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On commutativity of rings with generalized derivations

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Abstract Let R be a prime ring, extended centroid C, Utumi quotient ring U, and $m, n \geq 1$ are fixed positive integers, F a generalized derivation associated with a nonzero derivation d of R. We study the case when one of the following holds: (i) $F(x)\circ_m d(y) = (x \circ y)^n$ and (ii) $(F(x) \circ d(y))^m = (x \circ y)^n$, for all x, y in some appropriate subset of R. We also examine the case where R is a semiprime ring.

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1. Introduction

In all that follows, unless specifically stated otherwise, R will be an associative ring, $Z(R)$ the center of R, Q its Martindale quotient ring and U its Utumi quotient ring. The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [1], for the definitions and related properties of these objects). For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stands for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. Given $x, y \in R$, we set $x\circ_0 y = x$, $x\circ_1 y = x \circ y = xy + yx$, and inductively $x\circ_m y = (x\circ_{m-1} y) \circ y$ for $m > 1$. Recall that a ring R is prime if $xRy = \{0\}$ implies either $x = 0$ or $y = 0$, and R is semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + yd(x)$ holds for all $x, y \in R$. In particular d is an inner derivation induced by an element $q \in R$, if $d(x) = [q, x]$ holds for all $x \in R$. If R is a ring and $S \subseteq R$, a mapping $f : R \to R$ is called strong commutativity-preserving (scp) on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$.

Many results in the literature indicate that the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R. Derivations with certain properties investigated in various papers (see for Refs. [2–4]). Starting from these results, many authors studied generalized derivations in the context of prime and semiprime rings. By a generalized inner derivation on R, one usually means an additive mapping $F : R \to R$ if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping F, it is easy to see that $F(xy) = F(x)y +$

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$x[y, b] = F(x)y + xI_b(y)$. This observation leads to the definition given in [5]: an additive mapping $F : R \to R$ is called generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Familiar examples of generalized derivations are derivations and generalized inner derivations, and the latter includes left multipliers (i.e., an additive mapping $f(xy) = f(x)y$ for all $x, y \in R$). Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$ is a generalized derivation, where c is a fixed element of R and d is a derivation of R.

In [6], Lee extended the definition of a generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \to U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where I is a dense right ideal of R and d is a derivation from I into U. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation on U, and thus all generalized derivations of R will be implicitly assumed to be defined on the derivation F on dense right ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U (see Theorem 3, in [6]). More related results about generalized derivations can be found [7,8].

During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations (see [9], where further references can be found). In [9], Ashraf and Rehman prove that if R is a prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. In [10], Argaç and Inceboz generalized the above result as following: Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer, if R admits a nonzero derivation d with the property $(d(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative. In [8, Theorem 2.3], Quadri et al., discussed the commutativity of prime rings with generalized derivations. More precisely, Quadri et al., prove that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. In 2012 Huang [11], generalized the result obtained by Quadri et al., and he proved that if R is a prime ring, I a nonzero ideal of R, n a fixed positive integer and F a generalized derivation associated with a nonzero derivation d such that $(F(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative.

In 1994 Bell and Daif [12], initiated the study of strong commutativity-preserving maps and prove that a nonzero right ideal I of a semiprime ring is central if R admits a derivation which is scp on I. In 2002 Ashraf and Rehman [9], prove that if R is a 2-torsion free prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(x) \circ d(y) = x \circ y$ for all $x, y \in I$, then R is commutative. The present paper is motivated by the previous results and we here generalized the result obtained in [9,11]. Moreover, we continue this line of investigation by examining what happens if a ring R satisfies the identity.

(i) $(F(x) \circ d(y))^m = (x \circ y)^n$ for all $x, y \in I$.
(ii) $F(x) \circ_m d(y) = (x \circ y)^n$ for all $x, y \in I$.

We obtain some analogous results for semiprime rings in the case $I = R$.

