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Original Article

Picard, Adomian and predictor–corrector methods for an initial value problem of arbitrary (fractional) orders differential equation

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Abstract We study the two analytical methods, the classical method of successive approximations (Picard method), Adomian decomposition method (ADM) see (Abbaoui and Cherruault, 1994; Adomian et al., 1992; Adomian, 1995) [1–3] and the (numerical method) predictor corrector method (PECE) for an initial value problem of arbitrary (fractional) orders differential equation (FDE). The existence and uniqueness of the solution will be proved and the convergence will be discussed for each method. Some examples will be studied.

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1. Introduction

Let $\alpha \in [0, 1)$. In this paper, we study the existence and uniqueness of the solution of the initial value problem

$$\frac{dx}{dt} + D^{\alpha}x\ (t) = f(t, x(t)), \quad 0 < \alpha < 1, \tag{1}$$

$$x(0) = \tilde{x}. \tag{2}$$

We apply the three methods Adomian, Picard and predictor–corrector to obtain numerical solution of the problem (1) and (2).

Now, the definition of the fractional-order integral and differential operators are given by the following.

Definition 1. Let β be a positive real number, the fractional-order integral of order β of th function f is defined on the interval $[0, T]$ by

$$I_0^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds$$

and the fractional-order derivative of the function $f \in C^1[0, T]$ of order $\alpha \in (0, 1]$ is defined by

$$D^{\alpha} f(t) = I^{1-\alpha}\frac{df}{dt}.$$

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2. Uniqueness theorem

Now, the initial value problem (1) and (2) will be investigated under the following assumptions:

(i) $f : J = [0, T] \times D \to R$ is continuous where D is a closed subset of R;
(ii) f satisfies the Lipschitz condition with Lipschitz constant L
i.e

$$|f(t, x) - f(t, y)| \le L|x - y|, \quad \forall (t, x), (t, y) \in I \times D.$$

Let C = C(J) be the space of all real-valued functions which are continuous on J.

Definition 2. By a solution of the problem (1) and (2) we mean a function $x \in C[0, T]$. This function satisfies the problem (1) and (2).

Let $x(t)$ be a solution of the initial value problem (1) and (2). Integrating (1) we obtain

$$x(t) - \tilde{x} + I^{1-\alpha}\big(x(t) - \tilde{x}\big) = If(t, x),$$

then we have

$$x(t) = \tilde{x}\left(1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right) - \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds + \int_0^t f(s, x(s))ds. \tag{3}$$

Now let $x \in C(J)$ be a solution of the integral Eq. (3), then

$$\frac{dx}{dt} = 0 + \tilde{x}\frac{(1-\alpha)t^{-\alpha}}{\Gamma(2-\alpha)} - \frac{d}{dt}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds + \frac{d}{dt}\int_0^t f(s, x(s))ds = \tilde{x}\frac{t^{-\alpha}}{\Gamma(1-\alpha)} - \tilde{x}\frac{t^{-\alpha}}{\Gamma(1-\alpha)} - I^{1-\alpha}\frac{dx}{dt} + f(t, x)$$

and

$$\frac{dx}{dt} + D^{\alpha}x(t) = f(t, x(t))$$

also

$$\tilde{x} = \tilde{x}\left(1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right)_{t=0} - \left(\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds\right)_{t=0} + \left(\int_0^t f(s, x(s))ds\right)_{t=0}$$

then the problem (1) and (2) and the integral Eq. (3) are equivalent.

Comparison between analytical methods is studied in many papers, for examples [4–7].

Define the operator F as

$$(Fx)(t) = \tilde{x}\left(1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right) - \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds + \int_0^t f(s, x(s))ds, \quad \alpha > 0,\ \forall x \in C.$$

Theorem 1. *Let the assumptions (i)–(ii) be satisfied if* $LT + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} < 1$ *, then the initial value problem* (1) *has a unique solution* $x \in C$.

