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Original Article

Congruences and *d*-filters of principal *p*-algebras

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Abstract The concept of *d*-filters is introduced in *p*-algebras. Some properties of *d*-filters are studied. It is proved that the class $F^d(L)$ of all *d*-filters of a *p*-algebra L is a bounded complete lattice. A characterization of *d*-filters of a principal *p*-algebra is given. Also many properties of congruences induced by the *d*-filters are derived. A relationship between the *d*-filters of a principal *p*-algebra L and the congruences in $[\Phi, \nabla]$ is established.

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1. Introduction

The notion of pseudo-complements was introduced in semi-lattices and distributive lattices by O. Frink [1] and G. Birkhoff [2]. The pseudo-complements in Stone algebras were studied and discussed by O. Frink [1], R. Balbes [3] and G. Gratzer [4] etc. Recently, the concept of Boolean filter of bounded pseudo-complemented distributive lattices was introduced by M. Sambasiva Rao and K. P. Shum in [5]. A. Badawy and K. P. Shum [6] introduced and characterized the congruences and Boolean filters of quasi-modular *p*-algebras. Also A. Badawy and M. Sambasiva Rao [7] studied σ-ideals of distributive *p*-algebras. A. Badawy and M. Atallah [8] introduced the notion of Boolean filters of principal *p*-algebras

In this paper, we further study the *d*-filters in a *p*-algebra L and many properties of *d*-filters are also given. We will give a characterization theorem of *d*-filters of a principal *p*-algebra L. We also notice that the set $F^d(L)$ of all *d*-filters of a *p*-algebra forms a complete lattice. The relationship between the *d*-filters and the congruences in $[\Phi, \nabla]$ of a principal *p*-algebra L is introduced. We also prove that the Boolean algebras $B(L)$ and $Con_B(L) = \{\theta_a : a \in B(L)\}$ are isomorphic, where θ_a is the congruence on L induced by a *d*-filter $[a)^d$ for a closed element a of L. Moreover, we show that the Boolean algebra $Con_B(L)$ can be embedded into the interval $[\Phi, \nabla]$ of $Con(L)$. It is proved that the lattice of all *d*-filters of a finite principal *p*-algebra L is isomorphic to the sublattice $[\Phi, \nabla]$ of $Con(L)$.

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2. Preliminaries

In this section, we cite some known definitions and basic results which can be found in the papers [1,9–13].

A p-algebra is a universal algebra $(L, \vee, \wedge, ^*, 0, 1)$, where $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation * is defined by $x \wedge a = 0 \Leftrightarrow x \leq a^*$.

It is known that the class of all p-algebras is equational. A quasi-modular p-algebra is a p-algebra satisfying the identity

$$((x \wedge y) \vee z^{**}) \wedge x = (x \wedge y) \vee (z^{**} \wedge x).$$

An element a of a p-algebra L is called closed if $a^{**} = a$. Then $B(L) = \{a \in L : a = a^{**}\}$ is the set of all closed elements of L. It is known that $(B(L), \triangledown, \wedge, 0, 1)$, where $a \triangledown b = (a^* \wedge b^*)^*$, forms a Boolean algebra. The set $D(L) = \{x \in L : x^* = 0\} = \{x \vee x^* : x \in L\}$ of all dense elements of L is a filter of L.

For an arbitrary lattice L, the set $F(L)$ of all filters of L ordered by the set inclusion forms a lattice. It is known that $F(L)$ is modular (distributive) if and only if L is a modular (distributive) lattice. Let $a \in L$ and $[a)$ be the principal filter of L generated by $a : [a) = \{x \in L : x \geq a\}$.

An equivalent relation θ on a p-algebra $(L; \vee, \wedge, ^*)$ is called a congruence relation if

(1) θ is a lattice congruence, i.e., for all $(x, y), (x_1, y_1) \in \theta$ implies $(x \wedge x_1, y \wedge y_1), (x \vee x_1, y \vee y_1) \in \theta$,
(2) $(x, y) \in \theta$ implies $(x^*, y^*) \in \theta$.

