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ORIGINAL ARTICLE

Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives

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Abstract Estimates for second and third Maclaurin coefficients of certain bi-univalent functions in the open unit disk defined by convolution are determined. Certain special cases are also indicated.

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1. Introduction and definitions

Let \mathcal{A} be the class of functions f which are analytic univalent functions in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ with normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ and having form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \quad (z \in \mathbb{D}). \quad (1.1)$$

Familiar subclasses of starlike and convex functions for which either of the quantity $Re \{zf'(z)/f(z)\} > 0$ or $\{1 + zf''(z)/f'(z)\} > 0$. The class consisting these two functions are given by \mathcal{S}^* and \mathcal{C} , respectively. For a constant $\beta \in (-\pi/2, \pi/2)$, a function f is univalent on \mathbb{D} and satisfies the condition that $Re \{e^{i\beta} zf'(z)/f(z)\} > 0$ in \mathbb{D} . We denote this class by \mathcal{S}^*_β .

The Koebe one-quarter theorem [2] ensures that the image of \mathbb{D} under every univalent function $f \in \mathcal{A}$ contains a disk of

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radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, ($z \in \mathbb{D}$) and

$$f^{-1}(f(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{D} if both $f(z)$ and $f^{-1}(z)$ are univalent in \mathbb{D} . We denote the class of bi-univalent functions by σ . Lewin [4] investigated the class σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [1] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients. Recently, Srivastava et al. [8] and Frasin and Aouf [3] introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients.

Let f and g be analytic functions in \mathbb{D} , we say that f is subordinate to g , written as $f < g$, if there exists a Schwarz function $w(z)$ in \mathbb{D} , with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbb{D}$), such that $f(z) = g(w(z))$. In particular, when g is univalent, then the above subordination is equivalent to $f(0) = 0$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. For functions $f, g \in \mathcal{A}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

Definition 1.1 (cf. [6,7], see also [9]). Let the function f be analytic in a simple connected region of the z -plane containing the origin. The fractional derivative or order ' λ ' is defined by

$$(D_z^\lambda f)(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1), \quad (1.2)$$

where multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Using Definition 1.1 and its known extension involving fractional derivative and fractional integrals, Owa and Srivastava [6] introduced the fractional differ-integral operator $\Omega_z^\lambda : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$(\Omega_z^\lambda f)(z) = \Gamma(2-\lambda) z^\lambda (D_z^\lambda f)(z) \quad (\lambda \neq 2, 3, 4, \dots; z \in \mathbb{D}). \quad (1.3)$$

Note that $(\Omega_z^0 f)(z) = z f'(z)$ and $(\Omega_z^1 f)(z) = f(z)$.

Motivated by the work of Srivastava et al. [8] and Mishra and Gochhayat [5], we introduce a new subclass of bi univalent functions $\mathcal{S}\mathcal{B}_\sigma^{\lambda,\beta}(h)$.

Definition 1.2. Let

$$h : \mathbb{D} \rightarrow \mathbb{C},$$

be a convex univalent function such that

$$h(0) = 1 \quad \text{and} \quad h(\bar{z}) = \overline{h(z)}, \quad (z \in \mathbb{D}; \operatorname{Re}(h(z)) > 0).$$

A function $f(z)$ is said to be in the class $\mathcal{S}\mathcal{B}_\sigma^{\lambda,\beta}(h)$, if the following conditions are satisfied:

$$f \in \sigma \quad \text{and} \quad e^{i\beta} \frac{(\Omega_z^\lambda f)(z)}{z} \prec h(z) \cos \beta + i \sin \beta, \quad (z \in \mathbb{D}), \quad (1.4)$$

and

$$e^{i\beta} \frac{(\Omega_z^\lambda g)(w)}{w} \prec h(w) \cos \beta + i \sin \beta, \quad (w \in \mathbb{D}); \quad (1.5)$$

where $\beta \in (-\pi/2, \pi/2)$, $\lambda \neq 2, 3, \dots$ and $g = f^{-1}$.