Proof. Firstly we prove that $F : C \to C$ is continuous.
let $x \in C(J),\ t_1,\ t_2 \in J$ such that $|t_2 - t_1| < \delta$

$$Fx(t_2) - Fx(t_1) = \tilde{x}\frac{t_2^{1-\alpha}}{\Gamma(2-\alpha)} - \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds + \int_0^{t_2} f(s, x(s))ds - \tilde{x}\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)} + \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds - \int_0^{t_1} f(s, x(s))ds$$

$$|Fx(t_2) - Fx(t_1)| \le |\tilde{x}|\frac{\left|t_2^{1-\alpha} - t_1^{1-\alpha}\right|}{\Gamma(2-\alpha)} - \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}|x(s)|ds + \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)}|x(s)|ds + \int_{t_1}^{t_2} |f(s, x(s))|ds$$

$$\begin{aligned}\|Fx(t_2) - Fx(t_1)\| &= \max_{t\in J}|Fx(t_2) - Fx(t_1)| \\ &\le \|\tilde{x}\|\frac{\left|t_2^{1-\alpha} - t_1^{1-\alpha}\right|}{\Gamma(2-\alpha)} - \|x\|\int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}ds + \|x\|\int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)}ds + k\int_{t_1}^{t_2} ds \\ &\le \|\tilde{x}\|\frac{\left|t_2^{1-\alpha} - t_1^{1-\alpha}\right|}{\Gamma(2-\alpha)} + \frac{\|x\|}{\Gamma(2-\alpha)}\left|t_2^{1-\alpha} - t_1^{1-\alpha}\right| + k|t_2 - t_1| \\ &\le \frac{(\|\tilde{x}\| + \|x\|)}{\Gamma(2-\alpha)}\left|t_2^{1-\alpha} - t_1^{1-\alpha}\right| + k|t_2 - t_1| \le \epsilon\end{aligned}$$

where

$$|f(t, x(t))| \le k,$$

This proves that $F : C[0, T] \to C[0, T]$.

Now we prove that F is contraction, for this we have

$$Fx - Fy = -\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds + \int_0^t f(s, x(s))ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} y(s)ds - \int_0^t f(s, y(s))ds$$

$$\begin{aligned}|Fx - Fy| &\le \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}|x(s) - y(s)|ds + \int_0^t |f(s, x(s)) - f(s, y(s))|ds \\ &\le \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}|x(s) - y(s)|ds + L\int_0^t |x(s) - y(s)|ds \\ &\le \|x - y\|\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}ds + LT\|x - y\|\end{aligned}$$

$$\|Fx - Fy\| \le \|x - y\|\left(LT + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\right).$$

Since $LT + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} < 1$, then F is a contraction and F has a unique fixed point, thus there exists a unique solution $x \in C(J)$ of the initial value problem (1) and (2). □

3. Method of successive approximations (Picard method)

Applying Picard method to integral Eq. (3), then the solution is constructed by the sequence

$$x_n(t) = \tilde{x}\left(1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right) - \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x_{n-1}(s)ds + \int_0^t f(s, x_{n-1}(s))ds, \quad n = 1,\ 2,\ \ldots \tag{4}$$

$$x_0 = \tilde{x}\left(1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right)$$

All the functions $x_n(t)$ are continuous functions and x_n can be written as a sum of successive differences

$$x_n = x_0 + \sum_{j=1}^{n} (x_j - x_{j-1})$$

This means that convergence of the sequence x_n is equivalent to convergence the infinite series $\sum(x_j - x_{j-1})$ and the solution will be

$$x(t) = \lim_{n\to\infty} x_n(t)$$

i.e, if the finite series $\sum(x_j - x_{j-1})$ converges, then the sequence $x_n(t)$ will be converge to $x(t)$. To prove the uniform convergence of $\{x_n(t)\}$, we shall consider the associated series

$$\sum_{j=1}^{\infty} (x_j - x_{j-1})$$

from (4) for $n = 1$, we get

$$|x_1(t) - x_0| = \left| - \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x_0(s)ds + \int_0^t f(s, x_0(s)ds \right| \leq x_0 \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + Mt$$

and

$$|x_1(t) - x_0| \leq x_0 \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + MT \tag{5}$$

from (4)

