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Original Article

A note on the qualitative behaviors of non-linear Volterra integro-differential equation

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Abstract This paper considers a scalar non-linear Volterra integro-differential equation. We establish sufficient conditions which guarantee that the solutions of the equation are stable, globally asymptotically stable, uniformly continuous on $[0, \infty)$, and belongs to $L^1[0, \infty)$ and $L^2[0, \infty)$ and have bounded derivatives. We use the Lyapunov's direct method to prove the main results. Examples are also given to illustrate the importance of our results. The results of this paper are new and complement previously known results.

AMS (MOS) Mathematics Subject Classification: 34K20; 45D05; 45M05

1. Introduction

In 2009, Becker [1] considered the scalar linear homogeneous Volterra integro-differential equation

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)ds, \tag{1}$$

for $t \geq 0$, where a and b are real-valued and continuous functions on the respective domains $[0, \infty)$ and $\Omega := \{(t,s) : 0 \leq s \leq t < \infty\}$. The author studied the asymptotic behaviors of solutions of Eq. (1) by using the Lyapunov functionals.

In this paper, instead of Eq. (1), we consider the non-linear Volterra integro-differential equation of the form

$$x'(t) = -a(t)h(x(t)) + \int_0^t b(t,s)g(x(s))ds, \tag{2}$$

for $t \geq 0$, where $a : [0, \infty) \to [0, \infty)$, $h : \mathfrak{R} \to \mathfrak{R}$, $g : \mathfrak{R} \to \mathfrak{R}$ and $b : \Omega \to \mathfrak{R}$ are continuous functions on their respective domains, $\Omega := \{(t,s) : 0 \leq s \leq t < \infty\}$, that $h(0) = g(0) = 0$, and $h(x)$ and $g(x)$ are differentiable at $x = 0$.

We can write Eq. (2) in the form of

$$x'(t) = -a(t)h_1(x(t))x(t) + \int_0^t b(t,s)g_1(x(s))x(s)ds,$$

where

$$h_1(x) = \begin{cases} \dfrac{h(x)}{x}, & x \neq 0 \\ h'(0), & x = 0 \end{cases}$$

and

$$g_1(x) = \begin{cases} \dfrac{g(x)}{x}, & x \neq 0 \\ g'(0), & x = 0. \end{cases}$$

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In view of the mentioned information, it follows that the equation discussed by Becker [1], Eq. (1), is a special case of our equation, Eq. (2). That is, our equation, Eq. (2), includes Eq. (1) discussed in [1]. As Becker [1] studied the asymptotic behavior of solutions of linear Volterra integro-differential equation, we investigate the same topic for the nonlinear case. This case shows the novelty of this paper and is an improvement on the topic in the literature. Our results will also be different from that obtained in the literature (see [2–11,13–19] and the references thereof). Namely, the equation considered and the assumptions to be established here are different from that in the mentioned papers above. It should be noted that this paper has also a contribution to the subject in the literature, and it may be useful for researchers working on the qualitative behaviors of solutions for Volterra integro-differential equation.

We give some basic information related Eq. (2) and the non-homogeneous equation

$$x'(t) = -a(t)h(x(t)) + \int_0^t b(t,s)g(x(s))ds + f(t), \tag{3}$$

where $f : [0,\infty) \to \Re$ is a continuous function. It is worth mentioning that the following basic notations and definitions were taken from Becker [1].

We use the following notation throughout this paper (see [1]):

$C[t_0, t_1]$ (resp. $C[t_0, \infty)$) will denote the set of all continuous real-valued functions on $[t_0, t_1]$ (resp. $[t_0, \infty)$).

For $\varphi \in C[0, t_0]$, $|\varphi|_{t_0} := \sup\{|\varphi(t)| : 0 \le t \le t_0\}$.