Explicitly we shall prove the following theorems:

Theorem 1.1. *Let R be a prime ring, I a nonzero ideal of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ d(y))^m = (x \circ y)^n$ for all $x, y \in I$, then R is commutative.*

Theorem 1.2. *Let R be a prime ring, I a nonzero ideal of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x) \circ_m d(y) = (x \circ y)^n$ for all $x, y \in I$, then R is commutative.*

Theorem 1.3. *Let R be a semiprime ring, U the left Utumi quotient ring of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ d(y))^m = (x \circ y)^n$ for all $x, y \in R$, then R is commutative.*

Theorem 1.4. *Let R be a semiprime ring, U the left Utumi quotient ring of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x) \circ_m d(y) = (x \circ y)^n$ for all $x, y \in R$, then R is commutative.*

2. The results in prime rings

We will make frequent use of the following result due to Kharchenko [13] (see also [14]):

Let R be a prime ring, d a nonzero derivation of R and I a nonzero two sided ideal of R. Let $f(x_1, \dots x_n, d(x_1, \dots x_n))$ be a differential identity in I, that is

$$f(r_1, \dots r_n, d(r_1), \dots, d(r_n)) = 0 \text{ for all } r_1, \dots, r_n \in I.$$

One of the following holds:

(1) Either d is an inner derivation in Q, the Martindale quotient ring of R, in the sense that there exists $q \in Q$ such that $d = ad(q)$ and $d(x) = ad(q)(x) = [q, x]$, for all $x \in R$, and I satisfies the generalized polynomial identity

$$f(r_1, \dots, r_n, [q, r_1], \dots, [q, r_n]) = 0;$$

(2) or, I satisfies the generalized polynomial identity

$$f(x_1, \dots, x_n, y_1, \dots, y_n) = 0.$$

Theorem 1.1. *Let R be a prime ring, I a nonzero ideal of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ d(y))^m = (x \circ y)^n$ for all $x, y \in I$, then R is commutative.*

Proof. If $F = 0$, then $(x \circ y)^n = 0$ for all $x, y \in I$, which can be rewritten as $(xy + yx)^n = 0$. If $char(R) \neq 2$, then $(2x^2)^n = 0$ for all $x \in I$. This is a contradiction by Xu [15]. If $char(R) = 2$, then $(xy + yx)^n = 0 = [x, y]^n$ for all $s, y \in I$. Thus by Herstein [16, Theorem 2], we have $I \subseteq Z(R)$, and so R is commutative by Mayne [17]. Hence, onward we will assume that $F \neq 0$ and $(F(x) \circ d(y))^m = (x \circ y)^n$ for all $x, y \in I$. By Lee [6, Theorem 3], every generalized derivation of R will be implicitly assumed to be defined on dense right ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Therefore, I satisfies the polynomial identity

$$((ax + d(x)) \circ d(y))^m = (x \circ y)^n \text{ for all } x, y \in I.$$

Which is rewritten as, for all $x, y \in I$

$$((ax \circ d(y)) + (d(x) \circ d(y)))^m = (x \circ y)^n.$$

In light of Kharchenko's theory [13], we divide the proof into two cases:

Case 1. If d is Q-outer, then I satisfies the polynomial identity

$$((ax \circ t) + (s \circ t))^m = (x \circ y)^n, \text{ for all } x, y, s, t \in I.$$

In particular for $x = 0$, I satisfies the blended component $(st + ts)^n = 0$ for all $s, t \in I$, then R is commutative, by using the argument presented above.

Case 2. Let d is Q-inner induced by an element $q \in Q$, that is, $d(x) = [q, x]$ for all $x \in R$. Then for any $x, y \in I$,

$$((ax \circ [q, y]) + ([q, x] \circ [q, y]))^m = (x \circ y)^n.$$

By Chuang [18, Theorem 1], I and Q satisfy same generalized polynomial identities (GPIs), we have

$$((ax \circ [q, y]) + ([q, x] \circ [q, y]))^m = (x \circ y)^n, \text{ for all } x, y \in Q.$$