$$\begin{aligned}|x_2 - x_1| &\leq \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} |x_1(s) - x_0(s)|ds + \int_0^t |f(s, x_1(s)) - f(s, x_0(s))|ds \\ &\leq \left(\frac{x_0 T^{1-\alpha}}{\Gamma(2-\alpha)} + MT\right)\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right) + L\left(\frac{x_0 T^{1-\alpha}}{\Gamma(2-\alpha)} + MT\right)t \\ &\leq \left(\frac{x_0 T^{1-\alpha}}{\Gamma(2-\alpha)} + MT\right)\left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + LT\right)\end{aligned}$$

and

$$\begin{aligned}|x_3 - x_2| &\leq \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} |x_2(s) - x_1(s)|ds + \int_0^t |f(s, x_2(s)) - f(s, x_1(s))|ds, \\ &\leq \left(\frac{x_0 T^{1-\alpha}}{\Gamma(2-\alpha)} + MT\right)\left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + LT\right)^2\end{aligned}$$

Repeating this technique, we obtain the general estimate for the terms of the series:

$$|x_n - x_{n-1}| \leq \left(\frac{x_0 T^{1-\alpha}}{\Gamma(2-\alpha)} + MT\right)\left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + LT\right)^{n-1}$$

Since $\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + LT < 1$, then the uniform convergence of

$$\sum_{n=1}^{\infty} (x_n - x_{n-1})$$

is proved and so the sequence $\{x_n(t)\}$ is uniformly convergent.

4. Adomian decomposition method (ADM)

In this section, we shall study Adomian decomposition method (ADM) for the integral Eq. (3). The solution algorithm of the integral Eq. (3) using ADM is

$$x_0(t) = \tilde{x}\left(1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right) \tag{6}$$

$$x_n(t) = -\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x_{n-1}(s)ds + \int_0^t A_{n-1}(s)ds \tag{7}$$

where A_n Adomian polynomial of nonlinear term $f(s, x)$, which have the form

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(f\left(t, \sum_{i=0}^{\infty} \lambda^i x_i\right)\right)_{\lambda=0} \tag{8}$$

and the solution will be

$$x(t) = \sum_{i=0}^{\infty} x_i(t) \tag{9}$$

4.1. Convergence analysis

Theorem 2. *Let the solution of integral Eq. (3) be exist. If $|x_1(t)| < k$, k is a positive constant, then the series solution (9) converges.*

Proof. Define the sequence $\{S_p\}$ such that $S_p = \sum_{i=0}^{p} x_i(t)$ is the sequence of partial sums for the series $\sum_{i=0}^{\infty} x_i(t)$ and we have

$$f(t, x) = \sum_{i=0}^{\infty} A_i,$$

let S_p and S_q be two partial sums with $p > q$, we are going to prove that $\{S_p\}$ is the Caushy sequence in the Banach space E.

$$\begin{aligned} S_p - S_q &= -\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} S_{p-1}(s)ds + \int_0^t \sum_{i=0}^{p} A_{i-1}(s)ds \\ &+ \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} S_{q-1}(s)ds - \int_0^t \sum_{i=0}^{q} A_{i-1}(s)ds \\ &= -\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \left(S_{p-1}(s) - S_{q-1}(s)\right)ds \\ &+ \int_0^t \left(\sum_{i=0}^{q} A_{i-1}(s) + \sum_{i=q+1}^{p} A_{i-1}(s) - \sum_{i=0}^{q} A_{i-1}(s) \right) ds \end{aligned}$$

$$\begin{aligned} \|S_p - S_q\| &\le \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \left|S_{p-1}(s) - S_{q-1}(s)\right| ds \\ &+ \sup_{t\in I} \int_0^t \left| \sum_{i=q+1}^{p} A_{i-1}(s) \right| ds \\ &\le \|S_{p-1}(t) - S_{q-1}(t)\| \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds \\ &+ \sup_{t\in I} \int_0^t \left| \sum_{i=q}^{p-1} A_i(s) \right| ds \\ &\le \|S_{p-1}(t) - S_{q-1}(t)\| \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds \\ &+ \sup_{t\in I} \left| f(S, S_{p-1}) - f(S, S_{q-1}) \right| \int_0^t ds \\ &\le \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \|S_{p-1} - S_{q-1}\| + LT\|S_{p-1} - S_{q-1}\| \\ &\le \left(LT + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \right) \|S_{p-1} - S_{q-1}\| \\ &\le h\|S_{p-1} - S_{q-1}\| \end{aligned}$$