Through what follows, for a p-algebra L we shall denote by ∇ the universal congruence on L. The Cokernel of the lattice congruence θ on a lattice L is defined as

$$Coker\theta = \{x \in L : (x, 1) \in \theta\}.$$

The relation Φ of a p-algebra L is defined by $(x, y) \in \Phi \Leftrightarrow x^{**} = y^{**}$ and is called the Glivenko congruence relation. It is known that the Glivenko congruence is indeed a congruence on L such that $L/\Phi \cong B(L)$ holds.

We frequently use the following rules in the computations of p-algebras (see [10,13]):

(1) $0^{**} = 0$ and $1^{**} = 1$,
(2) $a \wedge a^* = 0$;
(3) $a \leq b$ implies $b^* \leq a^*$,
(4) $a \leq a^{**}$,
(5) $a^{***} = a^*$,
(6) $(a \vee b)^* = a^* \wedge b^*$,
(7) $(a \wedge b)^* \geq a^* \vee b^*$,
(8) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,
(9) $(a \vee b)^{**} = (a^* \wedge b^*)^* = (a^{**} \vee b^{**})^{**}$.

Haviar [14] introduced the class of principal p-algebras which contains all quasi-modular p-algebras having a smallest dense element.

Definition 2.1 ([14]). A p-algebra $(L; \vee, \wedge, ^*, 0, 1)$ is called a principal p-algebra, if it satisfies the following conditions:

(i) The filter $D(L)$ is principal, i.e., there exists an element $d \in L$ such that $D(L) = [d)$,
(ii) The element d is distributive, i.e., $(x \wedge y) \vee d = (x \vee d) \wedge (y \vee d)$ for all $x, y \in L$,
(iii) $x = x^{**} \wedge (x \vee d)$ for any $x \in L$.

Throughout this paper, d stands for a smallest dense element of a principal p-algebra L, unless otherwise mentioned.

3. Properties of *d*-filters

In this section, we introduce the concept of d-filter of a p-algebra. Some properties of d-filters in a p-algebra are derived. A characterization theorem of d-filters of a principal p-algebra will be given.

Definition 3.1. For any filter F of a p-algebra L, define an extension of F as the set

$$F^d = \{x \in L : x^{**} \geq f \text{ for some } f \in F\}$$

The following two Lemmas represent some basic properties of the set F^d.

Lemma 3.2. *The set F^d is a filter of a p-algebra L containing F.*

Proof. Clearly $1 \in F^d$. Let $x, y \in F^d$. Then $x^{**} \geq f$ and $y^{**} \geq g$ for some f, g of F. Hence $(x \wedge y)^{**} = x^{**} \wedge y^{**} \geq f \wedge g$. It follows that $x \wedge y \in F^d$ as $f \wedge g \in F$. Now, let $z \in L$ be such that $z \geq x \in F^d$. Then $z^{**} \geq x^{**} \geq f$ for some $f \in F$. Hence $z \in F^d$. Therefore F^d is a filter of L. Since $x^{**} \geq x$ for any $x \in F$, we have that $x \in F^d$ and $F \subseteq F^d$. □

Lemma 3.3. *For any two filters F, G of a p-algebra L, we have the following:*

(1) $F \subseteq G$ *implies* $F^d \subseteq G^d$,
(2) $(F \cap G)^d = F^d \cap G^d$,
(3) $(F^d)^d = F^d$.

Proof.

(1) Suppose that $F \subseteq G$. Let $x \in F^d$. Then, $x^{**} \geq f$ for some $f \in F$. It follows that $x \in G^d$ as $f \in G$.
(2) Obviously $(F \cap G)^d \subseteq F^d \cap G^d$. Conversely, let $x \in F^d \cap G^d$. Then $x^{**} \geq f$ and $x^{**} \geq g$ for some $f, g \in F$. Hence $x^{**} \geq f \vee g$. It yields that $x \in (F \cap G)^d$, where $f \vee g \in F \cap G$. Consequently $F^d \cap G^d \subseteq (F \cap G)^d$.
(3) By (1) above, $F^d \subseteq (F^d)^d$. Conversely, let $x \in (F^d)^d$. Then $x^{**} \geq f$ for some $f \in F^d$. Since $f \in F^d$, we have $f^{**} \geq f_1$ for some $f_1 \in F$. Hence $x^{**} \geq f^{**} \geq f_1$. Then $x \in F^d$ as $f_1 \in F$.

□

We now introduce the concept of d-filters in a p-algebra.