Remark 1.1. If we set $h(z) = 1 + Az/1 + Bz$, $-1 \leq B < A \leq 1$, then the class $\mathcal{S}\mathcal{B}_\sigma^{\lambda,\beta}(h)$ reduces to $\mathcal{S}\mathcal{B}_\sigma^{\lambda,\beta}(A, B)$ which is define as

$$f \in \sigma \quad \text{and} \quad e^{i\beta} \frac{(\Omega_z^\lambda f)(z)}{z} \prec \frac{1 + Az}{1 + Bz} \cos \beta + i \sin \beta, \quad (z \in \mathbb{D}), \quad (1.6)$$

and

$$e^{i\beta} \frac{(\Omega_z^\lambda g)(w)}{w} \prec \frac{1 + Aw}{1 + Bw} \cos \beta + i \sin \beta, \quad (w \in \mathbb{D}); \quad (1.7)$$

where $\beta \in (-\pi/2, \pi/2)$ and $g = f^{-1}$ and $\lambda \neq 2, 3, \dots$

Remark 1.2. Taking $\lambda = 0$ in above class, then we have $\mathcal{S}\mathcal{B}_\sigma^0(A, B)$ and if $f \in \mathcal{S}\mathcal{B}_\sigma^0(A, B)$

$$f \in \sigma \quad \text{and} \quad e^{i\beta} f'(z) \prec \frac{1 + Az}{1 + Bz} \cos \beta + i \sin \beta, \quad (z \in \mathbb{D}), \quad (1.8)$$

and

$$e^{i\beta} g'(w) \prec \frac{1 + Aw}{1 + Bw} \cos \beta + i \sin \beta, \quad (w \in \mathbb{D}); \quad (1.9)$$

where $\beta \in (-\pi/2, \pi/2)$ and $g = f^{-1}$.

Now substituting $A = 1 - 2\alpha$, $0 \leq \alpha < 1$, $B = -1$ and $\beta = 0$, we get known class $\mathcal{B}_\sigma(\beta)$ which is studied by Srivastva et al. [8].

Remark 1.3. Taking $\lambda = 1$ in the class $\mathcal{S}\mathcal{B}_\sigma^{\lambda,\beta}(A, B)$, we have $\mathcal{S}\mathcal{B}_\sigma^{1,\beta}(A, B)$ and if $f \in \mathcal{S}\mathcal{B}_\sigma^{1,\beta}(A, B)$, then

$$f \in \sigma \quad \text{and} \quad e^{i\beta} \frac{f(z)}{z} \prec \frac{1 + Az}{1 + Bz} \cos \beta + i \sin \beta, \quad (z \in \mathbb{D}), \quad (1.10)$$

and

$$e^{i\beta} \frac{g(w)}{w} \prec \frac{1 + Aw}{1 + Bw} \cos \beta + i \sin \beta, \quad (w \in \mathbb{D}); \quad (1.11)$$

where $\beta \in (-\pi/2, \pi/2)$ and $g = f^{-1}$.

Remark 1.4. Taking $\lambda = 0$ in above class $\mathcal{S}\mathcal{B}_\sigma^{\lambda,\beta}(A, B)$, we have $\mathcal{S}\mathcal{B}_\sigma^{0,\beta}(A, B)$ and if $f \in \mathcal{S}\mathcal{B}_\sigma^{0,\beta}(A, B)$, then

$$f \in \sigma \quad \text{and} \quad \operatorname{Re}(e^{i\beta} f'(z)) > \alpha \cos \beta, \quad (z \in \mathbb{D}), \quad (1.12)$$

and

$$\operatorname{Re}(e^{i\beta} g'(w)) > \alpha \cos \beta, \quad (w \in \mathbb{D}); \quad (1.13)$$

where $\beta \in (-\pi/2, \pi/2)$ and $g = f^{-1}$.

Remark 1.5. Taking $A = 1 - 2\alpha$, $B = -1$ in above class in class $\mathcal{S}\mathcal{B}_\sigma^{1,\beta}(A, B)$, it reduces to $\mathcal{S}\mathcal{B}_\sigma^{1,\beta}(\alpha)$ and if $f \in \mathcal{S}\mathcal{B}_\sigma^{1,\beta}(\alpha)$, then

$$f \in \sigma \quad \text{and} \quad \operatorname{Re}\left(e^{i\beta} \frac{f(z)}{z}\right) > \alpha \cos \beta, \quad (z \in \mathbb{D}), \quad (1.14)$$

and

$$\operatorname{Re}\left(e^{i\beta} \frac{g(w)}{w}\right) > \alpha \cos \beta \quad (w \in \mathbb{D}). \quad (1.15)$$

The object of the paper is to estimates for the coefficients a_2 and a_3 for functions in the class $\mathcal{S}\mathcal{B}_\sigma^{\lambda,\beta}(h)$ are obtained by employing the techniques used earlier by Srivastava et al. [8].

