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ORIGINAL ARTICLE

Characterization through distributional properties of dual generalized order statistics

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Abstract Distributional properties of two non-adjacent dual generalized order statistics have been used to characterize distributions. Further, one sided contraction and dilation for the dual generalized order statistics are discussed and then the results are deduced for generalized order statistics, order statistics, lower record statistics, upper record statistics and adjacent dual generalized order statistics.

1. Introduction

Kamps [6] introduced the concept of generalized order statistics (*gos*) as follows:

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (*iid*) random variables (*rv*) with the absolutely continuous distribution function (*df*) $F(x)$ and the probability density function (*pdf*) $f(x)$, $x \in (\alpha, \beta)$. Let $n \in N$, $n \geqslant 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$,

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such that $\gamma_r = k + (n-r) + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$. If $m_1 = m_2 = \cdots = m_{n-1} = m$, then $X(r,n,m,k)$ is called the rth *m-gos* and its *pdf* is given as:

$$f_{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}[\overline{F}(x)]^{\gamma_r - 1}\left[\frac{1-[\overline{F}(x)]^{m+1}}{m+1}\right]^{r-1} f(x),$$
$$\alpha < x < \beta, \tag{1.1}$$

where $\gamma_r = k + (n-r)(m+1)$ and $c_{r-1} = \prod_{i=1}^{r}\gamma_i$.

Based on the generalized order statistics (*gos*), Burkschat et al. [4] introduced the concept of the dual generalized order statistics (*dgos*) where the *pdf* of the rth *m-dgos* $X^*(r,n,m,k)$ is given as

$$f_{X^*(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1}\left[\frac{1-[F(x)]^{m+1}}{m+1}\right]^{r-1} f(x),$$
$$\alpha < x < \beta,$$

which is obtained just by replacing $\overline{F}(x) = 1 - F(x)$ by $F(x)$.

Ahsanullah [1] has characterized uniform distribution under random contraction for adjacent *dgos*. Khan and Shah [7] have characterized distributions using distributional properties of non-adjacent lower records, upper records and order statistics. In this paper, distributional properties of the *dgos* have been used to characterize a general form of distributions for non- adjacent *dgos* under random translation, dilation and contraction, thus generalizing the results of Ahsanullah [1]. Further, results in terms of lower records, upper records and order statistics are deduced. One may also refer to Alzaid and Ahsanullah [2], Beutner and Kamps [3], Wesolowski and Ahsanullah [8] and Castaño-Martínez et al. [5] for the related results.

Remark 1.1. It may be seen that if Y is a measureable function of X with the relation $Y = h(X)$, then $Y^*(r,n,m,k) = h(X^*(r,n,m,k))$ and $Y(r,n,m,k) = h(X(r,n,m,k))$, if h is increasing function (e.g., $Y_{r:n} = h(X_{r:n})$ and $Y_{L(r)} = h(X_{L(r)})$, where $X_{r:n}$ and $X_{L(r)}$ are the r^{th} order statistic and lower record, respectively). Moreover, $Y(r,n,m,k) = h(X^*(r,n,m,k))$ and $Y^*(r,n,m,k) = h(X(r,n,m,k))$, if h is decreasing function (e.g., $Y_{n-r+1:n} = h(X_{r:n})$ and $Y_{U(r)} = h(X_{L(r)})$, where $Y_{U(r)}$ is the rth upper record).

Remark 1.2. The following elementary facts will be needed in the next section:

