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Original Article

On projection-invariant submodules of QTAG-modules

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Abstract [illegible]module [illegible]er an associative ring R with unity is a *QTAG*-module if every finitely genera[illegible] subm[illegible]ule of a[illegible] homomorphic image of M is a direct sum of uniserial modules. Here we [illegible]y proje[illegible]on-invaria[illegible] submodule of *QTAG*-module. A submodule N of a *QTAG*-module M is sa[illegible] be [illegible]ariant in M if $f(N) \subseteq N$, for all idempotent endomorphisms f in End(M). [illegible]early, [illegible] invariant submodules are projection-invariant. Mehdi et. al. characterized fully invariant submo[illegible] and characteristic submodules with the help of their socles. Here we investigate the [illegible]cles of proj[illegible]n-invariant submodules of *QTAG*-modules.

[illegible] Mathematics Subject Classification: 16 K 20; 13 C 12; 13 C 13

1. In[illegible]uction [illegible]d prelimi[illegible]es

All the[illegible]s R considered here are associative with unity and modu[illegible] M are unital *QTAG*-modules. An element $x \in M$ is uniform, i[illegible] is a non-zero uniform (hence uniserial) module and for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M, e(x) = d(xR)$ and $H_M(x) = \sup\{d(\frac{yR}{xR}) | y \in M,\ x \in yR$ and y uniform$\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height. A *QTAG*-module M is said to be separable, if $M^1 = 0$. A family $\mathcal{N}$ of submodules of M is called a nice system in M if.

(i) $0 \in \mathcal{N}$;
(ii) If $\{N_i\}_{i \in I}$ is any subset of $\mathcal{N}$, then $\Sigma_I N_i \in \mathcal{N}$;
(iii) Given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated [1].

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A h-reduced $QTAG$-module M is called totally projective if it has a nice system. A submodule $B \subseteq M$ is a basic submodule of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible.

For a $QTAG$-module M, there is a chain of submodules $M^0 \supset M^1 \supset M^2 \cdots \supset M^\tau = 0$, for some ordinal τ. $M^{\sigma+1} = (M^\sigma)^1$, where M^σ is the σth-*Ulm* submodule of M. A fully invariant submodule $L \subset M$ is a large submodule of M, if $L + B = M$ for every basic submodule B of M. It was proved that several results which hold for TAG-modules also hold good for $QTAG$-modules [2]. Notations and terminology are followed from [3].

The *Ulm*-sequence of x is defined as $U(x) = (H(x), H(x_1), H(x_2), \dots)$. This is analogous to the U-sequences in groups [4]. These sequences are partially ordered because $U(x) \le U(y)$ if $H(x_i) \le H(y_i)$ for every i. Transitive and fully transitive $QTAG$-modules are defined with the help of U-sequences. *Ulm* invariants and *Ulm* sequences play an important role in the study of $QTAG$-modules. Using these concepts transitive and fully transitive modules were defined in [5]. A $QTAG$-module M is fully transitive if for $x, y \in M$, $U(x) \le U(y)$, there is an endomorphism f of M such that $f(x) = f(y)$ and it is transitive if for any two elements $x, y \in M$, with $U(x) \le U(y)$, there is an automorphism f of M such that $f(x) = f(y)$.

2. Main results

Mehdi et al. characterized fully invariant submodules and characteristic submodules with the help of their socles and defined socle-regular and strongly socle-regular $QTAG$-modules [6,7]. We start by recalling their definitions:

A $QTAG$-module M is said to be socle-regular (respectively strongly socle-regular) if for all fully invariant (respectively characteristic) submodules K of M, there exists an ordinal σ (depending on K) such that $\mathrm{Soc}(K) = \mathrm{Soc}(H_\sigma(M))$. It is self evident that strongly socle-regular $QTAG$-modules are themselves socle-regular.

Definition 2.1. A submodule N of a $QTAG$-module M is said to be projection-invariant in M if $f(N) \subseteq N$ for all idempotent endomorphisms $f \in \mathrm{End}(M)$. Clearly, fully invariant submodules are projection-invariant but the converse is not true in general [8].