2. Main result

In order to prove our main result for the functions class $f \in \mathcal{S}\mathcal{B}_\sigma^k(\beta)$, we first recall the following lemma:

Lemma 2.1 (Theorem 3.3, p. 11, 13). Let the function $\varphi(z)$ given $\varphi(z) = \sum_{n=1}^{\infty} B_n z^n$ be convex in \mathbb{D} . Suppose also that the function $h(z)$ given by

$$h(z) = \sum_{n=1}^{\infty} h_n z^n,$$

is holomorphic in \mathbb{D} . If $h(z) \prec \varphi(z)$ ($z \in \mathbb{D}$), then

$$|h_n| \leq |B_1| \quad (n \in \mathbb{N}). \tag{2.1}$$

Theorem 2.2. *If $f \in \mathcal{A}$ satisfies (1.1), is in the class $\mathcal{SB}_\sigma^{\lambda, \beta}(h)$. Then*

$$|a_2| \leq \sqrt{\frac{|B_1| \cos \beta (2 - \lambda)(3 - \lambda)}{12}}, \tag{2.2}$$

and

$$|a_3| \leq \left(\frac{|B_1|(2 - \lambda)}{2}\right)^2 + \frac{|B_1| \cos \beta (2 - \lambda)(3 - \lambda)}{12}, \tag{2.3}$$

where $\beta \in (-\pi/2, \pi/2)$ and $\lambda \neq 2, 3, \dots$

Proof. From (1.4) and (1.5)

$$e^{i\beta} \frac{(\Omega_z^\lambda f)(z)}{z} = p(z) \cos \beta + i \sin \beta, \quad (z \in \mathbb{D}), \tag{2.4}$$

and

$$e^{i\beta} \frac{(\Omega_w^\lambda f)(w)}{w} = q(w) \cos \beta + i \sin \beta, \quad (w \in \mathbb{D}), \tag{2.5}$$

where $p(z) \prec h(z)$ and $q(w) \prec h(w)$ and have following forms:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots \quad z \in \mathbb{D}, \tag{2.6}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \dots \quad w \in \mathbb{D}. \tag{2.7}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$e^{i\beta} \left(\frac{2}{2 - \lambda}\right) a_2 = p_1, \tag{2.8}$$

$$e^{i\beta} \frac{6}{(2 - \lambda)(3 - \lambda)} a_3 = p_2, \tag{2.9}$$

$$-e^{i\beta} \left(\frac{2}{2 - \lambda}\right) a_2 = q_1, \tag{2.10}$$

and

$$e^{i\beta} \frac{6}{(2 - \lambda)(3 - \lambda)} (2a_2^2 - a_3) = q_2. \tag{2.11}$$

From (2.8) and (2.10), we get

$$p_1 = -q_1, \tag{2.12}$$

and

$$e^{2i\beta} \left(\frac{8}{2 - \lambda}\right)^2 a_2^2 = (p_1^2 + q_1^2) \cos^2 \beta. \tag{2.13}$$

Adding (2.9) and (2.11), it follows:

$$a_2^2 = \frac{(2 - \lambda)(3 - \lambda)}{12} (p_2 + q_2) e^{-i\beta} \cos \beta. \tag{2.14}$$

Again from (2.9) and (2.11)

$$a_3 - a_2^2 = \frac{(2 - \lambda)(3 - \lambda)}{12} (p_2 - q_2) e^{-i\beta} \cos \beta. \tag{2.15}$$

Substituting value of a_2^2 from (2.13) in (2.14), we get

$$a_3 = \frac{(2 - \lambda)(3 - \lambda)}{12} (p_2 - q_2) e^{-i\beta} \cos \beta + \frac{(2 - \lambda)^2}{8} (p_1^2 + q_1^2) e^{-2i\beta} \cos^2 \beta. \tag{2.16}$$

Since $p(z), q(w) \in h(\mathbb{D})$. According to Lemma 2.1, we find that

$$|p_k| = \left|\frac{p^{(k)}(0)}{k!}\right| \leq |B_1| \quad (k \in \mathbb{N}), \tag{2.17}$$

and

$$|q_k| = \left|\frac{q^{(k)}(0)}{k!}\right| \leq |B_1| \quad (k \in \mathbb{N}). \tag{2.18}$$