In case the center C of Q is infinite, we have

$$((ax \circ [q, y]) + ([q, x] \circ [q, y]))^m = (x \circ y)^n,$$

for all $x, y \in Q \otimes_C \overline{C}$, where $\overline{C}$ is algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [19, Theorems 2.5 and 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according to C is finite or infinite. Thus we may assume that R is centrally closed over C (i.e., $RC = R$) which is either finite or algebraically closed and

$$((ax \circ [q, y]) + ([q, x] \circ [q, y]))^m = (x \circ y)^n \text{ for all } x, y \in R, \quad (1)$$

By Martindale [20, Theorem 3], RC (and so R) is a primitive ring having nonzero socle H with $\mathcal{D}$ as the associated division ring. Hence by Jacobson's theorem [21, p.75], R is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $\mathcal{D}$ and H consists of the finite rank linear transformations in R. If $\mathcal{V}$ is a finite dimensional over $\mathcal{D}$. Then the density of R on $\mathcal{V}$ implies that $R \cong \mathcal{M}_k(\mathcal{D})$, where $k = dim_{\mathcal{D}}\mathcal{V}$. Suppose that $dim_{\mathcal{D}}\mathcal{V} \geq 2$, otherwise we are done.

Step 1. We want to show that v and qv are linearly $\mathcal{D}$-dependent for all $v \in \mathcal{V}$. If $qv = 0$ then $\{v, qv\}$ is linearly $\mathcal{D}$-dependent. Suppose on contrary that v and qv are linearly $\mathcal{D}$-independent for some $v \in \mathcal{D}$.

If $q^2v \notin Span_{\mathcal{D}}\{v, qv\}$ then $\{v, qv, q^2v\}$ are linearly $\mathcal{D}$-independent. By the density of R there exist $x_0, y_0 \in R$ such that

$$x_0 v = 0, \quad x_0 qv = qv, \quad x_0 q^2 v = 0$$

$$y_0 v = 0, \quad y_0 qv = v, \quad y_0 q^2 v = v.$$

The application of (1) implies that

$$v = ((ax_0 \circ [q, y_0]) + [q, x_0] \circ [q, y_0])^m v = (x_0 y_0 + y_0 x_0)^n v = 0, \text{ a contradiction.}$$

If $q^2v \in Span_{\mathcal{D}}\{v, qv\}$ then $q^2v = v\alpha + qv\gamma$ for some $\alpha, 0 \neq \gamma \in \mathcal{D}$. In view of the density of R, there exist $x_0, y_0 \in R$ such that

$$x_0 v = 0, \quad x_0 qv = qv$$

$$y_0 v = 0, \quad y_0 qv = v.$$

It follows from the relation (1) that

$$0 = ((ax_0 \circ [q, y_0]) + [q, x_0] \circ [q, y_0])^m v = (x_0 y_0 + y_0 x_0)^n v = v\gamma^m \neq 0,$$

and we arrive at a contradiction. So we conclude that v and qv are linearly $\mathcal{D}$-dependent for all $v \in \mathcal{V}$.

Step 2. We show here that there exists $\beta \in \mathcal{D}$ such that $qv = v\beta$, for any $v \in \mathcal{V}$. Note that the arguments in [22] are still valid in the present situation. For the sake of completeness and clearness we prefer to present it. In fact, choose $v, w \in \mathcal{V}$ linearly independent. By Step 1, there exist $\beta_v, \beta_w, \beta_{v+w} \in \mathcal{D}$ such that

$$qv = v\beta_v, qw = w\beta_w, q(v + w) = (v + w)\beta_{v+w}$$

Moreover,

$$v\beta_v + w\beta = (v + w)\beta_{v+w}.$$

Hence

$$v(\beta_v - \beta_{v+w}) + w(\beta_w - \beta_{v+w}) = 0,$$

and because v, w are linearly $\mathcal{D}$-independent, we have $\beta_v = \beta_w = \beta_{v+w}$, that is, β does not depend on the choice of v. This completes the proof of Step 2.