$L^1[0,\infty)$ denotes the set of all real-valued functions f that are Lebesgue measurable on $[0,\infty)$ and for which the Lebesgue integral $\int_0^\infty |f|$ is finite. However, we use it to denote those functions in $L^1[0,\infty)$ that are also continuous on $[0,\infty)$. For such a function, say g, the improper Riemann integral $\int_0^\infty |g(t)|dt$ converges, i.e., $\lim_{t\to\infty} \int_0^t |g(s)|ds$ exists and is finite. In short, by $g \in L^1[0,\infty)$ we mean that g is continuous and absolutely Riemann integrable on $[0,\infty)$.

$L^2[0,\infty)$ will denote the set of all continuous real-valued functions that are square integrable on $[0,\infty)$. That is, $h \in L^2[0,\infty)$ will mean that h is continuous on $[0,\infty)$ and $h^2 \in L^1[0,\infty)$.

Definition 1. A solution of Eq. (2) (resp. (3)) on $[0, T)$, where $0 < T \le \infty$, with an initial value $x_0 \in \Re$ is a continuous function $x : [0, T) \to \Re$ that satisfies Eq. (2) (resp. (3)) on $[0, T)$ such that $x(0) = x_0$.

Definition 2. The zero solution of Eq. (2) is

(i) stable if for each $\varepsilon > 0$ and $t_0 \ge 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\varphi \in C[0, t_0]$ with $|\varphi|_{t_0} < \delta$ implies that $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \ge t_0$.

(ii) globally asymptotically stable (asymptotically stable in the large) if it is stable and if every solution of Eq. (2) approaches zero as $t \to \infty$.

(iii) uniformly stable if for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\varphi \in C[0, t_0]$ with $|\varphi|_{t_0} < \delta$ (any $t_0 \ge 0$) implies that $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \ge t_0$.

(iv) uniformly asymptotically stable if it is uniformly stable and if there exits an $\eta > 0$ with the following property: For each $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $\varphi \in C[0, t_0]$ with $|\varphi|_{t_0} < \eta$ (any $t_0 \ge 0$) implies that $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \ge t_0 + T$.

2. Main results

At the beginning, we obtain some sufficient conditions so that all of the solutions of Eq. (2) belong to $L^2[0,\infty)$. Then we will add more conditions that will drive these L^2 solutions tend to zero as $t \to \infty$.

Lemma 1. *If*

$$1 \le h_1(x) \le \alpha, \quad \sigma \le g_1(x) \le 1,$$

where α and σ, $\sigma \in (0,1)$, are positive constants,

$$a(t) - \int_0^t |b(t,s)|ds \ge 0$$

for all $t \ge 0$ and if

$$\alpha a(s) - \sigma \int_s^t |b(u,s)|du \ge 0$$

for all $t \ge s \ge 0$, then the zero solution of Eq. (2) is stable. In addition, if for some $t_1 \ge 0$ there is a constant $k > 0$ such that either

$$1 \le h_1(x) \le \alpha, \quad \sigma \le g_1(x) \le 1,$$
$$a(t) - \int_0^t |b(t,s)|ds \ge k$$

for all $t \ge t_1$ or

$$1 \le h_1(x) \le \alpha, \quad \sigma \le g_1(x) \le 1,$$
$$\alpha a(s) - \sigma \int_s^t |b(u,s)|du \ge k$$

for all $t \ge s \ge t_1$ holds, then every solution $x(t)$ of Eq. (2) belongs to $L^2[0,\infty)$.

Proof. We define the Lyapunov functional

$$V : [0,\infty) \times C[0,\infty) \to [0,\infty)$$

by

$$V(t,\psi(.)) := \psi^2(t) + \int_0^t \{a(s)h_1(\psi(s)) - \int_s^t |b(u,s)g_1(\psi(s))|du\}\psi^2(s)ds. \tag{4}$$

It is clear from (4) that $V(t, 0) = 0$ *and* $V(t,\psi(.)) \ge \psi^2(t)$ for all $t \ge 0$.

