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ORIGINAL ARTICLE

Direct and inverse theorems for Bernstein polynomials with inner singularities

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Abstract We introduce a new type of Bernstein polynomials, which can be used to approximate the functions with inner singularities. The direct and inverse results of the weighted approximation of this new type of combinations are obtained.

1. Introduction

The set of all continuous functions, defined on the interval I, is denoted by $C(I)$. For any $f \in C([0,1])$, the corresponding Bernstein polynomials are defined as follows:

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \ k = 0,1,2,\ldots,n, \ x \in [0,1].$$

Let $\bar{w}(x) = |x-\xi|^{\alpha}$, $0 < \xi < 1$, $\alpha > 0$ and $C_{\bar{w}} := \{f \in C([0,1] \setminus \{\xi\}) : \lim_{x \to \xi} (\bar{w}f)(x) = 0\}$. The norm in $C_{\bar{w}}$ is defined by $\|f\|_{C_{\bar{w}}} := \|\bar{w}f\| = \sup_{0 \leqslant x \leqslant 1} |(\bar{w}f)(x)|$. Define

$$W_{\phi}^2 := \{f \in C_{\bar{w}} : f' \in A.C.((0,1)), \ \|\bar{w}\phi^2 f''\| < \infty\},$$
$$W_{\bar{w},\lambda}^2 := \{f \in C_{\bar{w}} : f' \in A.C.((0,1)), \ \|\bar{w}\varphi^{2\lambda} f''\| < \infty\}.$$

For $f \in C_{\bar{w}}$, the weighted modulus of smoothness is defined by

$$\omega_{\phi}^2(f,t)_{\bar{w}} := \sup_{0 < h \leqslant t} \sup_{0 \leqslant x \leqslant 1} |\bar{w}(x) \Delta_{h\phi(x)}^2 f(x)|,$$

where

$$\Delta_{h\phi}^2 f(x) = f(x + h\phi(x)) - 2f(x) + f(x - h\phi(x)),$$

and $\varphi(x) = \sqrt{x(1-x)}$, $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$.

Recently Felten showed the following two theorems in [1]:

Theorem A. *Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\phi : [0,1] \to R$, $\phi \neq 0$ be an admissible step-weight function of the Ditzian–Totik modulus of smoothness [4] such that ϕ^2 and φ^2/ϕ^2 are concave. Then, for $f \in C[0,1]$ and $0 < \alpha < 2$,*

$$|B_n(f,x) - f(x)| \leqslant \omega_{\phi}^2\left(f, n^{-1/2} \frac{\varphi(x)}{\phi(x)}\right).$$

Theorem B. *Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\phi : [0,1] \to R$, $\phi \neq 0$ be an admissible step-weight function of the Ditzian–Totik*

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modulus of smoothness such that ϕ^2 *and* φ^2/ϕ^2 *are concave. Then, for* $f \in C[0,1]$ *and* $0 < \alpha < 2$,

$$|B_n(f,x) - f(x)| = O\left(\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^{\alpha}\right)$$

implies $\omega_\phi^2(f,t) = O(t^\alpha)$.

Approximation properties of Bernstein polynomials have been studied very well [2–5]. In order to approximate the functions with singularities, Della Vecchia et al. [3] introduced some kinds of modified Bernstein polynomials. Throughout the paper, C denotes a positive constant independent of n and x, which may be different in different cases.

Let ϕ: $[0,1] \to R$, $\phi \neq 0$ be an admissible step-weight function of the Ditzian–Totik modulus of smoothness, that is, ϕ satisfies the following conditions:

(I) For every proper subinterval $[a,b] \subseteq [0,1]$ there exists a constant $C_1 \equiv C(a,b) > 0$ such that $C_1^{-1} \leqslant \phi(x) \leqslant C_1$ for $x \in [a,b]$.

(II) There are two numbers $\beta(0) \geqslant 0$ and $\beta(1) \geqslant 0$ for which

$$\phi(x) \sim \begin{cases} x^{\beta(0)}, & \text{as } x \to 0+, \\ (1-x)^{\beta(1)}, & \text{as } x \to 1-. \end{cases}$$

($X \sim Y$ means $C^{-1}Y \leqslant X \leqslant CY$for some C).

Combining conditions (**I**) and (**II**) on ϕ, we can deduce that

$$C^{-1}\phi_2(x) \leqslant \phi(x) \leqslant C\phi_2(x), \quad x \in [0,1],$$

where $\phi_2(x) = x^{\beta(0)}(1-x)^{\beta(1)}$.

2. The main results

Let

$$\psi(x) = \begin{cases} 10x^3 - 15x^4 + 6x^5, & 0 < x < 1, \\ 0, & x \leqslant 0, \\ 1, & x \geqslant 1. \end{cases}$$

Obviously, ψ is non-decreasing on the real axis, $\psi \in C^2((-\infty, +\infty))$, $\psi^{(i)}(0) = 0$, $i = 0, 1, 2$. $\psi^{(i)}(1) = 0$, $i = 1,2$ and $\psi(1) = 1$. Further, let

$$x_1 = \frac{[n\xi - 2\sqrt{n}]}{n}, \quad x_2 = \frac{[n\xi - \sqrt{n}]}{n}, \quad x_3 = \frac{[n\xi + \sqrt{n}]}{n},$$
$$x_4 = \frac{[n\xi + 2\sqrt{n}]}{n},$$

and

$$\bar{\psi}_1(x) = \psi\left(\frac{x - x_1}{x_2 - x_1}\right), \quad \bar{\psi}_2(x) = \psi\left(\frac{x - x_3}{x_4 - x_3}\right).$$

Consider

$$P(x) := \frac{x - x_4}{x_1 - x_4}f(x_1) + \frac{x_1 - x}{x_1 - x_4}f(x_4),$$

the linear function joining the points $(x_1,f(x_1))$ and $(x_4,f(x_4))$. And let

$$\overline{F}_n(f,x) := \overline{F}_n(x) = f(x)(1 - \bar{\psi}_1(x) + \bar{\psi}_2(x)) + \bar{\psi}_1(x)(1 - \bar{\psi}_2(x))P(x).$$