(i) if $Y = \log X \sim Gum(\alpha)$ (i.e., $F_Y(y) = e^{-e^{-\alpha y}}, -\infty < y < \infty, \alpha > 0$), then $X \sim in\ W(\alpha)$ (i.e., $F_X(x) = e^{-x^{-\alpha}}$, $0 < x < \infty, \alpha > 0$).
(ii) if -log $X \sim Gum(\alpha)$, then $X \sim Wei(\alpha)$, (i.e., $F_X(x) = 1 - e^{-x^{\alpha}}, 0 < x < \infty, \alpha > 0$).
(iii) if $Y = \log X \sim exp(\alpha)$ (i.e., $F_Y(y) = 1 - e^{-\alpha y}, 0 < y < \infty, \alpha > 0$), then $X \sim Par(\alpha)$, (i.e., $F_X(x) = 1 - x^{-\alpha}$, $1 < x < \infty$, $\alpha > 0$).
(iv) if -log $X \sim exp(\alpha)$,then $X \sim pow(\alpha)$, (i.e., $F_X(x) = x^{\alpha}$, $0 < x < 1$, $\alpha > 0$).
(v) if $Y = \log X \sim genexp(\alpha)$ (i.e.,$F_Y(y) = [1-(m+1)e^{-\alpha y}]^{\frac{1}{m+1}}, \frac{1}{\alpha}\log(m+1) < y < \infty, \alpha > 0$), then $X \sim gen$-$Par(\alpha)$ (i.e., $F_X(x) = [1-(m+1)x^{-\alpha}]^{\frac{1}{m+1}}, (m+1)^{\frac{1}{\alpha}} < x < \infty, \alpha > 0$).
(vi) if -log $X \sim genexp(\alpha)$, then $X \sim genpow(\alpha)$ (i.e., $F_X(x) = 1 - [1-(m+1)x^{\alpha}]^{\frac{1}{m+1}}, 0 < x < (m+1)^{\frac{1}{\alpha}}, \alpha > 0$).

2. Characterizing results

Theorem 2.1. *Let $X^*(s,n,m,k)$ be the s^{th} m-dgos from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then for $1 \leqslant r < s \leqslant n$,*

$$X^*(r+j,n,m,k) \stackrel{d}{=} X^*(s,n,m,k) + Y_{s-r-j:s-1}, \quad j = 0, 1, \tag{2.1}$$

where $Y_{s-r-j:s-1}$ is the $(s-r-j)$th order statistic from a sample of size $(s-1)$ drawn from $exp(\alpha)$ distribution and is independent of $X^*(s,n,m,k)$ if and only if $X_1 \sim genexp(\alpha)$ and $X \stackrel{d}{=} Y$ denotes that X and Y have the same df.

Proof. To prove the necessary part, let the moment generating function (*mgf*) of $X^*(r,n,m,k)$ be $M_{X^*_{(r)}}(t)$, then $X^*(r,n,m,k) \stackrel{d}{=} X^*(s,n,m,k) + Y$, implies that $M_{X^*_{(r)}}(t) = M_{X^*_{(s)}}(t) \cdot M_Y(t)$.

Since for the *genexp*(α) distribution, we have

$$M_{X^*_{(r)}}(t) = \frac{C_{r-1}}{(r-1)!}\frac{1}{(m+1)^{r-\frac{t}{\alpha}}}\frac{\Gamma(r-\frac{t}{\alpha})\Gamma(\frac{\gamma_r}{m+1})}{\Gamma(r-\frac{t}{\alpha}+\frac{\gamma_r}{m+1})}.$$

Therefore,

$$M_Y(t) = \frac{M_{X^*_{(r)}}(t)}{M_{X^*_{(s)}}(t)} = \frac{\Gamma(s)}{\Gamma(r)}\frac{\Gamma(r-\frac{t}{\alpha})}{\Gamma(s-\frac{t}{\alpha})}.$$

But this is the *mgf* of $Y_{s-r:s-1}$, which is the $(s-r)^{th}$ order statistic from a sample of size $(s-1)$ drawn from $exp(\alpha)$.

To prove the sufficiency part, we have for $s \geqslant r+1$,

$$f_{X^*(r,n,m,k)}(x) = \int_0^x f_{X^*(s,n,m,k)}(y) \cdot f_{Y_{s-r:s-1}}(x-y)dy$$
$$= \frac{\alpha(s-1)!}{(r-1)!(s-r-1)!}\int_0^x [e^{-\alpha(x-y)}]^r[1 - e^{-\alpha(x-y)}]^{s-r-1} \times f_{X^*(s,n,m,k)}(y)dy. \tag{2.2}$$

Differentiate both the sides of (2.2) *w.r.t.* x, to get

$$\frac{d}{dx}f_{X^*(r,n,m,k)}(x) = \alpha r\left[f_{X^*(r+1,n,m,k)}(x) - f_{X^*(r,n,m,k)}(x)\right]$$

or, $f_{X^*(r,n,m,k)}(x) = \alpha r\left[F_{X^*(r+1,n,m,k)}(x) - F_{X^*(r,n,m,k)}(x)\right]$