Definition 3.4. A filter F of a p-algebra L is called an d-filter of L if it satisfies the condition, $F = F^d$.

From Lemma 3.3(2), we can observe that the intersection of two d-filters of a p-algebra is again a d-filter. But, in general, the supremum of two d-filters need not be a d-filter. However, in the following, we obtain the class $F^d(L)$ of all d-filters of L that is a bounded lattice.

Theorem 3.5. *For any p-algebra L, the class $F^d(L)$ forms a complete lattice on its own.*

Proof. For any two d-filters F, G of L, define the ordering $\leq$ on $F^d(L)$ such that $F \leq G \Leftrightarrow F \subseteq G$. Then clearly $(F^d(L), \leq)$ is a partially ordered set. Now, consider the following:

$$F \cap G = (F \cap G)^d \text{ and } F \sqcup G = (F \vee G)^d.$$

Clearly by Lemma 3.3(2), $(F \cap g)^d$ is the infimum of both F an G in $F^d(L)$. Clearly $(F \vee G)^d$ is an upper bound for F an G in $F^d(L)$. Suppose that K is a d-filter of L such that $F \subseteq K$ and $G \subseteq K$. Let $x \in (F \vee G)^d$. Then $x^{**} \geq f \wedge g$ for some $f \in F \subseteq K$ and $x \in G \subseteq K$. Hence $x \in K^d = K$. Therefore, $(F \vee G)^d$ is the supremum of both H and G in $F^d(L)$. Then $(F^d(L), \cap, \sqcup, [d), L)$ is a bounded lattice, where $[1)^d = [d)$ and $L^d = [0)^d = L$ are the smallest and greatest members of $F^d(L)$, respectively. By the extension of the properties $F \cap G = (F \cap G)^d$ and $F \sqcup G = (F \vee G)^d$, the lattice $(F^d(L), \cap, \sqcup, [d), L)$ is a complete. □

In the following theorem, we characterize the d-filters of a principal p-algebra.

Theorem 3.6. *Let F be a filter of a principal p-algebra L with the smallest dense element d. Then the following conditions are equivalent:*

(1) *F is a d-filter,*
(2) *$x^{**} \in F$ implies $x \in F$,*
(3) *For $x, y \in L$, $x^* = y^*$ and $x \in F$ imply $y \in F$,*
(4) *$d \in F$.*

Proof.

(1)⇒(2): Let F be a d-filter of L. Suppose $x^{**} \in F$. Since $(x \vee d)^{**} = 1 \in F$ (as $x \vee d \in D(L)$), we have $x \vee d \in F^d = F$. Then $x^{**} \wedge (x \vee d) \in F$. By Definition 2.1(iii), we get $x \in F$.

(2)⇒(3): Assume the condition (2). Let $x, y \in L$, $x^* = y^*$ and $x \in F$. Then, $y^{**} = x^{**} \in F$. Thus by condition (2), we obtain $y \in F$.

(3)⇒(4): Assume the condition (3). Since $d^* = 0 = 1^*$, we get by (3) that $d \in F$.

(4)⇒(1): Assume $d \in F$. We always have $F \subseteq F^d$. Conversely, let $x \in F^d$. Then $x^{**} \geq f$ for some $f \in F$. Hence $x^{**} \in F$. Since $x \vee d \geq d \in F$, we obtain $x \vee d \in F$. Thus, by Definition 2.1(iii), $x = x^{**} \wedge (x \vee d) \in F$ and $F^d \subseteq F$. Then F is a d-filter of L. □

4. Congruences on a principal *p*-algebra

In this section we investigate the relationships between the set of all d-filters and congruences of a principal p-algebra.

Definition 4.1. A congruence θ of a p-algebra L is called a closed congruence if $(x, x^{**}) \in \theta$ for all $x \in L$.

We first state the following proposition.

Proposition 4.2. *Let L be a principal p-algebra L with the smallest dense element d. Define the relation θ_d on L such that*

$$(x, y) \in \theta_d \text{ if and only if } x \wedge d = y \wedge d$$

Then we have the following:

(1) *θ_d is a closed congruence on L and $Coker\theta_d = [d)$,*
(2) *The quotient set L/θ_d is a Boolean lattice*

Proof.