From the above definitions it follows that

$$\overline{F}_n(f,x) = \begin{cases} f(x), & x \in [0,x_1] \cup [x_4,1], \\ f(x)(1 - \bar{\psi}_1(x)) + \bar{\psi}_1(x)P(x), & x \in [x_1,x_2], \\ P(x), & x \in [x_2,x_3], \\ P(x)(1 - \bar{\psi}_2(x)) + \bar{\psi}_2(x)f(x), & x \in [x_3,x_4]. \end{cases}$$

Evidently, $\overline{F}_n$ is a positive linear polynomials which depends on the functions values $f(k/n)$, $0 \leqslant k/n \leqslant x_2$ or $x_3 \leqslant k/n \leqslant 1$, it reproduces linear functions, and $\overline{F}_n \in C^2([0,1])$ provided $f \in W_\phi^2$. Now for every $f \in C_{\bar{w}}$ define the Bernstein type polynomials

$$\begin{aligned}\overline{B}_n(f,x) &:= B_n(\overline{F}_n(f),x) \\ &= \sum_{k/n \in [0,x_1] \cup [x_4,1]} p_{n,k}(x) f\left(\frac{k}{n}\right) + \sum_{x_2 < k/n < x_3} p_{n,k}(x) P\left(\frac{k}{n}\right) \\ &\quad + \sum_{x_1 < k/n < x_2} p_{n,k}(x)\left\{f\left(\frac{k}{n}\right)\left(1 - \bar{\psi}_1\left(\frac{k}{n}\right)\right) + \bar{\psi}_1\left(\frac{k}{n}\right)P\left(\frac{k}{n}\right)\right\} \\ &\quad + \sum_{x_3 < k/n < x_4} p_{n,k}(x)\left\{P\left(\frac{k}{n}\right)\left(1 - \bar{\psi}_2\left(\frac{k}{n}\right)\right) + \bar{\psi}_2\left(\frac{k}{n}\right)f\left(\frac{k}{n}\right)\right\}.\end{aligned} \tag{2.1}$$

Obviously, $\overline{B}_n$ is a positive linear polynomials, $\overline{B}_n(f)$ is a polynomial of degree at most n, it preserves linear functions, and depends only on the function values $f(k/n)$, $k/n \in [0,x_2] \cup [x_3,1]$. Now we state our main results as follows:

Theorem 1. *If* $\alpha > 0$, *for any* $f \in C_{\bar{w}}$, *we have*

$$\left\|\bar{w}\overline{B}_n''(f)\right\| \leqslant Cn^2\|\bar{w}f\|. \tag{2.2}$$

Theorem 2. *For any* $\alpha > 0$, $\min\{\beta(0),\beta(1)\} \geqslant \frac{1}{2}$, $0 < \xi < 1$, *we have*

$$|\bar{w}(x)\phi^2(x)\overline{B}_n''(f,x)| \leqslant \begin{cases} Cn\|\bar{w}f\|, & f \in C_{\bar{w}}, \\ C\|\bar{w}\phi^2 f''\|, & f \in W_\phi^2. \end{cases} \tag{2.3}$$

Theorem 3. *For* $f \in C_{\bar{w}}$, $0 < \xi < 1$, $\alpha > 0$, $\min\{\beta(0),\beta(1)\} \geqslant \frac{1}{2}$, $\alpha_0 \in (0,2)$, *we have*

$$\bar{w}(x)|f(x) - \overline{B}_n(f,x)| = O\left(\left(n^{-\frac{1}{2}}\phi^{-1}(x)\delta_n(x)\right)^{\alpha_0}\right) \Longleftrightarrow \omega_\phi^2(f,t)_{\bar{w}} = O(t^{\alpha_0}).$$

3. Lemmas

Lemma 1. [7] *For any non-negative real u and v, we have*

$$\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^{-u}\left(1 - \frac{k}{n}\right)^{-v} p_{n,k}(x) \leqslant Cx^{-u}(1-x)^{-v}. \tag{3.1}$$

Lemma 2. [3] *For any* $\alpha \geq 0$, $f \in C_{\bar{w}}$, *we have*

$$\|\bar{w}\overline{B}_n(f)\| \leqslant C\|\bar{w}f\|. \tag{3.2}$$

Lemma 3. [6] *Let* $\min\{\beta(0),\beta(1)\} \geqslant \frac{1}{2}$, *then for* $0 < t < \frac{1}{4}$ *and* $t < x < 1 - t$, *we have*

$$\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\phi^{-2}\left(x+\sum_{k=1}^{2}u_k\right)du_1\ du_2 \leqslant Ct^2\phi^{-2}(x). \tag{3.3}$$

Proof From the definition of $\phi(x)$, it is enough to prove (3.3) for $t<x\leqslant\frac{1}{2}$ since the proof for $\frac{1}{2}<x<1-t$ is very similar. Obviously, we have

$$\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{1}{x+\sum_{k=1}^{2}u_k}\ du_1\ du_2 \leqslant Ct^2x^{-1}.$$

Therefore, by the Hölder inequality, we have

$$\begin{aligned}&\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\phi^{-2}\left(x+\sum_{k=1}^{2}u_k\right)du_1\ du_2\\ &\leqslant C(16/7)^{2\beta(1)}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{1}{\left(x+\sum_{k=1}^{2}u_k\right)^{2\beta(0)}}\ du_1\ du_2\\ &\leqslant C(16/7)^{2\beta(1)}t^{2(1-2\beta(0))}\left(\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{1}{x+\sum_{k=1}^{2}u_k}\ du_1\ du_2\right)^{2\beta(0)}\\ &\leqslant C(16/7)^{2\beta(1)}t^2x^{-2\beta(0)}.\quad\square\end{aligned}$$

Lemma 4 [3]. *If $\gamma\in R$, then*

$$\sum_{k=0}^{n}p_{n,k}(x)|k-nx|^{\gamma} \leqslant Cn^{\frac{\gamma}{2}}\varphi^{\gamma}(x). \tag{3.4}$$

Lemma 5. *Let $A_n(x):=\bar{w}(x)\sum_{|k-n\xi|\leqslant\sqrt{n}}p_{n,k}(x)$. Then $A_n(x)\leqslant Cn^{-\frac{\alpha}{2}}$ for $0<\xi<1$ and $\alpha>0$.*