Now, since (Ahsanullah [1])

$$\left[F_{X^*(r+1,n,m,k)}(x) - F_{X^*(r,n,m,k)}(x)\right] = \frac{F(x)}{\gamma_{r+1}f(x)}f_{X^*(r+1,n,m,k)}(x).$$

Therefore, we have $\frac{(m+1)[F(x)]^m f(x)}{[1-(F(x))^{m+1}]} = \alpha$, which implies that $F(x) = [1-(m+1)e^{-\alpha x}]^{\frac{1}{m+1}}$. Hence the proof. □

Remark 2.1. Let $X_{r:n}$ be the rth order statistic from a sample of size n drawn from a continuous population with the *pdf* $f(x)$ and the *df* $F(x)$, then for $1 \leqslant r < s \leqslant n$,

$$X_{s-j:n} \stackrel{d}{=} X_{r:n} + X_{s-r-j:n-r}, \quad j = 0, 1, \tag{2.3}$$

where $X_{s-r-j:n-r}$ is independent of $X_{r:n}$ *if* and only *if* $X_1 \sim exp(\alpha)$.

This can be established by noting that order statistic appear in the generalized order statistics (*gos*) model as well as in dual generalized order statistics (*dgos*) model. Therefore at $m = 0$, (2.1) may be written as

$$X_{n-r-j+1:n} \stackrel{d}{=} X_{n-s+1:n} + X_{s-r-j:s-1}, \qquad j = 0, 1; \quad 1 \leqslant r < s \leqslant n,$$

which implies

$$X_{s-j:n} \stackrel{d}{=} X_{r:n} + X_{s-r-j:n-r}, \qquad j = 0, 1; \quad 1 \leqslant r < s \leqslant n,$$

obtained by replacing $(n - s + 1)$ by r and $(n - r + 1)$ by s as given by Khan and Shah [7].

Remark 2.2. Alzaid and Ahsanullah [2] have proved that

$$X_{r:n} \stackrel{d}{=} X_{r-1:n} + V$$

where $V \sim exp(n - r + 1)$ *if* and only *if* $X_1 \sim exp\ (1)$.

Remark 2.3. Castaño-Martínez *et al*. [5] have shown that

$$X_{s:n} \stackrel{d}{=} X_{r:n} + V$$

where $V \stackrel{d}{=} -\log W$ with $W \sim Be(n - s + 1, s - r)$ *if* and only *if* $X_1 \sim exp(1)$.

Remark 2.4. As $m \to -1$, $genexp(\alpha)$ tends to the $Gum(\alpha)$ and $X^*(r, n, m, k)$ to $X_{L(r)}$, the rth lower records. Therefore, we have

$$X_{L(r+j)} \stackrel{d}{=} X_{L(s)} + Y_{s-r-j:s-1}, \qquad j = 0, 1; \quad 1 \leqslant r < s,$$

where $Y_{s-r-j:s-1}$ is the $(s - r - j)^{th}$ order statistic from a sample of size $(s - 1)$ drawn from $exp(\alpha)$ distribution and is independent of $X_{L(s)}$ *if* and only *if* $X_1 \sim Gum(\alpha)$, as obtained by Khan and Shah [7].

Remark 2.5. Alzaid and Ahsanullah [2] have shown that

$$X_{L(r)} \stackrel{d}{=} X_{L(r+1)} + V$$

where $V \sim exp(r)$ *if* and only *if* $X_1 \sim Gum(1)$.

Corollary 2.1. *Let $X^*(s, n, m, k)$ be the s^{th} m-dgos from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then for $1 \leqslant r < s \leqslant n$,*

$$X^*(r + j, n, m, k) \stackrel{d}{=} X^*(s, n, m, k) \cdot Y_{s-r-j:s-1}, \quad j = 0, 1, \tag{2.4}$$

where $Y_{s-r-j:s-1}$ is the $(s - r - j)$th order statistic from a sample of size $(s - 1)$ drawn from $Par(\alpha)$ distribution and is independent of $X^*(s,n,m,k)$ if and only if $X_1 \sim genPar(\alpha)$.