Using above Eq. (2.12) and using (2.17) and (2.18), we have

$$|a_2|^2 \leq \frac{(2 - \lambda)(3 - \lambda)}{12} (|q_2| + |p_2|) \cos \beta \leq \frac{|B_1| \cos \beta (2 - \lambda)(3 - \lambda)}{6}, \tag{2.19}$$

which gives (2.2). Now using (2.13) and (2.15) and from (2.17) and (2.18), we can easily get (2.3). This is the end of the Theorem 2.2. \square

By setting, $h(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.2, we get the following corollary:

Corollary 2.3. *Let $f \in \mathcal{A}$ be in the class $\mathcal{SB}_\sigma^{\lambda, \beta}(A, B)$. Then*

$$|a_2| \leq \sqrt{\frac{(2 - \lambda)(3 - \lambda)(A - B) \cos \beta}{6}}, \tag{2.20}$$

and

$$|a_3| \leq \frac{(2 - \lambda)^2 (A - B)^2 \cos^2 \beta}{4} + \frac{(2 - \lambda)(3 - \lambda)(A - B) \cos \beta}{6}, \tag{2.21}$$

where $\beta \in (-\pi/2, \pi/2)$ and $\lambda \neq 2, 3, \dots$. Again putting, $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leq \alpha < 1$ in Theorem 2.2, we have

Corollary 2.4. *Let $f \in \mathcal{A}$ be in the class $\mathcal{SB}_\sigma^{\lambda, \beta}(\alpha)$. Then*

$$|a_2| \leq \sqrt{\frac{(2 - \lambda)(3 - \lambda)(1 - \alpha) \cos \beta}{3}} \tag{2.22}$$

and

$$|a_3| \leq (2 - \lambda)^2 (1 - \alpha)^2 \cos^2 \beta + \frac{(2 - \lambda)(3 - \lambda)(1 - \alpha) \cos \beta}{3}, \tag{2.23}$$

where $\beta \in (-\pi/2, \pi/2)$ and $\lambda \neq 2, 3, \dots$

If we take $\lambda = 0$ in Corollary 2.3, it gives

Corollary 2.5. *Let $f \in \mathcal{A}$ be in the class $\mathcal{SB}_\sigma^{0, \beta}(A, B)$. Then*

$$|a_2| \leq \sqrt{(A - B) \cos \beta}, \tag{2.24}$$

and

$$|a_3| \leq (A - B)^2 \cos^2 \beta + (A - B) \cos \beta, \quad (2.25)$$

where $\beta \in (-\pi/2, \pi/2)$.

If we take $\lambda = 1$ in Corollary 2.3, we obtain

Corollary 2.6. Let $f \in \mathcal{A}$ be in the class $\mathcal{S}\mathcal{B}_\sigma^{1,\beta}(A, B)$. Then

$$|a_2| \leq \sqrt{\frac{(A - B) \cos \beta}{3}}, \quad (2.26)$$

and

$$|a_3| \leq \frac{(A - B)^2}{4} \cos^2 \beta + \frac{(A - B) \cos \beta}{3}, \quad (2.27)$$

where $\beta \in (-\pi/2, \pi/2)$.

Upon putting $A = 1 - 2\alpha$, $0 \leq \alpha < 1$ and $B = -1$, above Corollaries 2.4 and 2.5, we get following results

Corollary 2.7. Let $f \in \mathcal{A}$ be in the class $\mathcal{S}\mathcal{B}_\sigma^{0,\beta}(\alpha)$. Then

$$|a_2| \leq \sqrt{2(1 - \alpha) \cos \beta}, \quad (2.28)$$

and

$$|a_3| \leq 4(1 - \alpha)^2 \cos^2 \beta + 2(1 - \alpha) \cos \beta, \quad (2.29)$$

where $\beta \in (-\pi/2, \pi/2)$.

Put $\lambda = 1$ in Corollary 2.3, we obtain

Corollary 2.8. Let $f \in \mathcal{A}$ be in the class $\mathcal{S}\mathcal{B}_\sigma^{1,\beta}(\alpha)$. Then

$$|a_2| \leq \sqrt{\frac{2(1 - \alpha) \cos \beta}{3}}, \quad (2.30)$$

and

$$|a_3| \leq (1 - \alpha)^2 \cos^2 \beta + \frac{2(1 - \alpha) \cos \beta}{3}, \quad (2.31)$$

where $\beta \in (-\pi/2, \pi/2)$.

Remark 2.1. On taking $\beta = 0$ in Corollary 2.8, we obtain a known result due to Srivastava et al. [8].

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