Proof 2. If $|x-\xi|\leqslant\frac{3}{\sqrt{n}}$, then the statement is trivial. Hence assume $0\leqslant x\leqslant\xi-\frac{3}{\sqrt{n}}$ (the case $\xi+\frac{3}{\sqrt{n}}\leqslant x\leqslant 1$ can be treated similarly). Then for a fixed x the maximum of $p_{n,k}(x)$ is attained for $k=k_n:=[n\xi-\sqrt{n}]$. By using Stirling's formula, we get

$$\begin{aligned}p_{n,k_n}(x) &\leqslant C\frac{\left(\frac{n}{e}\right)^n\sqrt{n}x^{k_n}(1-x)^{n-k_n}}{\left(\frac{k_n}{e}\right)^{k_n}\sqrt{k_n}\left(\frac{n-k_n}{e}\right)^{n-k_n}\sqrt{n-k_n}}\\ &\leqslant\frac{C}{\sqrt{n}}\left(\frac{nx}{k_n}\right)^{k_n}\left(\frac{n(1-x)}{n-k_n}\right)^{n-k_n}\\ &=\frac{C}{\sqrt{n}}\left(1-\frac{k_n-nx}{k_n}\right)^{k_n}\left(1+\frac{k_n-nx}{n-k_n}\right)^{n-k_n}.\end{aligned}$$

Now from the inequalities

$$k_n-nx=\left[n\xi-\sqrt{n}\right]-nx>n(\xi-x)-\sqrt{n}-1\geqslant\frac{1}{2}n(\xi-x),$$

and

$$1-u\leqslant e^{-u-\frac{1}{2}u^2},\ 1+u\leqslant e^{u},\ u\geqslant 0,$$

it follows that the second inequality is valid. To prove the first one we consider the function $\lambda(u)=e^{-u-\frac{1}{2}u^2}+u-1$. Here $\lambda(0)=0,\ \lambda'(u)=-(1+u)e^{-u-\frac{1}{2}u^2}+1,\ \lambda'(0)=0,\ \lambda''(u)=u(u+2)e^{-u-\frac{1}{2}u^2}\geqslant 0$, whence $\lambda(u)\geqslant 0$ for $u\geqslant 0$. Hence

$$\begin{aligned}p_{n,k_n}(x) &\leqslant\frac{C}{\sqrt{n}}\\ &\quad\times\exp\left\{k_n\left[-\frac{k_n-nx}{k_n}-\frac{1}{2}\left(\frac{k_n-nx}{k_n}\right)^2\right]+k_n-nx\right\}\\ &=\frac{C}{\sqrt{n}}\exp\left\{-\frac{(k_n-nx)^2}{2k_n}\right\}\leqslant e^{-Cn(\xi-x)^2}.\end{aligned}$$

Thus $A_n(x)\leqslant C(\xi-x)^{\alpha}e^{-Cn(\xi-x)^2}$. An easy calculation shows that here the maximum is attained when $\xi-x=\frac{C}{\sqrt{n}}$ and the lemma follows. $\square$

Lemma 6. *For $0<\xi<1$, $\alpha,\beta>0$, we have*

$$\bar{w}(x)\sum_{|k-n\xi|\leqslant\sqrt{n}}|k-nx|^{\beta}p_{n,k}(x)\leqslant Cn^{\frac{\beta-\alpha}{2}}\varphi^{\beta}(x). \tag{3.5}$$

Proof 3. By (3.4) and the Lemma 5, we have

$$\begin{aligned}&\bar{w}(x)^{\frac{1}{2n}}\left(\bar{w}(x)\sum_{|k-n\xi|\leqslant\sqrt{n}}p_{n,k}(x)\right)^{\frac{2n-1}{2n}}\left(\sum_{|k-n\xi|\leqslant\sqrt{n}}|k-nx|^{2n\beta}p_{n,k}(x)\right)^{\frac{1}{2n}}\\ &\leqslant Cn^{\frac{\beta-\alpha}{2}}\varphi^{\beta}(x).\qquad\square\end{aligned}$$

Lemma 7. *For any $\alpha>0$, $f\in W_{\phi}^2$, $\min\{\beta(0),\beta(1)\}\geqslant\frac{1}{2}$, we have*

$$\bar{w}(x)|f(x)-P(f,x)|_{[x_1,x_4]}\leqslant C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^2\|\bar{w}\phi^2f''\|. \tag{3.6}$$

Proof If $x\in[x_1,x_4]$, for any $f\in W_{\phi}^2$, we have

$$f(x_1)=f(x)+f'(x)(x_1-x)+\int_{x_1}^{x}(t-x_1)f''(t)dt,$$

$$f(x_4)=f(x)+f'(x)(x_4-x)+\int_{x_4}^{x}(t-x_4)f''(t)dt,$$

$$\delta_n(x)\sim\frac{1}{\sqrt{n}},\ n=1,2,\ldots.$$

So

$$\begin{aligned}\bar{w}(x)|f(x)-P(f,x)| &\leqslant\bar{w}(x)\left|\frac{x-x_4}{x_1-x_4}\right|\int_{x_1}^{x}|(t-x_1)f''(t)|dt\\ &\quad+\bar{w}(x)|\frac{x_1-x}{x_1-x_4}|\int_{x_4}^{x}|(t-x_4)f''(t)|dt\\ &:=I_1+I_2.\end{aligned}$$

Whence t between x_1 and x, we have $\frac{|t-x_1|}{\bar{w}(t)}\leqslant\frac{|x-x_1|}{\bar{w}(x)}$, then

$$\begin{aligned}I_1 &\leqslant Cn^{\frac{1}{2}}\|\bar{w}\phi^2f''\||(x-x_1)(x-x_4)|\int_{x_1}^{x}\phi^{-2}(t)dt\\ &\leqslant C\left(\frac{\varphi(x)}{\sqrt{n}\phi(x)}\right)^2\|\bar{w}\phi^2f''\|\leqslant C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^2\|\bar{w}\phi^2f''\|.\end{aligned}$$

Analogously, we have

$$I_2\leqslant C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^2\|\bar{w}\phi^2f''\|.$$

