ln(B+kI)}{(A+B+3kI)\ln(A+B+3kI)}} = 1, \tag{2.4}$$

and

$$\tau = \frac{1}{e\rho}\limsup_{k\to\infty}(A+B+3kI)(\|U_{A+B+3kI}\|)^{\frac{\rho}{A+B+3kI}}$$

$$= \frac{1}{e}\limsup_{k\to\infty}(A+B+3kI)\left(\frac{1}{3^{A+B+3kI}k!(A+kI)!(B+kI)!}\right)^{\frac{1}{A+B+3kI}}$$

$$= \frac{1}{3e}\limsup_{k\to\infty}(A+B+3kI)$$

$$\times\left(\frac{1}{\sqrt{2\pi k}\left(\frac{k}{e}\right)^k\sqrt{2\pi(A+kI)}\left(\frac{A+kI}{e}\right)^{A+kI}\sqrt{2\pi(B+kI)}\left(\frac{B+kI}{e}\right)^{B+kI}}\right)^{\frac{1}{A+B+3kI}}$$

$$\leqslant \frac{1}{3}\limsup_{k\to\infty}(A+B+3kI)\left(\frac{1}{k^k(A+kI)^{(A+kI)}(B+kI)^{(B+kI)}}\right)^{\frac{1}{A+B+3kI}} = 1. \tag{2.5}$$

Summarizing, the following result has been established.

Theorem 2.1. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ such that $A+(k+1)I$ and $B+(k+1)I$ are invertible for every integer $k \geqslant -1$. Then the Humbert matrix function is an entire function and the order and type of the Humbert matrix function is equal one.*

3. An integral representation

In this section, we provide integral expressions of Humbert matrix function $J_{A,B}(z)$ by the following theorems:

Theorem 3.1. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ such that $\gamma(A) > \gamma(B) > -1$. Then for complex number z it follows that*

$$J_{A,A}(z) = 3\left(\frac{z}{3}\right)^{A-B}\Gamma^{-1}(A-B)\int_0^1 (1-t^3)^{A-B-I}t^{-(A-2(B+I))}J_{A,B}(zt)\,\mathrm{d}t, \tag{3.1}$$

and

$$J_{B,B}(z) = 3\left(\frac{z}{3}\right)^{B-A}\Gamma^{-1}(B-A)\int_0^1 (1-t^3)^{B-A-I}t^{-(B-2(A+I))}J_{A,B}(zt)\,\mathrm{d}t. \tag{3.2}$$

where $\gamma(B) > \gamma(A) > -1$.

Proof. Let $\mathbf{I} = \int_0^1 (1-t^3)^{A-B-I}t^{-(A-2(B+I))}J_{A,B}(zt)\,\mathrm{d}t$, then from the expression of Humbert matrix function, we have

$$J_{A,B}(zt) = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!}\left(\frac{zt}{3}\right)^{A+B+3kI},$$

the integral becomes

$$\mathbf{I} = \int_0^1 (1-t^3)^{A-B-I}t^{-(A-2(B-I))}$$
$$\times\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!}\left(\frac{zt}{3}\right)^{A+B+3kI}\mathrm{d}t$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!}\left(\frac{z}{3}\right)^{A+B+3kI}$$
$$\times\int_0^1 (1-t^3)^{A-B-I}t^{3B+2I+3kI}\,\mathrm{d}t.$$

Letting $t^3 = s$, then

$$\mathbf{I} = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!}\left(\frac{z}{3}\right)^{A+B+3kI}$$
$$\times\int_0^1 (1-s)^{A-B-I}s^{B+kI}\,\mathrm{d}s. \tag{3.3}$$

From the properties of Beta and Gamma matrix function in [9], we get

$$\int_0^1 (1-s)^{A-B-I}s^{B+kI}\,\mathrm{d}s = \Gamma(A-B)\Gamma(B+(k+1)I)\Gamma^{-1}(A+(k+1)I), \tag{3.4}$$

then

$$\mathbf{I} = \frac{\Gamma(A-B)\left(\frac{z}{3}\right)^{B-A}}{3}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(A+(k+1)I)}{k!}\left(\frac{z}{3}\right)^{2A+3kI}$$
$$= \frac{\Gamma(A-B)\left(\frac{z}{3}\right)^{B-A}}{3}J_{A,A}(z).$$