let $p = q + 1$ then

$$\begin{aligned} \|S_{q+1} - S_q\| &\le h\|S_q - S_{q-1}\| \le h^2\|S_{q-1} - S_{q-2}\| \\ &\le h^3\|S_{q-2} - S_{q-3}\| \le \cdots \le h^q\|S_1 - S_0\| \end{aligned}$$

from triangle inequality, we have

$$\begin{aligned} \|S_p - S_q\| &\le \|S_{q+1} - S_q\| + \|S_{q+2} - S_{q+1}\| \\ &\le \cdots \le \|S_p - S_{p-1}\| \le \left(h^q + h^{q+1} + \cdots + h^{p-1}\right)\|S_1 - S_0\| \\ &\le h^q\left(1 + h + \cdots + h^{p-q-1}\right)\|S_1 - S_0\| \\ &\le h^q\left(\frac{1 - h^{p-q}}{1-h}\right)\|x_1\| \end{aligned}$$

where $0 < h < 1$ and $p > q$ imply that $(1 - h^{p-q}) \le 1$. Consequently,

$$\|S_p - S_q\| \le \frac{h^q}{1-h}\|x_1\| \le \frac{h^q}{1-h}\sup_{t\in I}|x_1(t)|$$

but $|x_1(t)| < k$ and as $q \to \infty$, then $\|S_p - S_q\| \to 0$ and hence, $\{S_p\}$ is a caushy sequence in this Banach space E and the series $\sum_{i=0}^{\infty} x_i(t)$ converges. □

4.2. Error analysis

Theorem 3. *The maximum absolute truncation error of the series solution (9) of the integral Eq. (3) is estimated to be*

$$\sup_{t\in I}\left|x(t) - \sum_{i=0}^{q} x_i(t)\right| \le \frac{h^q}{1-h}\sup_{t\in I}|x_1(t)|$$

Proof. From theorem (2), we have

$$\|S_p - S_q\| \le \frac{h^q}{1-h}\sup_{t\in I}|x_1(t)|$$

but $S_p = \sum_{i=0}^{p} x_i(t)$ as $p \to \infty$, then $S_p \to x(t)$ *so*,

$$\|x(t) - S_q\| \le \frac{h^q}{1-h}\sup_{t\in I}|x_1(t)|$$

So, the maximum absolute truncation error in the interval I is

$$\sup_{t\in I}\left|x(t) - \sum_{i=0}^{q} x_i(t)\right| \le \frac{h^q}{1-h}\sup_{t\in I}|x_1(t)|$$

and this completes the proof. □

5. Predictor–corrector method (PECE)

An Adams-type predictor–corrector method has been introduced in [8–12].

In this section, we use an Adams-type predictor–corrector method for the integral Eq. (3).

The product trapezoidal quadrature formula is used with t_j $(j = 0, 1, \ldots, k+1)$, taken with respect to the weight function $(t_{k+1} - u)^{-\alpha}$. In other words, one applies the approximation

$$\begin{aligned} \int_{t_0}^{t_{k+1}} (t_{k+1} - u)^{-\alpha} g(u)du &\approx \int_{t_0}^{t_{k+1}} (t_{k+1} - u)^{-\alpha} g_{k+1}(u)du \\ &= \sum_{j=0}^{k+1} a_{j,k+1} g(t_j), \end{aligned}$$

where

$$a_{j,k+1} = \begin{cases} \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)}\left[k^{2-\alpha} - (k+\alpha-1)(k+1)^{1-\alpha}\right] & if\ j = 0, \\ \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)} & if\ j = k+1 \end{cases}$$

and h is a step size, and for $1 \le j \le k$

$$\begin{aligned} a_{j,k+1} = \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)} &\left[(k-j+2)^{2-\alpha} - 2(k-j+1)^{2-\alpha} \right. \\ &\left. +(k-j)^{2-\alpha}\right] \end{aligned}$$

for $\alpha = 0$ we have

$$a\prime_{j,k+1} = \begin{cases} \frac{h}{2} & \text{if } j = 0,\ j = k+1 \\ \frac{h}{2}\left[(k-j+2)^2 - 2(k-j+1)^2 + (k-j)^2\right] & \text{if } 1 \le j \le k \end{cases}$$