Now the lemma follows from combining these results together. □

Lemma 8. *If* $f \in W_\phi^2$, $\min\{\beta(0), \beta(1)\} \geqslant \frac{1}{2}$, *then*

$$\left\|\bar{w}\phi^2\overline{F}_n''\right\| = O(\|\bar{w}\phi^2 f''\|). \tag{3.7}$$

Proof Again, it is sufficient to estimate $\left(\bar{w}\phi^2\overline{F}_n''\right)(x)$ for $x \in [x_3, x_4]$, and the same as $x \in [x_1, x_2]$. For $x \in [x_2, x_3]$, $\overline{F}_n''(x) = 0$, while for $x \in [0, x_1] \cup [x_4, 1]$, $\overline{F}_n(x) = f(x)$. Thus for $x \in [x_3, x_4]$, then $\overline{F}_n(x) = P(x) + \bar{\psi}_2(x)(f(x) - P(x))$ and

$$\begin{aligned}
\overline{F}_n''(x) &= n\psi''\left[n^{\frac{1}{2}}(x - x_3)\right](f(x) - P(x)) \\
&\quad + 2n^{\frac{1}{2}}\psi'\left[n^{\frac{1}{2}}(x - x_3)\right](f(x) - P(x))' \\
&\quad + \psi\left[n^{\frac{1}{2}}(x - x_3)\right]f''(x) \\
&:= I_1(x) + I_2(x) + I_3(x).
\end{aligned}$$

From the proof of Lemma 7, we have

$$\begin{aligned}
|\bar{w}(x)\phi^2(x)I_1(x)| &= O(n\phi^2(x)\psi''\left[n^{\frac{1}{2}}(x - x_3)\right]\bar{w}(x)(f(x) - P(x))) \\
&= O\left(n\phi^2(x) \cdot \left(\frac{\varphi(x)}{\sqrt{n}\phi(x)}\right)^2 \|\bar{w}\phi^2 f''\|\right) \\
&= O(\|\bar{w}\phi^2 f''\|).
\end{aligned}$$

For $I_3(x)$, it is obvious that

$$|\bar{w}(x)\phi^2(x)I_3(x)| = O(\|\bar{w}\phi^2 f''\|).$$

Finally

$$\begin{aligned}
|\bar{w}(x)\phi^2(x)I_2(x)| &= O\left(n^{\frac{1}{2}}\bar{w}(x)\phi^2(x)|f'(x) - P'(x)|\right) \\
&= O\left(n^{\frac{1}{2}}\bar{w}(x)\phi^2(x)\left|f'(x) - n^{\frac{1}{2}}\int_{x_1}^{x_4} f'(t)dt\right|\right) \\
&= O\left(n^{\frac{1}{2}}\bar{w}(x)\phi^2(x)\left|n^{\frac{1}{2}}\int_{x_1}^{x_4}\int_t^x f''(u)du\, dt\right|\right) \\
&= O\left(n^{\frac{1}{2}}\bar{w}(x)\phi^2(x)\left|\int_{x_1}^{x_4} f''(u)du\right|\right) \\
&= O(\|\bar{w}\phi^2 f''\|). \qquad \square
\end{aligned}$$

4. Proof of theorem

4.1. Proof of Theorem 1

Case 1. If $f \in C_{\bar{w}}$, when $x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$, by [2], we have

$$\begin{aligned}
\left|\bar{w}(x)\overline{B}_n''(f, x)\right| &\leqslant n\varphi^{-2}(x)\bar{w}(x)\left|\overline{B}_n(f, x)\right| \\
&\quad + \bar{w}(x)\varphi^{-4}(x)\sum_{k=0}^n p_{n,k}(x)|k \\
&\quad - nx|\left|\overline{F}_n\left(\frac{k}{n}\right)\right| + \bar{w}(x)\varphi^{-4}(x)\sum_{k=0}^n p_{n,k}(x) \\
&\quad \times (k - nx)^2\left|\overline{F}_n\left(\frac{k}{n}\right)\right| \\
&:= A_1 + A_2 + A_3.
\end{aligned} \tag{4.1}$$

By (3.2), we have

$$A_1(x) = n\varphi^{-2}(x)\bar{w}(x)\left|\overline{B}_n(f, x)\right| \leqslant Cn^2\|\bar{w}f\|. \tag{4.2}$$

and

$$\begin{aligned}
A_2 &= \bar{w}(x)\varphi^{-4}(x)\left[\sum_{k/n \in A}\left|k - nx\right|\left|\overline{F}_n\left(\frac{k}{n}\right)\right|p_{n,k}(x)\right. \\
&\quad \left. + \sum_{x_2 \leqslant k/n \leqslant x_3}\left|k - nx\right|\left|P\left(\frac{k}{n}\right)\right|p_{n,k}(x)\right] := \sigma_1 + \sigma_2.
\end{aligned}$$

thereof A:=$[0, x_2] \cup [x_3, 1]$. If $\frac{k}{n} \in A$, when $\frac{\bar{w}(x)}{\bar{w}\left(\frac{k}{n}\right)} \leqslant C(1 + n^{-\frac{\alpha}{2}}|k - nx|^\alpha)$, we have $|k - n\xi| \geqslant \frac{\sqrt{n}}{2}$, by (3.4), then

$$\begin{aligned}
\sigma_1 &\leqslant C\|\bar{w}f\|\varphi^{-4}(x)\sum_{k=0}^n p_{n,k}(x)|k - nx|[1 + n^{-\frac{\alpha}{2}}|k - nx|^\alpha] \\
&= C\|\bar{w}f\|\varphi^{-4}(x)\sum_{k=0}^n p_{n,k}(x)|k - nx| \\
&\quad + Cn^{-\frac{\alpha}{2}}\|\bar{w}f\|\varphi^{-4}(x)\sum_{k=0}^n p_{n,k}(x)|k - nx|^{1+\alpha} \\
&\leqslant Cn^{\frac{1}{2}}\varphi^{-3}(x)\|\bar{w}f\| + Cn^{\frac{1}{2}}\varphi^{-3+\alpha}(x)\|\bar{w}f\| \leqslant Cn^2\|\bar{w}f\|.
\end{aligned}$$