It is easy to see that N is projection-invariant in M if and only if, $f(N) = N \cap f(M)$ for every projection $f \in \mathrm{End}(M)$. Projection-invariant submodules satisfies the property of being distributed across the direct sum *i.e.*, if $M = P \oplus Q$ and N is projection-invariant, then $N = (P \cap N) \oplus (Q \cap N)$ [8].

Motivated by the concepts of socle-regular and strongly socle-regular $QTAG$-modules we make the following definition:

Definition 2.2. A $QTAG$-module M is said to be projectively socle-regular if for each projection-invariant submodule N of M, there is an ordinal σ (depending on N) such that $\mathrm{Soc}(N) = \mathrm{Soc}(H_\sigma(M))$.

It is obvious that projectively socle-regular $QTAG$-modules are socle-regular.

Let us recall the terminology used in [6]:

For a submodule N of M, put $\sigma = \min\{H_M(x) | x \in \mathrm{Soc}\ (N)\}$ and denote $\sigma = \inf(\mathrm{Soc}(N))$. Here $\mathrm{Soc}(N) \subseteq \mathrm{Soc}(H_\sigma(M))$.

Proposition 2.1. *If N is a projection-invariant submodule of a $QTAG$-module M and $\inf(\mathrm{Soc}(N)) = k$, a positive integer, then $\mathrm{Soc}(N) = \mathrm{Soc}(H_k(M))$. Consequently, if M is separable, then M is projectively socle-regular.*

Proof. Suppose that N is a projection-invariant submodule of M and $\inf(\mathrm{Soc}(N)) = k < \omega$. It remains to show that $\mathrm{Soc}(H_k(M)) \subseteq \mathrm{Soc}(N)$. As $\inf(\mathrm{Soc}(N)) = k$, there is an element $x \in \mathrm{Soc}(N)$ such that $H_M(x) = k$ and so $d(\frac{yR}{xR}) = k$, for $y \in M$. Since every element of exponent one and finite height can be embedded in a direct summand by [9] yR is a summand of M containing x. Therefore $M = yR \oplus M'$, for some submodule M' of M. If z is an arbitrary element of $\mathrm{Soc}(H_k(M))/\mathrm{Soc}(H_{k+1}(M))$, then there exists $u \in H^{k+1}(M)$ such that $d(\frac{uR}{zR}) = k$ and hence $M = uR \oplus M''$. Now, $d(uR) = d(yR) = k + 1$, implying that $uR \cong yR$. Then we have that $u = ry + m'$, for some $r \in R$ and $m' \in M'$. We may define $\phi : yR \oplus M' \to M'$ by $\phi(y) = m'$, $\phi(M') = 0$. Now, ϕ is the difference of two idempotent endomorphisms of M and we define $\theta : M \to M$ by $\theta(m) = r(\psi(m)) + \phi(m)$, where ψ is the projection map given by $\psi(y) = y$, $\psi(M') = 0$. Here θ is a sum of idempotents and $\theta(y) = ry + m' = u$. Since $\theta(x) = v$ such that $d(\frac{vR}{\theta(yR)}) = k$ and $vR = zR$ as $d(\frac{uR}{zR}) = k$ and $x \in N$, which is projection-invariant submodule of M, we conclude that $z \in N$. Hence $\mathrm{Soc}(H_k(M))/\mathrm{Soc}(H_{k+1}(M)) \subseteq \mathrm{Soc}(N)$. However, if $s \in \mathrm{Soc}(H_{k+1}(M))$, then $z + s \in \mathrm{Soc}(H_k(M))$ and so by the argument above, $z + s \in \mathrm{Soc}(N)$. Thus we have that $\mathrm{Soc}(H_k(M)) \subseteq \mathrm{Soc}(N)$ and we are done. □

Corollary 2.1. *If M is a $QTAG$-module such that $d(H_\omega(M)) = 1$, then M is projectively socle-regular.*