This yields the corrector formula, i.e. the fractional variant of the one-step Adams–Moulton method

$$\begin{aligned} x_{k+1} &= x_0(t_{k+1})\left(1 + \frac{t_{k+1}^{1-\alpha}}{\Gamma(2-\alpha)}\right) \\ &- \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{k} a_{j,k+1} x(t_j) + a_{k+1,k+1} x_{k+1}^p\right] \\ &+ \sum_{j=0}^{k} a\prime_{j,k+1} f(t_j, x(t_j)) + a\prime_{k+1,k+1} f(t_{k+1}, x_{k+1}^p) \end{aligned}$$

The remaining problem is the determination of the predictor formula that is need to calculate the value x^p_{k+1}.

The idea used to generalize the one-step Adam–Bashforth method is the same as the one described above for the Adams–Moulton technique: the integral on the right-hand side of Eq. (3) is replaced by the product rectangle rule, i.e,

$$\int_{t_0}^{t_{k+1}} (t_{k+1}-u)^{-\alpha} g(u)du \approx \sum_{j=0}^{k} b_{j,k+1} g(t_j),$$

where

$$b_{j,k+1} = \frac{h^{1-\alpha}}{1-\alpha}\left[(k-j+1)^{1-\alpha} - (k-j)^{1-\alpha}\right]$$

for $\alpha = 0$ we have

$$b\prime_{j,k+1} = h.$$

Thus, the predictor x^p_{k+1} is determined by the fractional Adams–Bashforth method:

$$x^p_{k+1} = x_0(t_{k+1})\left(1+\frac{t^{1-\alpha}_{k+1}}{\Gamma(2-\alpha)}\right) - \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{k} b_{j,k+1} x(t_j)\right] + \sum_{j=0}^{k} b\prime_{j,k+1} f(t_j, x(t_j))$$

6. Numerical examples

In this section, we shall study some numerical examples applying Picard, ADM and Predictor–corrector methods and comparing the results.

Example 1. Consider the following FDE

$$\frac{dx}{dt} + D^{\frac{1}{2}} x(t) = 1 + \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} - t^2 + x^2, \quad x(0) = 0 \tag{10}$$

which has the exact solution $x(t) = t$.

Applying Picard method to Eq. (10), we get

$$x_n(t) = \left(t + \frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}} - \frac{t^3}{3}\right) - \int_0^t \frac{(t-s)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} x_{n-1}(s)ds + \int_0^t x^2_{n-1}(s)ds, \quad n = 1, 2, \ldots$$

$$x_0(t) = \left(t + \frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}} - \frac{t^3}{3}\right)$$

and the solution will be

$$x(t) = x_n(t).$$

Applying ADM to Eq. (10), we get

$$x_0(t) = \left(t + \frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}} - \frac{t^3}{3}\right)$$

$$x_i(t) = -\int_0^t \frac{(t-s)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} x_{i-1}(s)ds + \int_0^t A_{i-1}(s)ds, \quad i \geq 1.$$

where A_i are Adomian polynomials of the nonlinear term x^2, and the solution will be

$$x(t) = \sum_{i=0}^{q} x_i(t)$$

Applying PECE to Eq. (10), we have

$$x_0 = 0, \quad f(t,x) = 1 + \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} - t^2 + x^2$$

Table 1 shows a comparison between the absolute error of Picard (when $n = 3$), ADM solutions (when $q = 3$) and PECE solutions.

Example 2. Consider the following FDE

$$\frac{dx}{dt} + D^{\frac{2}{3}} x(t) = 2t - t^4 + \frac{2}{\Gamma\left(\frac{7}{3}\right)} t^{\frac{4}{3}} + x^2, \quad x(0) = 0 \tag{11}$$

which has the exact solution $x(t) = t^2$.