For σ_2, P is a linear function. We note $\left|P\left(\frac{k}{n}\right)\right| \leqslant max(|P(x_1)|, |P(x_4)|) := P(a)$. If $x \in [x_1, x_4]$, we have $\bar{w}(x) \leqslant \bar{w}(a)$. So, if $x \in [x_1, x_4]$, by (3.4), then

$$\sigma_2 \leqslant C\bar{w}(a)P(a)\varphi^{-4}(x)\sum_{k=0}^n p_{n,k}(x)|k - nx| \leqslant Cn^2\|\bar{w}f\|.$$

If $x \notin [x_1, x_4]$, then $\bar{w}(a) > n^{-\frac{\alpha}{2}}$, by (3.5), we have

$$\begin{aligned}
\sigma_2 &\leqslant C\bar{w}(x)\varphi^{-4}(x)\sum_{x_2 \leqslant k/n \leqslant x_3}|P(a)||(k - nx)|p_{n,k}(x) \\
&\leqslant Cn^{\frac{\alpha}{2}}\|\bar{w}f\|\varphi^{-4}(x)\bar{w}(x)\sum_{x_2 \leqslant k/n \leqslant x_3}|k - nx|p_{n,k}(x) \\
&\leqslant Cn^2\|\bar{w}f\|.
\end{aligned}$$

So

$$A_2 \leqslant Cn^2\|\bar{w}f\|. \tag{4.3}$$

Similarly

$$A_3 \leqslant Cn^2\|\bar{w}f\|. \tag{4.4}$$

It follows from combining with (4.1)–(4.4) that the inequality is proved.

Case 2. When $x \in \left[0, \frac{1}{n}\right]$ (The same as $x \in \left[1 - \frac{1}{n}, 1\right]$), by [4], then

$$\overline{B}_n''(f, x) = n(n-1)\sum_{k=0}^{n-2}\overrightarrow{\Delta}_{\frac{1}{n}}^2\overline{F}_n(k/n)p_{n-2,k}(x).$$

We have

$$\begin{aligned}
\left|\bar{w}(x)\overline{B}_n''(f, x)\right| &\leqslant Cn^2\bar{w}(x)\sum_{k=0}^{n-2}\left|\overrightarrow{\Delta}_{\frac{1}{n}}^2\overline{F}_n(k/n)\right|p_{n-2,k}(x) \\
&= Cn^2\bar{w}(x)\left[\sum_{k/n \in A}p_{n-2,k}(x)\left|\overrightarrow{\Delta}_{\frac{1}{n}}^2\overline{F}_n(k/n)\right|\right. \\
&\quad \left. + \sum_{x_2 \leqslant k/n \leqslant x_3}p_{n-2,k}(x)\left|\overrightarrow{\Delta}_{\frac{1}{n}}^2 P(k/n)\right|\right].
\end{aligned}$$

We can deal with it in accordance with Case 1, and prove it immediately, then the theorem is done. □

4.2. Proof of Theorem 2

(1) We prove the first inequality of Theorem 2.
Case 1. If $0 \leqslant \varphi(x) \leqslant \frac{1}{\sqrt{n}}$, by (2.2), we have

$$|\bar{w}(x)\phi^2(x)\overline{B}_n''(f,x)| = \varphi^2(x)\cdot\frac{\phi^2(x)}{\varphi^2(x)}|\bar{w}(x)\overline{B}_n''(f,x)| \leqslant Cn\|\bar{w}f\|.$$

Case 2. If $\varphi(x) > \frac{1}{\sqrt{n}}$, by [4], we have

$$\overline{B}_n''(f,x) = B_n''(\overline{F}_n,x) = (\varphi^2(x))^{-1}\sum_{i=0}^{2}Q_i(x,n)n^i \times\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^i\overline{F}_n\left(\frac{k}{n}\right)p_{n,k}(x),\quad (\varphi^2(x))^{-1}Q_i(x,n)n^i \leqslant C[n/\varphi^2(x)]^{1+i/2}.$$

So

$$\begin{aligned}&\left|\bar{w}(x)\phi^2(x)\overline{B}_n''(f,x)\right|\\ &\leqslant C\bar{w}(x)\phi^2(x)\sum_{i=0}^{2}\left(\frac{n}{\varphi^2(x)}\right)^{1+i/2}\sum_{k=0}^{n}\left|x-\frac{k}{n}\right|^i\left|\overline{F}_n\left(\frac{k}{n}\right)\right|p_{n,k}(x)\\ &= C\bar{w}(x)\phi^2(x)\sum_{i=0}^{2}\left(\frac{n}{\varphi^2(x)}\right)^{1+i/2}\sum_{k/n\in A}\left|x-\frac{k}{n}\right|^i\left|\overline{F}_n\left(\frac{k}{n}\right)\right|p_{n,k}(x)\\ &\quad + C\bar{w}(x)\phi^2(x)\sum_{i=0}^{2}\left(\frac{n}{\varphi^2(x)}\right)^{1+i/2}\\ &\quad\times\sum_{x_2\leqslant k/n\leqslant x_3}\left|x-\frac{k}{n}\right|^i\left|P\left(\frac{k}{n}\right)\right|p_{n,k}(x) := \sigma_1+\sigma_2.\end{aligned}$$

where $A{:=}[0,x_2]\cup[x_3,1]$. Working as in the proof of Theorem 1, We can get $\sigma_1 \leqslant Cn\|\bar{w}f\|$, $\sigma_2 \leqslant Cn\|\bar{w}f\|$. By bringing these facts together, we can immediately get the first inequality of Theorem B.