For any $t_0 \ge 0$ and initial function $\varphi \in C[0, t_0]$, *let* $x(t) = x(t, t_0, \varphi)$ denote the unique solution of Eq. (2) on $[0,\infty)$ such that $x(t) = \varphi(t)$ *for* $0 \le t \le t_0$. For brevity, let $V(t) = V(t, x(.))$, that is, the value of the functional V along the solution $x(t)$ *at* t. Taking the derivative of V with respect to t, we have

$$V'(t) = 2x(t)x'(t) + a(t)h_1(x(t))x^2(t) - \int_0^t |b(t,s)g_1(x(s))|x^2(s)ds$$
$$= 2x(t)[-a(t)h_1(x(t))x(t) + \int_0^t b(t,s)g_1(x(s))x(s)ds]$$
$$+a(t)h_1(x(t))x^2(t) - \int_0^t |b(t,s)g_1(x(s))|x^2(s)ds \leq$$
$$-a(t)h_1(x(t))x^2(t) + 2\int_0^t |b(t,s)||g_1(x(s))||x(t)||x(s)|ds$$
$$-\int_0^t |b(t,s)||g_1(x(s))|x^2(s)ds \leq -a(t)h_1(x(t))x^2(t)$$
$$+\int_0^t |b(t,s)||g_1(x(s))|(x^2(t)+x^2(s))ds$$
$$-\int_0^t |b(t,s)||g_1(x(s))|x^2(s)ds = -a(t)h_1(x(t))x^2(t)$$
$$+\int_0^t |b(t,s)||g_1(x(s))|x^2(t)ds \leq$$
$$-\{a(t) - \int_0^t |b(t,s)|ds\}x^2(t). \tag{5}$$

It follows that the assumptions of Lemma 1 imply

$$V'(t) \leq 0.$$

This last estimate, together with $V(t) \geq x^2(t)$, gives

$$x^2(t) \leq V(t) \leq V(t_0) \tag{6}$$

for all $t \geq t_0$. It is clear that

$$V(t_0) = \varphi^2(t_0) + \int_0^{t_0} \{a(s)h_1(\varphi(s)) - \int_s^{t_0} |b(u,s)g_1(\varphi(s))|du\}\varphi^2(s)ds$$
$$\leq \varphi^2(t_0) + \int_0^{t_0} \{\alpha a(s) - \sigma \int_s^{t_0} |b(u,s)|du\}\varphi^2(s)ds$$
$$\leq |\varphi|^2_{t_0} M(t_0),$$

where

$$M(t_0) := 1 + \int_0^{t_0} [\alpha a(s) - \sigma \int_s^{t_0} |b(u,s)|du\}ds.$$

Then

$$|x(t)| \leq |\varphi|_{t_0}\sqrt{M(t_0)} \tag{7}$$

for all $t \geq t_0$. This implies the zero solution of the considered equation is stable. Namely, for $\varepsilon > 0$, *let* $\delta = \frac{\varepsilon}{\sqrt{M(t_0)}}$. Then, for $\varphi \in C[0, t_0]$ *with* $|\varphi|_{t_0} < \delta$,

$$|x(t)| \leq \delta\sqrt{M(t_0)} = \varepsilon \tag{8}$$

for all $t \geq t_0$. If the assumptions $a(s) - \int_0^t |b(u,s)|du \geq k$ also holds, then we can easily get

$$V'(t) \leq -kx^2(t)$$

for all $t \geq \tau$, *where* $\tau := \max\{t_0, t_1\}$. By integrating the last estimate, we obtain

$$V(t) - V(\tau) \leq -k\int_\tau^t x^2(s)ds.$$

so that

$$x^2(t) \leq V(t) \leq V(\tau) - k\int_\tau^t x^2(s)ds \tag{9}$$

for all $t \geq \tau$. If, on the other hand, the assumption $\alpha a(s) - \sigma\int_s^t |b(u,s)|du \geq k$ holds, then (4) and (6) together yield

$$x^2(t) + k\int_{t_1}^t x^2(s)ds \leq V(t) \leq V(t_0) \tag{10}$$

for all $t \geq t_1$. Either one, (9) or (10) implies that $x^2 \in L^1[0,\infty)$.