(1) It is clear that θ_d is a lattice congruence on L. Let $(x, y) \in \theta_d$. Then $x \wedge d = y \wedge d$. Hence $x^{**} = x^{**} \wedge d^{**} = (x \wedge d)^{**} = (y \wedge d)^{**} = y^{**} \wedge d^{**} = y^{**}$ as $d_d^{**} = 1$. It follows that $x^* = y^*$. Hence $x^* \wedge d = y^* \wedge d$ and $(x^*, y^*) \in \theta_d$. Therefore θ_d is a congruence on L. By Definition 2.1(iii), we have

$$x \wedge d = x^{**} \wedge (x \vee d) \wedge d = x^{**} \wedge d.$$

Then we deduce that $(x, x^{**}) \in \theta_d$. Now

$$\begin{aligned} Coker\,\theta_d &= \{x \in L : (x, 1) \in \theta_d\} \\ &= \{x \in L : x \wedge d = 1 \wedge d = d)\} \\ &= \{x \in L : x \geq d)\} \\ &= [d). \end{aligned}$$

(2) It is known that $(L/\theta_d, \vee, \wedge, [0]\theta_d, [1]\theta_d)$ is a bounded lattice, where $L/\theta_d = \{[x]\theta_d : x \in L\}$, $[x]\theta_d \vee [y]\theta_d = [x \vee y]\theta_d$ and $[x]\theta_d \wedge [y]\theta_d = [x \wedge y]\theta_d$. By (1), θ_d is a closed congruence. Hence $[x]\theta_d = [x^{**}]\theta_d$ for every $x \in L$. This deduces immediately that L/θ_d is distributive. Since $x \wedge x^* = 0$ and $(x \vee x^*, 1) \in \theta_d$ (as $(x \vee x^*) \wedge d = d = 1 \wedge d$ for all $x \in L$), we get $[x]\theta_d \wedge [x^*]\theta_d = [x \wedge x^*]\theta_d = [0]\theta_d$ and $[x]\theta_d \vee [x^*]\theta_d = [x \vee x^*]\theta_d = [1]\theta_d$, respectively. It follows that the congruence class $[x^*]\theta_d$ is the complement of $[x]\theta_d$ in L/θ_d. Therefore L/θ_d is a Boolean lattice. □

Lemma 4.3. *Let θ be a closed congruence on a principal p-algebra L with the smallest dense element d. Then $Coker\theta$ is a d-filter of L.*

Proof. Obviously $Coker\theta = \{x \in L : (x, 1) \in \theta\}$ is a filter of L. Since θ is a closed congruence, we get $(d, 1) = (d, d^{**}) \in \theta$. Hence $d \in Coker\theta$. By Theorem 3.6(4), $Coker\theta$ is a d-filter of L. □

From Proposition 4.2(1) and Lemma 4.3, we have the following Corollary

Corollary 4.4. *The filter $[d)$ is a d-filter of L.*

For a d-filter F of a principal p-algebra L, define a relation θ_F on L as follows:

$$(x, y) \in \theta_F \Leftrightarrow x^{**} \wedge a = y^{**} \wedge a \text{ for some } a \in F \cap B(L).$$

We now establish the following theorem for a d-filter of L.

Theorem 4.5. *Let F be a d-filter of a principal p-algebra L with the smallest dense element d. Then the following statements hold:*

(1) *θ_F is a congruence on L such that $\Phi \subseteq \theta_F$,*
(2) *θ_F is a closed congruence on L,*
(3) *$Coker\theta_F = F$,*
(4) *$\theta_{[1)} = \Phi$ and $\theta_{[0)} = \nabla$ whenever F is identical with $[1)$, respectively, $[0)$,*
(5) *L/θ_F is a Boolean lattice.*

Proof.