Proof. Suppose N is a projection-invariant submodule of M. If $\mathrm{Soc}(N) \nsubseteq H_\omega(M)$, then $\inf(\mathrm{Soc}(N))$ is finite and by Proposition 2.1 above we obtain that $\mathrm{Soc}(N) = \mathrm{Soc}(H_k(M))$ for some integer k. So we may assume that $\mathrm{Soc}(N) \subseteq H_\omega(M)$. Since the $H_\omega(M)$ is a uniserial module of decomposition length 1, either $N + \mathrm{Soc}(N) = 0$ whence $\mathrm{Soc}(N) = \mathrm{Soc}(H_{\omega+1}(M))$ or $\mathrm{Soc}(N) = \mathrm{Soc}(H_\omega(M))$ as required. □

The property of a $QTAG$-module M being projectively socle-regular is inherited by submodules of the form $H_\sigma(M)$.

Proposition 2.2. *If M is a projectively socle-regular $QTAG$-module, then so also is $H_\sigma(M)$, for all ordinals σ.*

Proof. Let $K = H_\sigma(M)$ and suppose that N is a projection invariant submodule of K. Let f be an arbitrary idempotent in $\mathrm{End}(M)$. Then $f^* = f|_K$ is an idempotent endomorphism of K. Thus $f(N) = f^*(N) \subseteq N$, since N is projection-invariant submodule of K. Consequently N is a projection-invariant submodule of M and so there is an ordinal ρ such that

Soc(N) = Soc$(H_\rho(M))$. Since N is contained in $H_\sigma(M)$, we infer that $\rho \geq \sigma$; say $\rho = \sigma + \gamma$. But then Soc(N) = Soc$(H_{\sigma+\gamma}(M))$ = Soc$(H_\rho(K))$, showing that K is also a projectively socle-regular $QTAG$-module. □

Theorem 2.1. *Let M be a QTAG-module. If k is a non-negative integer and $H_k(M)$ is projectively socle-regular QTAG-module, then M is a projectively socle-regular module.*

Proof. Let N be projection invariant submodule of M. If inf(Soc(N)) = k is finite, then by Proposition 2.1, Soc(N) = Soc$(H_k(M))$. Otherwise, if inf(Soc(N)) $\geq \omega$, then Soc$(N) \subseteq$ Soc$(H_\omega(M)) \subseteq H_k(M)$ and we consider an idempotent endomorphism f of $H_k(M)$. Since every endomorphisms of $H_k(M)$ lifts to an endomorphism of M, there is an endomorphism $\overline{f}$ of M such that $\overline{f}|_{H_k(M)} = f$. Also, there exists an idempotent g of M such that $g|_{H_k(M)} = \overline{f}|_{H_k(M)} = f$. If we define $\overline{g}: M \to M$ such that $\overline{g} = 0 + g$ then $\overline{g}$ is idempotent endomorphism of M with $\overline{g}|_{H_k(M)} = f$. Hence f(Soc(N)) = $\overline{g}$(Soc(N)) $\subseteq$ Soc(N). As $H_k(M)$ is projectively socle-regular we have that Soc(N) = Soc$(H_\sigma(H_k(M)))$, for some ordinal σ. Thus, Soc(N) = Soc$(H_\rho(M))$, where $\rho = k + \sigma$ and M is projectively socle-regular. □

Proposition 2.3. *Transitive, fully transitive and totally projective QTAG-modules satisfying the following condition are projectively socle-regular: If $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots$ [illegible] β_m are two disjoint finite sequence of ordinals such that th*[illegible] *Kaplansky invariants $f_M(\alpha_i) \neq 0$ for each positive intege*[illegible]*hen there is a direct decomposition $M = L \oplus K$ where $f_L(\alpha_i) =$* [illegible]*or $i = 1, 2, \ldots, n$ and $f_L(\beta_j) = 0$ for $j = 1, 2$,* [illegible] *m.*

Proof. If N is a projection invariant [illegible]modul[illegible] M, the[illegible] N is fully invariant in M. In view of [illegible]r. Thus Soc(N) = Soc$(H_\sigma(M))$, fo[illegible]e ord[illegible], whence M is projectively socle-regular as re[illegible]d. Since a t[illegible]y projective $QTAG$-module is both tran[illegible]e [illegible] fully transit[illegible] and satisfies the above condition, it is project[illegible] socle-regular. □

Theorem 2.2.