(2) If $f\in W_\phi^2$, by (2.1), then

$$\begin{aligned}\left|\bar{w}(x)\phi^2(x)\overline{B}_n''(f,x)\right| &\leqslant n^2\bar{w}(x)\phi^2(x)\sum_{k=0}^{n-2}\left|\overrightarrow{\Delta}_{\frac{1}{n}}^2\overline{F}_n\left(\frac{k}{n}\right)\right|p_{n-2,k}(x)\\ &= n^2\bar{w}(x)\phi^2(x)\sum_{k=1}^{n-3}\left|\overrightarrow{\Delta}_{\frac{1}{n}}^2\overline{F}_n\left(\frac{k}{n}\right)\right|p_{n-2,k}(x)\\ &\quad + n^2\bar{w}(x)\phi^2(x)\left|\overrightarrow{\Delta}_{\frac{1}{n}}^2\overline{F}_n(0)\right|p_{n-2,0}(x)\\ &\quad + n^2\bar{w}(x)\phi^2(x)\left|\overrightarrow{\Delta}_{\frac{1}{n}}^2\overline{F}_n\left(\frac{n-2}{n}\right)\right|p_{n-2,n-2}(x)\\ &:= I_1+I_2+I_3.\end{aligned}\tag{4.5}$$

By [4], if $0 < k < n-2$, we have

$$\left|\overrightarrow{\Delta}_{\frac{1}{n}}^2\overline{F}_n\left(\frac{k}{n}\right)\right| \leqslant Cn^{-1}\int_0^{\frac{2}{n}}\left|\overline{F}_n''\left(\frac{k}{n}+u\right)\right|du,\tag{4.6}$$

If $k = 0$, we have

$$\left|\overrightarrow{\Delta}_{\frac{1}{n}}^2\overline{F}_n(0)\right| \leqslant C\int_0^{\frac{2}{n}}u\left|\overline{F}_n''(u)\right|du,\tag{4.7}$$

Similarly

$$\left|\overrightarrow{\Delta}_{\frac{1}{n}}^2\overline{F}_n\left(\frac{n-2}{n}\right)\right| \leqslant Cn^{-1}\int_{1-\frac{2}{n}}^{1}(1-u)\left|\overline{F}_n''(u)\right|du.\tag{4.8}$$

By (4.6), then

$$\begin{aligned}I_1 &\leqslant Cn\bar{w}(x)\phi^2(x)\sum_{k=1}^{n-3}\int_0^{\frac{2}{n}}\left|\overline{F}_n''\left(\frac{k}{n}+u\right)\right|du p_{n-2,k}(x)\\ &= Cn\bar{w}(x)\phi^2(x)\sum_{k/n\in A}\int_0^{\frac{2}{n}}\left|\overline{F}_n''\left(\frac{k}{n}+u\right)\right|du p_{n-2,k}(x)\\ &\quad + Cn\bar{w}(x)\phi^2(x)\sum_{x_2\leqslant k/n\leqslant x_3}\int_0^{\frac{2}{n}}\left|P''\left(\frac{k}{n}+u\right)\right|du p_{n-2,k}(x)\\ &:= T_1+T_2.\end{aligned}$$

where $A{:=}[0,x_2]\cup[x_3,1]$, P is a linear function. If $k/n\in A$, when $\frac{\bar{w}(x)}{\bar{w}(k/n)} \leqslant C(1+n^{-\frac{\alpha}{2}}|k-nx|^\alpha)$, we have $|k-n\xi| \geqslant \frac{\sqrt{n}}{2}$, by (3.1), (3.4) and (3.7), then

$$\begin{aligned}T_1 &\leqslant C\bar{w}(x)\phi^2(x)\left\|\bar{w}\phi^2\overline{F}_n''\right\|\sum_{k/n\in A}p_{n-2,k}(x)\bar{w}^{-1}(k/n)\phi^{-2}(k/n)\\ &\leqslant C\phi^2(x)\left\|\bar{w}\phi^2\overline{F}_n''\right\|\sum_{k=0}^{n-2}p_{n-2,k}(x)\left(1+n^{-\frac{\alpha}{2}}|k-nx|^\alpha\right)\phi^{-2}(k/n)\\ &\leqslant C\left\|\bar{w}\phi^2\overline{F}_n''\right\| \leqslant C\left\|\bar{w}\phi^2f''\right\|.\end{aligned}$$

Working as the Theorem 1, we can get

$$T_2 \leqslant C\left\|\bar{w}\phi^2f''\right\|.$$

So, we can get

$$I_1 \leqslant C\left\|\bar{w}\phi^2f''\right\|.\tag{4.9}$$

By (3.7) and (4.7), we have

$$\begin{aligned}I_2 &\leqslant Cn^2\bar{w}(x)\phi^2(x)(1-x)^{n-2}\int_0^{\frac{2}{n}}u\left|\overline{F}_n''(u)\right|du\\ &\leqslant Cn^2\bar{w}(x)\phi^2(x)(1-x)^{n-2}\left\|\bar{w}\phi^2\overline{F}_n''\right\|\int_0^{\frac{2}{n}}u\bar{w}^{-1}(u)\phi^{-2}(u)du\\ &\leqslant C\left\|\bar{w}\phi^2\overline{F}_n''\right\| \leqslant C\left\|\bar{w}\phi^2f''\right\|.\end{aligned}\tag{4.10}$$

Similarly

$$I_3 \leqslant C\left\|\bar{w}\phi^2f''\right\|.\tag{4.11}$$

By bringing (4.5), (4.9)–(4.11) together, we can get the second inequality of TheoremB. □

Corollary 1. *For any $\alpha > 0$, $0\leqslant\lambda\leqslant 1$, we have*

$$\left|\bar{w}(x)\varphi^{2\lambda}(x)\overline{B}_n''(f,x)\right| \leqslant \begin{cases}Cn\{\max\{n^{1-\lambda},\varphi^{2(\lambda-1)}\}\}\|\bar{w}f\|, & f\in C_{\bar{w}},\\ C\|\bar{w}\varphi^{2\lambda}f''\|, & f\in W_{\bar{w},\lambda}^2.\end{cases}\tag{4.12}$$

4.3. Proof of Theorem 3

4.3.1. The direct theorem

We know

$$\overline{F}_n(t) = \overline{F}_n(x)+\overline{F}_n'(t)(t-x)+\int_x^t(t-u)\overline{F}_n''(u)du,\tag{4.13}$$

$$B_n(t-x,x) = 0.\tag{4.14}$$

According to the definition of W^2_ϕ, by (4.13) and (4.14), for any $g \in W^2_\phi$, we have $\overline{B}_n(g,x) = B_n(\overline{G}_n(g),x)$, then

$$\bar{w}(x)\left|\overline{G}_n(x) - B_n(\overline{G}_n,x)\right| = \bar{w}(x)\left|B_n(R_2(\overline{G}_n,t,x),x)\right|, \tag{4.15}$$

thereof $R_2(\overline{G}_n,t,x) = \int_x^t (t-u)\overline{G}''_n(u)du$.