(1) Clearly, θ_F is an equivalence relation on L. Now we prove that θ_F is a lattice congruence on L. Let $(x, y), (c, d) \in \theta_F$. Then $x^{**} \wedge a = y^{**} \wedge a$ and $c^{**} \wedge b = d^{**} \wedge b$ for some a, $b \in F \cap B(L)$. Now we have the following equalities.

$$\begin{aligned} (x \wedge c)^{**} \wedge (a \wedge b) &= x^{**} \wedge c^{**} \wedge a \wedge b \\ &= y^{**} \wedge d^{**} \wedge a \wedge b \\ &= (y \wedge d)^{**} \wedge (a \wedge b) \end{aligned}$$

Then $(x \wedge c, y \wedge d) \in \theta_F$. Now by distributivity of $B(L)$ we have

$$\begin{aligned}(x \vee c)^{**} \wedge (a \wedge b) &= (x^* \wedge c^*)^* \wedge (a \wedge b)\\ &= (x^{***} \wedge c^{***})^* \wedge (a \wedge b)\\ &= (x^{**} \triangledown c^{**}) \wedge (a \wedge b)\\ &= (x^{**} \wedge a \wedge b) \triangledown (c^{**} \wedge a \wedge b)\\ &= (y^{**} \wedge a \wedge b) \triangledown (d^{**} \wedge a \wedge b)\\ &= (y^{**} \triangledown d^{**}) \wedge (a \wedge b)\\ &= (y \vee d)^{**} \wedge (a \wedge b)\end{aligned}$$

Then $(x \vee c, y \vee d) \in \theta_F$ as $a \wedge b \in F \cap B(L)$. Now we show that θ_F preserves the operation *. Let $(x, y) \in \theta_F$. Then $x^{**} \wedge a = y^{**} \wedge a$ for some $a \in F \cap B(L)$. Now by the distributivity of $B(L)$ we have the following set of implications.

$$\begin{aligned}x^{**} \wedge a = y^{**} \wedge a &\Rightarrow (x^{**} \wedge a) \triangledown a^* = (y^{**} \wedge a) \triangledown a^*\\ &\Rightarrow (x^{**} \triangledown a^*) \wedge (a \triangledown a^*)\\ &= (y^{**} \triangledown a^*) \wedge (a \triangledown a^*)\\ &\Rightarrow x^{**} \triangledown a^* = y^{**} \triangledown a^*\\ &\Rightarrow (x^{***} \wedge a^{**})^* = (y^{***} \wedge a^{**})^*\\ &\Rightarrow (x^{***} \wedge a)^{**} = (y^{***} \wedge a)^{**}\\ &\Rightarrow x^{***} \wedge a = y^{***} \wedge a\\ &\Rightarrow (x^*, y^*) \in \theta_F\end{aligned}$$

It is immediate that θ_F is a congruence on L. Let $(x, y) \in \Phi$. Then $x^{**} = y^{**}$. Hence, $x^{**} \wedge a = y^{**} \wedge a$, for some $a \in F \cap B(L)$. Thus $(x, y) \in \theta_F$ and $\Phi \subseteq \theta_F$.

(2) Since $x^{****} \wedge a = x^{**} \wedge a$ for some $a \in F \cap B(L)$, $(x^{**}, x) \in \theta_F$, and thereby θ_F is closed congruence.

(3) It is known that $Coker\theta_F = [1]\theta_F$. Let $x \in Coker\theta_F$. Then we get the following implications:

$$\begin{aligned}x \in Coker\theta_F &\Rightarrow (x, 1) \in \theta_F\\ &\Rightarrow x^{**} \wedge a = 1^{**} \wedge a \text{ for some } a \in F \cap B(L)\\ &\Rightarrow x^{**} \wedge a = a \text{ as } 1^{**} = 1\\ &\Rightarrow x^{**} \geq a \in F\\ &\Rightarrow x^{**} \in F\\ &\Rightarrow x \in F \text{ as } F \text{ is a } d\text{-filter of } L.\end{aligned}$$

Then $Coker\theta_F \subseteq F$. Conversely, let $y \in F$. Then

$$\begin{aligned}y \in F &\Rightarrow y^{**} \wedge y^{**} = y^{**} = 1^{**} \wedge y^{**}\\ &\Rightarrow (y, 1) \in \theta_F \text{ as } y^{**} \in F \cap B(L)\\ &\Rightarrow y \in Coker\theta_F\end{aligned}$$

Then $F \subseteq Coker\theta_F$.