We have just proved that under the conditions of Lemma 1, the solution $x(t, t_0, \varphi)$ of Eq. (2) belongs to $L^2[0,\infty)$. It seems plausible that $x^2(t) \to 0$ *as* $t \to \infty$. However, by itself convergence of an improper Riemann integral of a function f *on* $[0,\infty)$ does not ensure that f *approaches* 0 *as* $t \to \infty$ (see [12]). But if f were also known to be uniformly continuous, then it would be according to the next lemma attributed to Barbălat [20]. The proof of Lemma 1 is completed. □

Lemma 2 (Barbălat's Lemma)**.** *If $f : [0,\infty) \to \Re$ is both uniformly continuous and Riemann integrable on $[0,\infty)$, then $f(t) \to 0$ as $t \to \infty$ (see [20]).*

Lemma 3. *If $f : [0,\infty) \to \Re$ is uniformly continuous on $[0,\infty)$ and if f^2 is Riemann integrable on $[0,\infty)$, then $f(t) \to 0$ as $t \to \infty$ (see [1]).*

Theorem 1. *If $1 \leq h_1(x) \leq \alpha$, $\sigma \leq g_1(x) \leq 1$, where α and σ, $\sigma \in (0,1)$, are positive constants,*

$$a(t) - \int_0^t |b(t,s)|ds \geq 0$$

for all $t \geq 0$,

$$a(s) - \int_s^t |b(u,s)|du \geq 0$$

for all $t \geq s \geq 0$, and if for some $t_1 \geq 0$ there are positive constants k and K such that either

$$k + \int_0^t |b(t,s)|ds \leq a(t) \leq K\alpha^{-1}$$

for all $t \geq t_1$ or

$$k + \int_s^t |b(u,s)|du \leq a(s) \leq K\alpha^{-1}$$

for all $t \geq s \geq t_1$, then all solutions of Eq. (2) are uniformly continuous on $[0,\infty)$ and belong to $L^2[0,\infty)$. Furthermore, the zero solution of Eq. (2) is globally asymptotically stable, (see also [15]).

Proof. We only need to show that all solutions of Eq. (2) tend to zero since the stability has already been established in Lemma 1. To this end, for any $t_0 \geq 0$ *and* $\varphi \in C[0, t_0]$, consider the corresponding solution $x(t) = x(t, t_0, \varphi)$. By (7), we have

$$\|x(t)| \leq |\varphi|_{t_0}\sqrt{M(t_0)}$$

for all $t \geq t_0$.

Consequently, since $a(t) \leq K\alpha^{-1}$ *for* $t \geq t_1, 1 \leq h_1(x) \leq \alpha$ $\sigma \leq g_1(x) \leq 1$, it follows that

$$|x'(t)| \le a(t)h_1(x(t))|x(t)| + \int_0^{t_0} |b(t,s)||g_1(\varphi(s))||\varphi(s)|ds$$
$$+ \int_{t_0}^{t} |b(t,s)||g_1(x(s))||x(s)|ds$$
$$\le 2K|\varphi|_{t_0}\sqrt{M(t_0)} + K|\varphi|_{t_0}$$

for $t \ge \tau$, *where* $\tau = \max\{t_0, t_1\}$. *Since* $x'(t)$ is bounded on $[\tau, \infty)$, $x(t)$ satisfies a Lipschitz condition on $[\tau, \infty)$. Consequently, it is uniformly continuous on $[\tau, \infty)$. This and the continuity of $x(t)$ *on* $[0, \infty)$ *imply* $x(t)$ is uniformly continuous on the entire interval $[0, \infty)$. By Lemma 1, $x^2(t) \in L^1[0, \infty)$. Therefore, by Lemma 3, it follows that $x(t) \to 0$ *as* $t \to \infty$.