Thus,

$$J_{A,A}(z) = 3\left(\frac{z}{3}\right)^{A-B}\Gamma^{-1}(A-B)\int_0^1 (1-t^3)^{A-B-I}t^{-(A-2(B-I))}J_{A,B}(zt)\,\mathrm{d}t.$$

Another integral representation of $J_{A,B}(z)$ can be established starting from the formula in [10, p. 115, No. (5.10.5)] and Lemma 2 of [9, p. 209] we find that

$$\Gamma^{-1}(A+(k+1)I) = \frac{1}{2\pi i}\int_C \exp(s)s^{-(A+(k+1)I)}\,\mathrm{d}s, \tag{3.5}$$

and

$$\Gamma^{-1}(B+(k+1)I) = \frac{1}{2\pi i}\int_{C'} \exp(t)t^{-(B+(k+1)I)}\,\mathrm{d}t, \tag{3.6}$$

and substituting the above expression into the series expression of the Humbert matrix function given in (2.1), it follows that

$$J_{A,B}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{3}\right)^{A+B+3kI}}{k!}\frac{1}{(2\pi i)^2}\int_{C'}\int_C \exp(s+t)s^{-(A+(k+1)I)}t^{-(B+(k+1)I)}\,\mathrm{d}s\,\mathrm{d}t.$$

Interchanging the order of the integral and summation,

$$J_{A,B}(z) = \frac{\left(\frac{z}{3}\right)^{A+B}}{(2\pi i)^2}\int_{C'}\int_C \exp(s+t)s^{-(A+I)}t^{-(B+I)}\sum_{k=0}^{\infty}\frac{\left(\frac{-z^3}{27st}\right)^k}{k!}\,\mathrm{d}s\,\mathrm{d}t,$$

that is,

$$J_{A,B}(z) = \frac{\left(\frac{z}{3}\right)^{A+B}}{(2\pi i)^2} \int_{C'} \int_{C} \exp(s+t) s^{-(A+I)} t^{-(B+I)} \exp\left(\frac{-z^3}{27st}\right) \mathrm{d}s\, \mathrm{d}t,$$
$$= \frac{\left(\frac{z}{3}\right)^{A+B}}{(2\pi i)^2} \int_{C'} \int_{C} \exp\left(s+t-\frac{z^3}{27st}\right) s^{-(A+I)} t^{-(B+I)} \mathrm{d}s\, \mathrm{d}t.$$

Therefore, the following result has been established. □

Theorem 3.2. *Let A and B be two matrices in $\mathbb{C}^{N\times N}$. Then the Humbert matrix function for complex variable z satisfies the following integral*

$$J_{A,B}(z) = \frac{\left(\frac{z}{3}\right)^{A+B}}{(2\pi i)^2} \int_{C'} \int_{C} \exp\left(s+t-\frac{z^3}{27st}\right) s^{-(A+I)} t^{-(B+I)} \mathrm{d}s\, \mathrm{d}t. \tag{3.7}$$

4. Recurrence relations

Some recurrence relations are carried out on the Humbert matrix function. We obtain the following:

Theorem 4.1. *The Humbert matrix function $J_{A,B}(z)$ satisfies the following relations:*

(i) $\left(\frac{d}{z\,dz}\right)^n \left\{z^{-(A+B)} J_{A,B}(z)\right\} = (-1)^n z^{-(A+B)-nI}\ J_{A+nI,B+nI}(z)$,
(ii) $J'_{A,B}(z) = J_{A,B-I}(z) - \frac{(2B-A)}{z}\ J_{A,B}(z)$,
(iii) $J'_{A,B}(z) = J_{A-I,B}(z) - \frac{(2A-B)}{z}\ J_{A,B}(z)$,
(iv) $\frac{(3A)}{z}\ J_{A,B}(z) = J_{A+I,B+I}(z) + J_{A-I,B}(z)$,
(v) $\frac{(3B)}{z}\ J_{A,B}(z) = J_{A+I,B+I}(z) + J_{A,B-I}(z)$.