(4) Since $[1) \cap B(L) = \{1\}$ and $[0) \cap B(L) = B(L)$, we deduce the following equalities:

$$\begin{aligned}\theta_{[1)} &= \{(x, y) \in L \times L : x^{**} \wedge 1 = y^{**} \wedge 1\}\\ &= \{(x, y) \in L \times L : x^{**} = y^{**}\}\\ &= \Phi,\\ \theta_{[0)} &= \{(x, y) \in L \times L : x^{**} \wedge 0 = y^{**} \wedge 0\}\\ &= \{(x, y) \in L \times L : x, y \in L\}\\ &= \nabla.\end{aligned}$$

(5) From (2) we have, $L/\theta_F = \{[x]\theta_F : x \in L\} = \{[x^{**}]\theta_F : x \in L\}$. Let $[x]\theta_F, [y]\theta_F, [z]\theta_F \in L/\theta_F$. Then

$$\begin{aligned}[x]\theta_F \wedge ([y]\theta_F \vee [z]\theta_F) &= [x \wedge (y \vee z)]\theta_F\\ &= [(x \wedge (y \vee z))^{**}]\theta_F\\ &= [x^{**} \wedge (y \vee z)^{**}]\theta_F\\ &= [x^{**} \wedge (y^{**} \triangledown z^{**})]\theta_F\\ &= [(x^{**} \wedge y^{**}) \triangledown (x^{**} \wedge z^{**})]\theta_F\\ &= [(x \wedge y)^{**} \triangledown (x \wedge z)^{**}]\theta_F\\ &= [((x \wedge y) \vee (x \wedge z))^{**}]\theta_F\\ &= [(x \wedge y) \vee (x \wedge z)]\theta_F\\ &= [x \wedge y]\theta_F \vee [x \wedge z]\theta_F\\ &= ([x]\theta_F \wedge [y]\theta_F) \vee ([x]\theta_F \wedge [z]\theta_F)\end{aligned}$$

This shows that L/θ_F is a distributive lattice. Clearly, $[0]\theta_F$ and $[1]\theta_F = F$ are the zero and the unit elements of L/θ_F. This shows that L/θ_F is a bounded distributive lattice. Now we proceed to show that every $[x]\theta_F$ of L/θ_F has a complement. Since $x \wedge x^* = 0$, $[x]\theta_F \wedge [x^*]\theta_F = [x \wedge x^*]\theta_F = [0]\theta_F$. Since F is a d-filter, $x \vee x^* \in F$. Hence, we have $[x]\theta_F \vee [x^*]\theta_F = [x \vee x^*]\theta_F = F$. Thus we have proved that L/θ_F is a Boolean lattice. □

Now, let $F = [a)^d$ for some $a \in B(L)$. Then $a \in F \cap B(L)$. For brevity, we write θ_a instead of $\theta_{[a)^d}$.

In the following Corollary, we state some congruence properties of a principal p-algebra.

Corollary 4.6. *Let L be a principal p-algebra. Then the following statements hold:*

(1) $(x, y) \in \theta_a \Leftrightarrow x^{**} \wedge a = y^{**} \wedge a$,
(2) $Coker\theta_a = [a)^d$ *and* $Ker\theta_a = (a^*]$,
(3) $\theta_1 = \Phi$ *and* $\theta_0 = \nabla$.

Proof.

(1) Let $(x, y) \in \theta_a$. Then

$$\begin{aligned}(x, y) \in \theta_a &\Rightarrow x^{**} \wedge b = y^{**} \wedge b \text{ for some } b \in [a) \cap B(L)\\ &\Rightarrow x^{**} \wedge b \wedge a = y^{**} \wedge b \wedge a\\ &\Rightarrow x^{**} \wedge a = y^{**} \wedge a \text{ as } b = b^{**} \geq a.\end{aligned}$$

Conversely, let $x^{**} \wedge a = y^{**} \wedge a$. Then $(x, y) \in \theta_a$ as $a \in [a) \cap B(L)$.

(2) By Theorem 4.5(3), we have $Coker\theta_a = [a)^d$. Now we prove the second equality in (2) as follows:

$$\begin{aligned}Ker\theta_a &= \{x \in L : (x, 0) \in \theta_a\}\\ &= \{x \in L : x^{**} \wedge a = 0^{**} \wedge a\}\\ &= \{x \in L : x^{**} \wedge a = 0\} \text{ as } 0^{**} = 0\\ &= \{x \in L : x \leq x^{**} \leq a^*\}\\ &= (a^*].\end{aligned}$$

(3) Using Theorem 4.5 (4), we get $\theta_1 = \theta_{[d)^d} = \Phi$ and $\theta_0 = \theta_{[0)^d} = \theta_L = \nabla$. □

By combining Lemma 4.3 and Theorem 4.5(1), (3) we establish the following characterization theorem of a d-filter of L.