The proof of Theorem 1 is completed. □

Example 1. Consider the non-linear Volterra integro-differential equation of the form

$$x'(t) = -\left(k + \frac{1}{1+t}\right)\left(x(t) + \frac{x(t)}{1+x^2(t)}\right)$$
$$+ \int_0^t \frac{\cos s}{(1+t)^3}\left(\frac{x(s)}{2} + \frac{x^3(s)}{1+2x^2(s)}\right)ds$$

for $t \ge 0$, *where* k is a positive real number.

When we compare this equation with Eq. (2), it follows that

$$a(t) = k + \frac{1}{1+t},$$
$$h(x) = x + \frac{x}{1+x^2}, \quad h_1(x) = 1 + \frac{1}{1+x^2}, \ (x \ne 0),$$
$$l1 \le h_1(x) = 1 + \frac{1}{1+x^2} \le 2 = \alpha,$$
$$b(t,s) = \frac{\cos s}{(1+t)^3},$$
$$g(x) = \frac{x}{2} + \frac{x^3}{2x^2+1},$$
$$g_1(x) = \frac{1}{2} + \frac{x^2}{2x^2+1}, \ (x \ne 0).$$
$$\sigma = \frac{1}{2} \le \frac{1}{2} + \frac{x^2}{2x^2+1} = g_1(x) \le 1,$$
$$k + \int_0^t |b(t,s)|ds = k + \int_0^t \frac{|\cos s|}{(1+t)^3}ds$$
$$\le k + \frac{t}{(1+t)^3} < k + \frac{t}{1+t} = a(t),$$

for all $t \ge 0$. Hence, the estimate

$$k + \int_0^t |b(t,s)|ds \le a(t) \le K\alpha^{-1}$$

holds with $K = 2(k+1)$. Further, it is clear that

$$\int_s^t |b(u,s)|du \le \int_s^t \frac{1}{(1+u)^3}du$$
$$< \frac{1}{2(1+s)^2} < \frac{1}{1+s} < k + \frac{1}{1+s} = a(s)$$

for all $(t,s) \in \Omega$.

Thus, all the assumptions of Theorem 1 hold. Hence, we can conclude that all solutions of the equation given are uniformly continuous on $[0, \infty)$ and belong to $L^2[0, \infty)$. Furthermore, the zero solution of the equation given is globally asymptotically stable.

Lemma 4. *If*

$$\int_s^t |b(u,s)|du \le \alpha^{-1}a(s) \tag{11}$$

for all $t \ge s \ge 0$, *then the zero solution of* Eq. (2) *is stable. Furthermore, if for some* $t_1 \ge 0$ *there is a constant* $k > 0$ *such that*

$$a(t) \ge k \tag{12}$$

for all $t \ge t_1$ *and a constant* $\lambda \in (0,1)$ *such that*

$$\int_s^t |b(u,s)|du \le \lambda\alpha a(s) \tag{13}$$

for all $t \ge s \ge t_1$, *then every solution* $x(t)$ *of* Eq. (2) *belongs to* $L^1[0, \infty)$.

Proof. Define the Lyapunov functional

$V : [0, \infty) \times C[0, \infty) \to [0, \infty)$

by

$$V(t, \psi(.)) := |\psi(t)| + \int_0^t \{a(s)h_1(\psi(s))$$
$$- \int_s^t |b(u,s)g_1(\psi(s))|du\}|\psi(s)|ds. \tag{14}$$

It is clear from the assumptions of Lemma 4 that $V(t, 0) = 0$ and $V(t, \psi(.)) \ge |\psi(t)|$ for all $t \ge 0$.