Proof.

(i) Applying mathematical induction for 1:

$$\left(\frac{d}{dz}\right)^n \left\{z^{-(A+B)} J_{A,B}(z)\right\}$$
$$= \frac{d}{dz}\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{3}\right)^{A+B+3kI} \Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!} (z)^{3kI}$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{1}{3}\right)^{A+B+3kI-I} \Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{(k-1)!} (z)^{3kI-I}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \left(\frac{1}{3}\right)^{A+B+3kI+2I} \Gamma^{-1}((A+I)+(k+1)I)\Gamma^{-1}((B+I)+(k+1)I)}{k!} (z)^{3kI+2I},$$

i.e.,

$$\left(\frac{d}{z\,dz}\right)\left\{z^{-(A+B)} J_{A,B}(z)\right\} = -z^{-(A+B)-I}\ J_{A+I,B+I}(z), \tag{4.1}$$

For $n = r-1$, we have

$$\left(\frac{d}{z\,dz}\right)^{r-1} \left\{z^{-(A+B)} J_{A,B}(z)\right\} = (-1)^{r-1} z^{-(A+B)-(r-1)I}\ J_{A+(r-1)I,B+(r-1)I}(z).$$

Then for $n = r$:

$$\left(\frac{d}{z\,dz}\right)^r \left\{z^{-(A+B)} J_{A,B}(z)\right\}$$
$$= \left(\frac{d}{z\,dz}\right)\left(\frac{d}{z\,dz}\right)^{r-1} \left\{z^{-(A+B)} J_{A,B}(z)\right\}$$
$$= \left(\frac{d}{z\,dz}\right)\left\{(-1)^{r-1} z^{-(A+B)-(r-1)I} J_{A+(r-1)I,B+(r-1)I}(z)\right\}$$
$$= (-1)^r z^{-(A+B)-\ rI}\ J_{A+rI,B+rI}(z). \tag{4.2}$$

Thus, the proof of relation (i) is completed.

(ii) In this case

$$\frac{d}{dz}\left(z^{2A-B} J_{A,B}(z)\right)$$
$$= \frac{d}{dz}\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{3}\right)^{A+B+3kI} \Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!} (z)^{3A+3kI}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{3}\right)^{A+B+3kI-I} \Gamma^{-1}(A+kI)\Gamma^{-1}(B+(k+1)I)}{k!} (z)^{3A+3kI-I}$$
$$= z^{2A-B} J_{A-I,B}(z).$$

By carrying out the differentiation of the product on the left-hand side, we have

$$(2A-B) z^{2A-B-I} J_{A,B}(z) + z^{2A-B} J'_{A,B}(z) = z^{2A-B} J_{A-I,B}(z).$$

Hence

$$J'_{A,B}(z) = J_{A-I,B}(z) - \frac{(2A-B)}{z}\ J_{A,B}(z). \tag{4.3}$$

(iii) We have

$$\frac{d}{dz}\left(z^{2B-BA} J_{A,B}(z)\right)$$
$$= \frac{d}{dz}\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{3}\right)^{A+B+3kI} \Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!} (z)^{3B+3kI}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{3}\right)^{A+B+3kI-I} \Gamma^{-1}(B+kI)\Gamma^{-1}(A+(k+1)I)}{k!} (z)^{3B+3kI-I}$$
$$= z^{2B-A} J_{A,B-I}(z). \tag{4.4}$$

By carrying out the differentiation of the product on the left-hand side, we have

$$(2B-A) z^{2B-A-I} J_{A,B}(z) + z^{2B-A} J'_{A,B}(z) = z^{2B-A} J_{A,B-I}(z).$$

Hence

$$J'_{A,B}(z) = J_{A,B-I}(z) - \frac{(2B-A)}{z}\ J_{A,B}(z). \tag{4.5}$$

(iv) From result (i), we have

$$J'_{A,B}(z) = \frac{(A+B)}{z}\ J_{A,B}(z) - J_{A+I,B+I}(z). \tag{4.6}$$

Subtracting (4.5) from result (ii) we obtain the required relationship.