Theorem 4.7. *A filter F of a principal p-algebra L is a cokernel of a congruence $\theta \in [\Phi, \nabla]$ if and only if F is a d-filter.*

Consider $Con_B(L) = \{\theta_a : a \in B(L)\}$, we observe that $Con_B(L)$ is a partially ordered set under set inclusion. We now study properties of the elements in the set $Con_B(L)$.

Theorem 4.8. *Let L be a principal p-algebra. Then for every $a, b \in B(L)$, the following statement hold in $Con_B(L)$:*

(1) $a \leq b$ *if and only if* $\theta_b \subseteq \theta_a$,
(2) *The set* $Con_B(L)$ *is a Boolean algebra on its own. Moreover* $Con_B(L) \cong B(L)$,
(3) $\theta_a \sqcup \theta_b = \theta_{a \wedge b}$ *and* $\theta_a \sqcap \theta_b = \theta_{a \triangledown b}$,
(4) $\theta_a \sqcap \theta_{a^*} = \Phi$ *and* $\theta_a \sqcup \theta_{a^*} = \nabla$.

Proof.

(1) Let $a \leq b$ and $(x, y) \in \theta_b$. Then $x^{**} \wedge b = y^{**} \wedge b$. Hence $x^{**} \wedge b \wedge a = y^{**} \wedge b \wedge a$. This leads to $x^{**} \wedge a = y^{**} \wedge a$. Thus $(x, y) \in \theta_a$ and $\theta_b \subseteq \theta_a$. Conversely, let $\theta_b \subseteq \theta_a$. Then we have $(b, 1) \in \theta_b \subseteq \theta_a$. This implies that $b \wedge a = 1 \wedge a = a$. Thus $a \leq b$.
(2) Define the mapping Ψ: $B(L) \to Con_B(L)$ as follows:

$\Psi(a) = \theta_a$ for all $a \in B(L)$.

By (1) above, Ψ is an order anti-isomorphism between $B(L)$ and $Con_B(L)$. This immediately implies that $Con_B(L)$ is a Boolean algebra. Now if we define the mapping f: $B(L) \to Con_B(L)$ by $f(a) = \theta_{a^*}$, then f is an isomorphism between Boolean algebras $B(L)$ and $Con_B(L)$.
(3) Since by (2) above Ψ is a anti-isomorphism, we have $\Psi(a \wedge b) = \Psi(a) \sqcup \Psi(b)$ and $\Psi(a \triangledown b) = \Psi(a) \sqcap \Psi(b)$, where $\sqcup$ and $\sqcap$ are the join and meet operations on $Con_B(L)$. Now

$$\theta_a \sqcup \theta_b = \Psi(a) \sqcup \Psi(b) = \Psi(a \wedge b) = \theta_{a \wedge b}$$

and

$$\theta_a \sqcap \theta_b = \Psi(a) \sqcap \Psi(b) = \Psi(a \triangledown b) = \theta_{a \triangledown b}.$$

(4) From (3) above we have

$$\theta_a \sqcap \theta_{a^*} = \theta_{a \triangledown a^*} = \theta_1 = \Phi$$

and

$$\theta_a \sqcup \theta_{a^*} = \theta_{a \wedge a^*} = \theta_0 = \nabla.$$

Therefore $Con_B(L) = (Con_B(L), \sqcup, \sqcap, ^-, \Phi, \nabla)$, where $\overline{\theta}_a = \theta_{a^*}$ is the complement of θ_a in $Con_B(L)$ and Φ, ∇ are the smallest and greatest elements of $Con_B(L)$, respectively. □

In the following Corollary an isomorphism between the sublattice $[\Phi, \nabla]$ of $Con(L)$ and the lattice $F^d(L)$ of all d-filters of L is obtained.

Corollary 4.9. *Let L be a finite principal p-algebra. Then* $[\Phi, \nabla] \cong F^d(L)$.

Proof. Since L is finite, the elements of $F^d(L)$ are principal filters and hence $Con_B(L) = [\Phi, \nabla]$. By the above Theorem 4.8, we deduce that $F^d(L) \cong [\Phi, \nabla]$. □

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