$$\begin{aligned}\bar{w}(x)&\left|\overline{G}_n(x) - B_n(\overline{G}_n,x)\right| \\ &\leqslant C\bar{w}(x)\sum_{k=1}^{n-1}p_{n,k}(x)\int_x^{\frac{k}{n}}\left|\frac{k}{n}-u\right|\left|\overline{G}''_n(u)\right|du \\ &\quad + C\bar{w}(x)p_{n,0}(x)\int_0^x u\left|\overline{G}''_n(u)\right|du \\ &\quad + C\bar{w}(x)p_{n,n}(x)\int_x^1 (1-u)\left|\overline{G}''_n(u)\right|du := I_1+I_2+I_3.\end{aligned} \tag{4.16}$$

If u between $\frac{k}{n}$ and x, we have

$$\frac{\left|\frac{k}{n}-u\right|}{\bar{w}^2(u)} \leqslant \frac{\left|\frac{k}{n}-x\right|}{\bar{w}^2(x)}, \quad \frac{\left|\frac{k}{n}-u\right|}{\phi^4(u)} \leqslant \frac{\left|\frac{k}{n}-x\right|}{\phi^4(x)}. \tag{4.17}$$

By (3.4) and (4.17), then

$$\begin{aligned}I_1 &\leqslant C\left\|\bar{w}\phi^2\overline{G}''_n\right\|\bar{w}(x)\sum_{k=1}^{n-1}p_{n,k}(x)\int_x^{\frac{k}{n}}\frac{\left|\frac{k}{n}-u\right|}{\bar{w}(u)\phi^2(u)}du \\ &\leqslant C\left\|\bar{w}\phi^2\overline{G}''_n\right\|\bar{w}(x)\sum_{k=1}^{n-1}p_{n,k}(x)\left(\int_x^{\frac{k}{n}}\frac{\left|\frac{k}{n}-u\right|}{\bar{w}^2(u)}du\right)^{\frac12}\left(\int_x^{\frac{k}{n}}\frac{\left|\frac{k}{n}-u\right|}{\phi^4(u)}du\right)^{\frac12} \\ &\leqslant Cn^{-2}\left\|\bar{w}\phi^2\overline{G}''_n\right\|\phi^{-2}(x)\sum_{k=0}^{n-1}p_{n,k}(x)(k-nx)^2 \\ &\leqslant Cn^{-1}\frac{\varphi^2(x)}{\phi^2(x)}\left\|\bar{w}\phi^2\overline{G}''_n\right\| \leqslant Cn^{-1}\frac{\delta_n^2(x)}{\phi^2(x)}\left\|\bar{w}\phi^2\overline{G}''_n\right\| \\ &= C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^2\left\|\bar{w}\phi^2\overline{G}''_n\right\|.\end{aligned} \tag{4.18}$$

For I_2, when u between $\frac{k}{n}$ and x, we let $k=0$, then $\frac{u}{\bar{w}(u)} \leqslant \frac{x}{\bar{w}(x)}$, and

$$\begin{aligned}I_2 &\leqslant C\left\|\bar{w}\phi^2\overline{G}''_n\right\|\bar{w}(x)p_{n,0}(x)\int_0^x u\bar{w}^{-1}(u)\phi^{-2}(u)du \\ &\leqslant C(nx)(1-x)^{n-1}\cdot n^{-1}\frac{\varphi^2(x)}{\phi^2(x)}\left\|\bar{w}\phi^2\overline{G}''_n\right\| \\ &\leqslant C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^2\left\|\bar{w}\phi^2\overline{G}''_n\right\|.\end{aligned} \tag{4.19}$$

Similarly, we have

$$I_3 \leqslant C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^2\left\|\bar{w}\phi^2\overline{G}''_n\right\|. \tag{4.20}$$

By bringing (4.18)–(4.20), we have

$$\bar{w}(x)\left|\overline{G}_n(x) - B_n(\overline{G}_n,x)\right| \leqslant C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^2\left\|\bar{w}\phi^2\overline{G}''_n\right\|. \tag{4.21}$$

By (3.6) and (4.21), when $g \in W^2_\phi$, then

$$\begin{aligned}\bar{w}(x)&\left|g(x) - \overline{B}_n(g,x)\right| \\ &\leqslant \bar{w}(x)\left|g(x) - \overline{G}_n(g,x)\right| + \bar{w}(x)\left|\overline{G}_n(g,x) - \overline{B}_n(g,x)\right| \\ &\leqslant \bar{w}(x)|g(x) - P(g,x)|_{[x_1,x_4]} + C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^2\left\|\bar{w}\phi^2\overline{G}''_n\right\| \\ &\leqslant C\left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^2\left\|\bar{w}\phi^2 g''\right\|.\end{aligned} \tag{4.22}$$

For $f \in C_{\bar{w}}$, we choose proper $g \in W^2_\phi$, by (3.2) and (4.22), then

$$\begin{aligned}\bar{w}(x)\left|f(x) - \overline{B}_n(f,x)\right| &\leqslant \bar{w}(x)|f(x)-g(x)| + \bar{w}(x)\left|\overline{B}_n(f-g,x)\right| \\ &\quad + \bar{w}(x)\left|g(x) - \overline{B}_n(g,x)\right| \\ &\leqslant C\left(\|\bar{w}(f-g)\| + \left(\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)^2\left\|\bar{w}\phi^2 g''\right\|\right) \\ &\leqslant C\omega^2_\phi\left(f,\frac{\delta_n(x)}{\sqrt{n}\phi(x)}\right)_{\bar{w}}. \quad \square\end{aligned}$$

4.3.2. The inverse theorem

The main-part K-functional is given by

$$K_{2,\phi}(f,t^2)_{\bar{w}} = \sup_{0<h\leqslant t}\inf_g\left\{\|\bar{w}(f-g)\| + t^2\left\|\bar{w}\phi^2 g''\right\|,\ g' \in A.C._{loc}\right\}.$$