Applying Picard method to Eq. (11), we get

$$x_n(t) = \left(t^2 + \frac{2}{\Gamma\left(\frac{10}{3}\right)} t^{\frac{7}{3}} - \frac{t^5}{5}\right) - \int_0^t \frac{(t-s)^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} x_{n-1}(s)ds + \int_0^t x^2_{n-1}(s)ds, \quad n = 1, 2, \ldots$$

$$x_0(t) = \left(t^2 + \frac{2}{\Gamma\left(\frac{10}{3}\right)} t^{\frac{7}{3}} - \frac{t^5}{5}\right)$$

and the solution will be

$$x(t) = x_n(t).$$

Applying ADM to Eq. (11), we get

$$x_0(t) = \left(t^2 + \frac{2}{\Gamma\left(\frac{10}{3}\right)} t^{\frac{7}{3}} - \frac{t^5}{5}\right)$$

Table 1 Absolute error.

t	$\lvert x_{Exact} - x_{ADM}\rvert$	$\lvert x_{Exact} - x_{Picard}\rvert$	$\lvert x_{Exact} - x_{PECE}\rvert$
0.1	0.000145565	0.000154652	0.00594708
0.2	0.000866898	0.00107201	0.00575016
0.3	0.00177981	0.00296762	0.00554446
0.4	0.00160474	0.00544699	0.00545929
0.5	0.00114624	0.00769084	0.00548365
0.6	0.00724476	0.0087612	0.00561132
0.7	0.0159463	0.00794191	0.00584349
0.8	0.0246302	0.00503403	0.0061882
0.9	0.0290786	0.000536499	0.00666045
1.0	0.0244311	0.00434315	0.00728311
1.1	0.00666801	0.00788871	0.00808866
1.2	0.0256868	0.00832175	0.00912196
1.3	0.070065	0.00454006	0.0104444
1.4	0.119007	0.00278959	0.0121396
1.5	0.160355	0.0097316	0.0143225
1.6	0.178695	0.00720761	0.171509
1.7	0.158033	0.02101	0.020844
1.8	0.085346	0.0998739	0.0257082
1.9	0.0455844	0.263316	0.0321751
2.0	0.230932	0.552159	0.0408581

Table 2 Absolute error.

t	$\lvert x_{Exact} - x_{ADM} \rvert$	$\lvert x_{Exact} - x_{Picard} \rvert$	$\lvert x_{Exact} - x_{PECE} \rvert$
0.1	0.0000996616	0.0000999158	0.0038934
0.2	0.000971827	0.000989338	0.00673041
0.3	0.00346046	0.00366554	0.00922231
0.4	0.00772792	0.00887093	0.0114889
0.5	0.0123999	0.0165692	0.0135783
0.6	0.014066	0.0254858	0.0155372
0.7	0.00780759	0.032944	0.0174238
0.8	0.0106784	0.0352666	0.0193149
0.9	0.041401	0.0289394	0.0213133
1.0	0.0760295	0.0124994	0.0235607
1.1	0.0960106	0.0112672	0.0262577
1.2	0.0761537	0.0337261	0.0296985
1.3	0.00391433	0.0409254	0.0343312
1.4	0.141526	0.0188498	0.0408647
1.5	0.293056	0.0321881	0.0504657
1.6	0.369605	0.0682921	0.0651329
1.7	0.261289	0.0428224	0.0884361
1.8	0.103847	0.562204	0.127046
1.9	0.683794	1.86137	0.194076
2.0	1.28107	4.28081	0.316867

$$x_i(t) = -\int_0^t \frac{(t-s)^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} x_{i-1}(s)ds + \int_0^t A_{i-1}(s)ds, \quad i \geq 1.$$

where A_i are Adomian polynomials of the nonlinear term x^2, and the solution will be

Applying PECE to Eq. (11), we have

$$x_0 = 0, \quad f(t,x) = 2t - t^4 + \frac{2}{\Gamma\left(\frac{7}{3}\right)} t^{\frac{4}{3}} + x^2.$$

Table 2 shows a comparison between the absolute error of Picard (when $n = 3$), ADM solutions (when $q = 3$) and PECE solutions.

7. Conclusion

In this paper, we present two analytical methods and numerical method to solve fractional differential equation and the comparison between them in the above two examples. We see from these examples that picard and ADM methods give more accurate solution than PECE in small intervals but PECE gives more accurate solution in large interval.

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