(v) Subtracting (4.5) from result (iii) we obtain the required relationship.

Finally, we can find some properties related with differentiation of the Humbert matrix function with respect to indexes A and B in the forms

$$\frac{\partial}{\partial A}J_{A,B}=\frac{\partial}{\partial A}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!}\left(\frac{z}{3}\right)^{A+B+3kI}$$
$$=J_{A,B}\left[\ln\left(\frac{z}{3}\right)-\Gamma'(A+(k+1)I)\Gamma^{-1}(A+(k+1)I)\right]$$
$$=\ln\left(\frac{z}{3}\right)J_{A,B}-J_{A,B}\psi(A+(k+1)I), \tag{4.7}$$

and

$$\frac{\partial}{\partial B}J_{A,B}=\frac{\partial}{\partial B}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(k+1)I)}{k!}\left(\frac{z}{3}\right)^{A+B+3kI}$$
$$=J_{A,B}\left[\ln\left(\frac{z}{3}\right)-\Gamma'(B+(k+1)I)\Gamma^{-1}(B+(k+1)I)\right]$$
$$=\ln\left(\frac{z}{3}\right)J_{A,B}-J_{A,B}\psi(B+(k+1)I). \quad \square \tag{4.8}$$

5. Humbert matrix differential equation

We know a matrix differential equation satisfied by any ${}_0F_2$ by specializing the result in [11]. The matrix differential equation:

$$[\theta(\theta I+A)(\theta I+B)-y]U=0; \quad \theta=y\frac{d}{dy}, \tag{5.1}$$

has $U={}_0F_2(-,A+I,B+I,y)$ as one solution. Eq. (5.1) can also be written

$$y^2\frac{d^3U}{dy^3}+(A+B+3I)y\frac{d^2U}{dy^2}+(A+I)(B+I)\frac{dU}{dy}-U=0, \tag{5.2}$$

we now put $y=-\frac{z^3}{27}$, therefore

$$\frac{dU}{dy}=-\frac{9}{z^2}\cdot\frac{dU}{dz},$$
$$\frac{d^2U}{dy^2}=\frac{81}{z^4}\cdot\frac{d^2U}{dz^2}-\frac{162}{z^5}\cdot\frac{dU}{dz},$$

and

$$\frac{d^3U}{dy^3}=-\frac{729}{z^6}\cdot\frac{d^3U}{dz^3}+\frac{428}{z^7}\cdot\frac{d^2U}{dz^2}-\frac{7290}{z^8}\cdot\frac{dU}{dz},$$

in (5.2) to obtain

$$z^2\frac{d^3U}{dz^3}+(3A+3B+3I)z\frac{d^2U}{dz^2}-[6A+6B-2I+z^2(A+I)(B+I)]\frac{dU}{dz}+z^2U=0, \tag{5.3}$$

in which primes denote differentiations with respect to z, one solution (5.3) is

$$U={}_0F_2\left(-,\ A+I,\ B+I,\ -\frac{z^3}{27}\right).$$

We seek an equation satisfied by $U=z^{A+B}W$. Hence in (5.3) we now put $W=z^{-(A+B)}U$ and arrive at the matrix differential equation.

$$z^3\frac{d^3W}{dz^3}+3z^2\frac{d^2W}{dz^2}$$
$$-\left[3(A+B)(A+B+I)+(6A+6B-2I)+z^2(A+I)(B+I)\right]z\frac{dW}{dz}$$
$$+\left[(A+B)(A+B+I)(2A+2B+I)+(A+B)(6A+6B-2I)\right.$$
$$\left.+z^2(A+B)(A+I)(B+I)+z^3\right]W=0 \tag{5.4}$$

of which one solution is $W=z^{A+B}{}_0F_2\left(-,\ A+I,\ B+I,\ -\frac{z^3}{27}\right)$. Eq. (5.4) is Humbert matrix differential equation.

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