By [4], we have

$$C^{-1}K_{2,\phi}(f,t^2)_{\bar{w}} \leqslant \omega^2_\phi(f,t)_{\bar{w}} \leqslant CK_{2,\phi}(f,t^2)_{\bar{w}}. \tag{4.23}$$

Proof Let $\delta > 0$, by (4.23), we choose proper g so that

$$\|\bar{w}(f-g)\| \leqslant C\omega^2_\phi(f,\delta)_{\bar{w}}, \quad \left\|\bar{w}\phi^2 g''\right\| \leqslant C\delta^{-2}\omega^2_\phi(f,\delta)_{\bar{w}}. \tag{4.24}$$

then

$$\begin{aligned}\left|\bar{w}(x)\Delta^2_{h\phi}f(x)\right| &\leqslant \left|\bar{w}(x)\Delta^2_{h\phi}\left(f(x) - \overline{B}_n(f,x)\right)\right| \\ &\quad + \left|\bar{w}(x)\Delta^2_{h\phi}\overline{B}_n(f-g,x)\right| + \left|\bar{w}(x)\Delta^2_{h\phi}\overline{B}_n(g,x)\right| \\ &\leqslant \sum_{j=0}^{2}C_2^j\left(n^{-\frac12}\frac{\delta_n(x+(1-j)h\phi(x))}{\phi(x+(1-j)h\phi(x))}\right)^{\alpha_0} \\ &\quad + \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\bar{w}(x)\overline{B}''_n\left(f-g,x+\sum_{k=1}^{2}u_k\right)du_1\,du_2 \\ &\quad + \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\bar{w}(x)\overline{B}''_n\left(g,x+\sum_{k=1}^{2}u_k\right)du_1\,du_2 \\ &:= J_1+J_2+J_3.\end{aligned} \tag{4.25}$$

Obviously

$$J_1 \leqslant C\left(\left(n^{-\frac12}\phi^{-1}(x)\delta_n(x)\right)^{\alpha_0}\right). \tag{4.26}$$

By (2.2) and (4.24), we have

$$\begin{aligned}J_2 &\leqslant Cn^2\|\bar{w}(f-g)\|\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}\int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}}du_1du_2 \\ &\leqslant Cn^2h^2\phi^2(x)\|\bar{w}(f-g)\| \leqslant Cn^2h^2\phi^2(x)\omega^2_\phi(f,\delta)_{\bar{w}}.\end{aligned} \tag{4.27}$$

By the second inequality of (4.12) and (4.24), we have

$$J_2 \leqslant Cn\|\bar{w}(f-g)\| \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \varphi^{-2}\left(x+\sum_{k=1}^{2} u_k\right) du_1\, du_2$$
$$\leqslant Cnh^2\phi^2(x)\varphi^{-2}(x)\|\bar{w}(f-g)\| \leqslant Cnh^2\phi^2(x)\varphi^{-2}(x)\omega_\phi^2(f,\delta)_{\bar{w}}. \tag{4.28}$$

By the second inequality of (2.3), (3.3)and (4.24), we have

$$J_3 \leqslant C\left\|\bar{w}\phi^2 g''\right\|\bar{w}(x) \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \bar{w}^{-1}\left(x+\sum_{k=1}^{2} u_k\right)\phi^{-2}$$
$$\times\left(x+\sum_{k=1}^{2} u_k\right) du_1\, du_2$$
$$\leqslant Ch^2\left\|\bar{w}\phi^2 g''\right\| \leqslant Ch^2\delta^{-2}\omega_\phi^2(f,\delta)_{\bar{w}}. \tag{4.29}$$

Now, by (4.25)–(4.29), there exists a constant $M > 0$ so that

$$\left|\bar{w}(x)\Delta_{h\phi}^2 f(x)\right| \leqslant C\left(\left(n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)}\right)^{\alpha_0} + \min\left\{n\frac{\phi^2(x)}{\varphi^2(x)}, n^2\phi^2(x)\right\}\right.$$
$$\left.\times h^2\omega_\phi^2(f,\delta)_{\bar{w}} + h^2\delta^{-2}\omega_\phi^2(f,\delta)_{\bar{w}}\right)$$
$$\leqslant C\left(\left(n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)}\right)^{\alpha_0} + h^2M^2\left(n^{-\frac{1}{2}}\frac{\varphi(x)}{\phi(x)} + n^{-\frac{1}{2}}\frac{n^{-1/2}}{\phi(x)}\right)^{-2}\right.$$
$$\left.\times\omega_\phi^2(f,\delta)_{\bar{w}} + h^2\delta^{-2}\omega_\phi^2(f,\delta)_{\bar{w}}\right)$$
$$\leqslant C\left(\left(n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)}\right)^{\alpha_0} + h^2M^2\left(n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)}\right)^{-2}\right.$$
$$\left.\times\omega_\phi^2(f,\delta)_{\bar{w}} + h^2\delta^{-2}\omega_\phi^2(f,\delta)_{\bar{w}}\right).$$

When $n \geqslant 2$, we have

$$n^{-\frac{1}{2}}\delta_n(x) < (n-1)^{-\frac{1}{2}}\delta_{n-1}(x) \leqslant \sqrt{2}n^{-\frac{1}{2}}\delta_n(x),$$

Choosing proper x, δ, $n \in N$, so that

$$n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)} \leqslant \delta < (n-1)^{-\frac{1}{2}}\frac{\delta_{n-1}(x)}{\phi(x)},$$

Therefore

$$\left|\bar{w}(x)\Delta_{h\phi}^2 f(x)\right| \leqslant C\left\{\delta^{\alpha_0} + h^2\delta^{-2}\omega_\phi^2(f,\delta)_{\bar{w}}\right\}.$$

Which implies

$$\omega_\phi^2(f,t)_{\bar{w}} \leqslant C\left\{\delta^{\alpha_0} + h^2\delta^{-2}\omega_\phi^2(f,\delta)_{\bar{w}}\right\}.$$

So, by Berens–Lorentz lemma in [4], we get

$$\omega_\phi^2(f,t)_{\bar{w}} \leqslant Ct^{\alpha_0}. \quad \square$$

We can obtain the similar results when the Bernstein polynomials have no singularities. Now,we can consider the combinations of Bernstein Polynomials with inner singularities as Theorem 3 with countable or uncountable singularities.

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