(*i*) If [illegible]M) i[illegible] direct [illegible]n of uniserial modules and [illegible]M) [illegible] project[illegible]cle-regular, then M is projec[illegible]vely so[illegible]-regular.

(*ii*) [illegible]ively socle-regular and $M/H_\sigma(M)$ is t[illegible] projective, then M is projectively socle-regular. Her[illegible] an ordinal strictly less than ω^2.

Proof. Let N be an arbitrary projection invariant submodule of M such that $N \subseteq H_\omega(M)$. We have to show that Soc(N) is a projection-invariant submodule of $H_\omega(M)$. Let f be an idempotent endomorphism of $H_\omega(M)$, then there is an idempotent endomorphism g of M such that $g|_{H_\omega(M)} = f$. But then f(Soc(N) = g(Soc$(N) \subseteq N$, because N is a projection-invariant submodule of M. Therefore Soc(N) is a projection-invariant submodule of $H_\omega(M)$. So Soc(N) = Soc$(H_\sigma(H_\omega(M)))$ = Soc$(H_{\omega+\sigma}(M))$ for some ordinal σ and so M is projectively socle-regular, which completes the proof of (*i*).

For (*ii*), we use the transfinite induction. If $\sigma \leq \omega$, the result holds from (*i*). Now suppose that the result holds for all ordinals less than σ. There are two possibilities: either σ is a successor or σ is a limit ordinal of the form $\omega \cdot n$. In the first case $\sigma = \beta + 1$, for some β. Let $X = H_\beta(M)$ and note that $H(X) = H_\sigma(M)$ is projectively socle-regular. If M is projectively socle-regular and N a projective-invariant submodule of M such that $H_\omega(N) = H_\omega(M)$, then N is projectively socle-regular. Therefore $X = H_\beta(M)$ is projectively socle-regular. Moreover as $\beta < \sigma$, it is easy to show that $M/H_\beta(M)$ is totally projective. Inductively M is projectively socle-regular. In the second case $\sigma = \beta + \omega$, for some β. Set $X = H_\beta(M)$ [illegible]$(X) = H_\sigma(M)$ is projectively socle-regular. Now X[illegible](X) [illegible]$M)/H_\sigma(M)$ and this is totally projective hen[illegible]s a direct s[illegible] of uniserial modules. It now follows from part [illegible]ove that X[illegible] $H_\beta(M)$ is projectively socle-regular. [illegible], M/H_β[illegible] is to[illegible]y projective and therefore inductivel[illegible] is pro[illegible]ective [illegible]gular. □

The next assert[illegible]on [illegible]tes tha[illegible]rtain submodules inherit projective[illegible]e-regu[illegible].

Propositi[illegible] *If M is a p*[illegible]*tively socle-regular QTAG-module an*[illegible] *N is a* [illegible]*ection-invariant submodule of M with the sam*[illegible]*Ulm-submo*[illegible] *then N is projectively socle-regular.*

[illegible]**oof.** Suppose Q is an arbitrary projection-invariant submod[illegible] of N. Sin[illegible]he projection-invariant property is obviously [illegible]itive, it f[illegible]ws that Q is a projection-invariant submodul[illegible] T[illegible]efore there is an ordinal σ such that Soc(Q) = Soc$(H_\sigma$[illegible]). If $\sigma \geq \omega$, it follows that Soc(Q) = Soc$(H_\sigma(N))$ [illegible] we are done. If now σ is a finite ordinal number, say n, the[illegible]oc(Q) = Soc$(H_n(M)) \supseteq$ Soc$(H_n(N))$ and so it is easy to check that Soc(Q) = Soc$(H_k(N))$ for some natural number k, as required. □

Corollary 2.2. *If M is a projectively socle-regular QTAG-module and L is a large submodule of M, then L is projectively socle-regular.*

In the end we would like to mention an open problem as follows:

Problem. *If M is a socle-regular QTAG-module with finite $H_\omega(M)$, does it follows that M is projectively socle-regular?*

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