Let now for $r \in R$, $v \in \mathcal{V}$. By Step 2, $qv = v\alpha$, $r(qv) = r(v\alpha)$, and also $q(rv) = (rv)\alpha$. Thus $0 = [q, r]v$, for any $v \in \mathcal{V}$, that is $[q, r]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible R-module, hence $[q, r] = 0$, for all $r \in R$, i.e., $q \in Z(R)$ and $d = 0$, which contradicts our hypothesis. This completes the proof of theorem. □

Theorem 1.2. *Let R be a prime ring, I a nonzero ideal of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x) \circ_m d(y) = (x \circ y)^n$ for all $x, y \in I$, then R is commutative.*

Proof. If $F = 0$, then $(x \circ y)^n = 0$. Using the same argument presenting in Theorem 1.1 we have done. Now suppose that $F \neq 0$ and $F(x) \circ_m d(y) = (x \circ y)^n$ for all $x, y \in I$. By Lee [6, Theorem 3], $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. This condition is a differential identity and therefore I satisfies

$$(ax + d(x)) \circ_m d(y) = (x \circ y)^n \text{ for all } x, y \in I.$$

Which is rewritten as, for all $x, y \in I$

$$(ax \circ_m d(y)) + (d(x) \circ_m d(y)) = (x \circ y)^n.$$

In light of Kharchenko's theory [13], either $d = ad(q)$ is the inner derivation induced by an element $a \in Q$, the Martindale quotient ring R, or I satisfies the polynomial identity

$$(ax \circ_m t) + (s \circ_m t) = (x \circ y)^n, \text{ for all } x, y, s, t \in I.$$

In the latter case set $t = 0$, to obtain the identity $(xy + yx)^n = 0$ for all $x, y \in I$, and R is commutative by Theorem 1.1. Assume now that $d = ad(q)$. Then,

$$(ax \circ_m [q, y]) + ([q, x] \circ_m [q, y]) = (x \circ y)^n,$$

for all $x, y \in I$. As in the proof Theorem 1.1, we see that

$$(ax \circ_m [q, y]) + ([q, x] \circ_m [q, y]) = (x \circ y)^n \text{ for all } x, y \in R, \quad (2)$$

where R is a primitive ring with $\mathcal{D}$ as the associated division ring. If $\mathcal{V}$ is finite dimensional over $\mathcal{D}$, then the density of R implies that $R \cong \mathcal{M}_k(\mathcal{D})$, where $k = dim_{\mathcal{D}}\mathcal{V}$.

Suppose that $dim_{\mathcal{D}}\mathcal{V} \geq 2$, otherwise we are done. We want to show that v and qv are linearly $\mathcal{D}$-dependent for all $v \in \mathcal{V}$. If $qv = 0$ then $\{v, qv\}$ is linearly $\mathcal{D}$-dependent. Suppose on contrary that v and qv are linearly $\mathcal{D}$-independent for some $v \in \mathcal{D}$.

If $q^2v \notin Span_{\mathcal{D}}\{v, qv\}$ then $\{v, qv, q^2v\}$ are linearly $\mathcal{D}$-independent. By the density of R there exist $x, y \in R$ such that

$$xv = 0, \quad xqv = qv, \quad xq^2v = 0$$
$$yv = 0, \quad yqv = v, \quad yq^2v = v.$$

The application of (2) implies that $v = (ax\circ_m[q, y])v + ([q, x]\circ_m[q, y])v = (xy + yx)^n v = 0$, and we arrive at a contradiction.

If $q^2v \in Span_{\mathcal{D}}\{v, qv\}$ then $q^2v = v\alpha + qv\gamma$ for some $\alpha, 0 \neq \gamma \in \mathcal{D}$. In view of the density of R, there exist $x, y \in R$ such that

$$xv = 0, \quad xqv = qv$$
$$yv = 0, \quad yqv = v.$$

It follows from the relation (2) that

$$0 = (ax\circ_m[q, y]) + ([q, x]\circ_m[q, y])v = (xy + yx)^n v$$
$$= (-1)^{m+1}2^{m-1}v\gamma \neq 0,$$

and we arrive at a contradiction. we conclude that v and qv are linearly $\mathcal{D}$-dependent for all $v \in \mathcal{V}$. Reasoning as in the proof of Theorem 1.1, we get required result. □

The following example demonstrates that R to be prime is essential in the hypothesis.

Example 2.1. Let S be any ring.