Proof. Here the product $X^*(s, n, m, k) \cdot Y_{s-r-j:s-1}$ in (2.4) is called random dilation of $X^*(s, n, m, k)$ (Beutner and Kamps [3]). Note that if

$$\log X^*(r, n, m, k) \stackrel{d}{=} \log X^*(s, n, m, k) + \log Y_{s-r:s-1}$$

then

$$X^*(r, n, m, k) \stackrel{d}{=} X^*(s, n, m, k) \cdot Y_{s-r:s-1}$$

in view of Remarks 1.1 and 1.2 and the result follows. □

Remark 2.6. In case of ordinary order statistics, i.e., at $m = 0$, we have

$$X_{s-j:n} \stackrel{d}{=} X_{r:n} \cdot X_{s-r-j:n-r}, \qquad j = 0, 1; \quad 1 \leqslant r < s \leqslant n,$$

where $X_{s-r-j:n-r}$ is independent of $X_{r:n}$ *if* and only *if* $X_1 \sim Par(\alpha)$, as obtained by Castaño-Martínez et al. [5] and Khan and Shah Imtiyaz [7].

Remark 2.7. As $m \to -1$, we get

$$X_{L(r+j)} \stackrel{d}{=} X_{L(s)} \cdot Y_{s-r-j:s-1}, \qquad j = 0, 1; \quad 1 \leqslant r < s,$$

where $Y_{s-r-j:s-1}$ is the $(s - r - j)$th order statistic from a sample of size $(s - 1)$ drawn from the $Par(\alpha)$ distribution and is independent of $X_{L(s)}$, the sth lower records *if* and only *if* $X_1 \sim inW(\alpha)$.

Corollary 2.2. *Let $X(s, n, m, k)$ be the sth m-gos from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then for $1 \leqslant r < s \leqslant n$,*

$$X(r + j, n, m, k) \stackrel{d}{=} X(s, n, m, k) \cdot Y_{r+j:s-1}, \quad j = 0, 1, \tag{2.5}$$

where $Y_{r+j:s-1}$ is the $(r + j)^{th}$ order statistic from a sample of size $(s - 1)$ drawn from $pow(\alpha)$ distribution and is independent of $X(s, n, m, k)$ if and only if $X_1 \sim genpow(\alpha)$.

Proof. Here the product $X(s, n, m, k) \cdot Y_{r+j:s-1}$ in (2.5) is called random contraction of $X(s - j, n, m, k)$ (Beutner and Kamps [3]). Since

$$-\log X^*(r, n, m, k) \stackrel{d}{=} -\log X^*(s, n, m, k) - \log Y_{s-r:s-1}$$

implies

$$X(r, n, m, k) \stackrel{d}{=} X(s, n, m, k) \cdot Y_{r:s-1}$$

in view of Remarks 1.1 and 1.2 and the result follows. □

Remark 2.8. Beutner and Kamps [3] have shown that for adjacent generalized order statistics

$$X(r, n, m, k) \stackrel{d}{=} X(r + 1, n, m, k) \cdot V$$

where $V \sim pow(r\alpha)$ *if* and only *if* $X_1 \sim genpow(\alpha)$.

Remark 2.9. We can get the corresponding characterizing results for the order statistics at $m = 0$ as:

Let $X_{r:n}$ be the r^{th} order statistic from a sample of size n drawn from continuous population with the *pdf* $f(x)$ and the *df* $F(x)$, then for $1 \leqslant r < s \leqslant n$,

$$X_{r+j:n} \stackrel{d}{=} X_{s:n} \cdot X_{r+j:s-1}, \quad j = 0, 1,$$

where $X_{r+j:s-1}$ is independent of $X_{s:n}$ *if* and only *if* $X_1 \sim pow(\alpha)$, as given by Khan and Shah [7].

For adjacent order statistics one may also refer to Ahsanullah [1] and Wesolowski and Ahsanullah [8].

Remark 2.10. The corresponding result for the lower records as $m \to -1$ is:

Let $X_{U(s)}$ be the sth upper record from a continuous population with the *pdf* $f(x)$ and the *df* $F(x)$, then

$$X_{U(r+j)} \stackrel{d}{=} X_{U(s)} \cdot Y_{r+j:s-1}, \quad j = 0, 1; 1 \leqslant r < s,$$

where $Y_{r+j:s-1}$ is the $(r + j)$th order statistic from a sample of size $(s - 1)$ drawn from $pow(\alpha)$ distribution and is independent of $X_{U(s)}$ *if* and only *if* $X_1 \sim Wei(\alpha)$, as obtained by Khan and Shah [7].

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