(i) Let $R = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in S\right\}$ and $I = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S\right\}$. Then R is a ring under usual operations and I is a nonzero ideal of R. We define a map $F : R \to R$ by $F(x) = 2e_{11}x - xe_{11}$. Then it is easy to see that F is a generalized derivation associated with a nonzero derivation $d(x) = e_{11}x - xe_{11}$. It is straightforward to check that for all positive integers m, n, F satisfies the properties, $(1)(F(x) \circ d(y))^m = (x \circ y)^n$ $(2)F(x)\circ_m d(y) = (x \circ y)^n$ for $x, y \in I$, however R is not commutative.

(ii) Let $R = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in Z_2\right\}$ and $I = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in Z_2\right\}$ be a nonzero ideal of R. Define a map $F : R \to R$ by $F(x) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. It is easy to see that F is a generalized derivation associated with a nonzero derivation $d(x) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. It is straightforward to check that F satisfies the properties, $(1)(F(x) \circ d(y))^m = (x \circ y)^n$ $(2)F(x)\circ_m d(y) = (x \circ y)^n$ for $x, y \in I$, but R is not commutative.

3. The results in semiprime rings

In all that follows, R will be semiprime ring, U is the left Utumi quotient ring of R. In order to prove the main result of this section we will make use of the following facts:

Fact 3.1 ([1, Proposition 2.5.1]). *Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U, and so any derivation of R can be defined on the whole U.*

Fact 3.2 ([23, p. 38]). *If R is semiprime then so is its left Utumi quotient ring. The extended centroid C of a semiprime ring coincides with the center of its left Utumi quotient ring.*

Fact 3.3 ([23, p. 42]). *Let B be the set of all the idempotents in C, the extended centroid of R. Assume R is a B-algebra orthogonal complete. For any maximal ideal P of B, PR forms a minimal prime ideal of R, which is invariant under any nonzero derivation of R.*

We will prove the following:

Theorem 1.3. *Let R be a semiprime ring, U the left Utumi quotient ring of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ d(y))^m = (x \circ y)^n$ for all $x, y \in R$, then R is commutative.*

Proof. Since R is semiprime and F is a generalized derivation of R, by Lee [6, Theorem 3], $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. We are given that

$$((ax \circ d(y)) + (d(x) \circ d(y)))^m = (x \circ y)^n,$$

for all $x, y \in R$. By Fact 3.2, $Z(U) = C$, the extended centroid of R, and, by Fact 3.1, the derivation d can be uniquely extended on U. By Lee [14], R and U satisfy the same differential identities. Then

$$((ax \circ d(y)) + (d(x) \circ d(y)))^m = (x \circ y)^n,$$

for all $x, y \in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. Since U is a B-algebra orthogonal complete [23, p.42], by Fact 3.3, MU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/MU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in U$. For any $\overline{x}, \overline{y} \in \overline{U}$,

$$\left((\overline{ax} \circ \overline{d}(\overline{y})) + (\overline{d}(\overline{x}) \circ \overline{d}(\overline{y}))\right)^m = (\overline{x} \circ \overline{y})^n.$$

It is obvious that $\overline{U}$ is prime. Therefore, by Theorem 1.1, we have either $\overline{U}$ is commutative or $\overline{d} = 0$ in $\overline{U}$. This implies that, for any maximal ideal M of B, either $d(U) \subseteq MU$ or $[U, U] \subseteq MU$. In any case $d(U)[U, U] \subseteq MU$, for all M, where MU runs over all prime ideals of U. Therefore $d(U)[U, U] \subseteq \bigcap_M MU = 0$, we obtain $d(U)[U, U] = 0$. Therefore $[U, U] = 0$ since $\bigcap_M MU = 0$. In particular, R is commutative. This completes the proof of the theorem. □

Using arguments similar to those used in the proof of the above theorem, we may conclude with the following (we omit the proof brevity). We can prove.

Theorem 1.4. *Let R be a semiprime ring, U the left Utumi quotient ring of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x)\circ_m d(y) = (x \circ y)^n$ for all $x, y \in R$, then R is commutative.*

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