For any $t_0 \ge 0$ and initial function $\varphi \in C[0, t_0]$, let $x(t) = x(t, t_0, \varphi)$ denote the solution of Eq. (2) on $[0, \infty)$ with $x(t) = \varphi(t)$ for $0 \le t \le t_0$. Then consider $V(t) := V(t, x(.))$, that is, the value of the functional V along the solution $x(t)$ at t and the derivative of $V$with respect to t. Since $x(t)$ is continuously differentiable on $[t_0, \infty)$, $x(t)$ has a right derivative $D_r|x(t)|$ given by

$$D_r|x(t)| = \begin{cases} x'(t)sgn\ x(t), & \text{if } x(t) \ne 0 \\ |x'(t)|, & \text{if } x(t) = 0 \end{cases}$$

for all $t \ge t_0$. Thus, the right derivative of V for $t \ge t_0$ is

$$D_rV(t) = D_r|x(t)| + \frac{d}{dt}\int_0^t [a(s)h_1(x(s))$$
$$- \int_s^t |b(u,s)g_1(x(s))|du]|x(s)|ds \le -a(t)h_1(x(t))|x(t)|$$
$$+ \int_0^t |b(t,s)||g_1(x(s))||x(s)|ds + a(t)h_1(x(t))|x(t)|$$
$$- \int_0^t |b(t,s)||g_1(x(s))||x(s)|ds = 0 \tag{15}$$

so that

$D_rV(t) \le 0$

Hence,

$$|x(t)| \le V(t) \le V(t_0) \tag{16}$$

for all $t \ge t_0$, where

$$V(t_0) = |\varphi(t_0)| + \int_0^{t_0} \{a(s)h_1(\varphi(s))$$
$$- \int_s^{t_0} |b(u,s)g_1(\varphi(s))|du\}|\varphi(s)|ds \le |\varphi(t_0)|$$
$$+ \int_0^{t_0} \{\alpha a(s) - \sigma \int_s^{t_0} |b(u,s)|du\}|\varphi(s)|ds \le M(t_0)|\varphi|_{t_0}$$

and

$$M(t_0) := 1 + \int_0^{t_0} \{\alpha a(s) - \sigma \int_s^{t_0} |b(u,s)|du\}ds.$$

For a given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{M(t_0)}$. Then, for $\varphi \in C[0, t_0]$ with $|\varphi|_{t_0} < \delta$, we have

$$|x(t)| \le V(t_0) \le M(t_0)|\varphi|_{t_0} < \delta M(t_0) = \varepsilon$$

for all $t \ge t_0$, which proves the stability.

Now suppose (12) and (13) also holds. In that case, let $\gamma := \sqrt{\lambda}$ and

$$V_\gamma(t) := |x(t)| + \int_0^t [\gamma a(s) h_1(x(s)) - \frac{1}{\gamma}\int_s^t |b(u,s) g_1(x(s))|du]|x(s)|ds.$$

By (13),

$$V_\gamma(t) \ge |x(t)|$$

for all $t \ge t_1$. And

$$\begin{aligned} D_r V_\gamma(t) &\le -a(t)h_1(x(t))|x(t)| + \int_0^t |b(t,s) g(x(s))||x(s)|ds \\ &\quad + \gamma a(t) h_1(x(t))|x(t)| - \frac{1}{\gamma}\int_0^t |b(t,s) g_1(x(s))||x(s)|ds \\ &\le -(1-\gamma) a(t) h_1(x(t))|x(t)| \end{aligned}$$

for all $t \ge \tau$, where $\tau = \max\{t_0, t_1\}$. Then, because of (12), we have

$$D_r V_\gamma(t) \le -k(1-\gamma)|x(t)|.$$

Integration along with $V_\gamma(t) \ge |x(t)|$ yields

$$|x(t)| \le V_\gamma(t) \le V_\gamma(\tau) - k(1-\gamma)\int_\tau^t |x(s)|ds$$

for all $t \ge \tau$. Therefore, the improper integral

$$\int_0^\infty |x(t)|dt$$

converges. The proof of Lemma 4 is completed. □

Theorem 2. *If*

$$1 \le h_1(x) \le \alpha, \quad \sigma \le g_1(x) \le 1,$$

where α and σ, $\sigma \in (0,1)$, are positive constants,

$$\int_0^t |b(t,s)|ds \le \alpha a(t)$$

for all $t \ge 0$,

$$\int_s^t |b(u,s)|du \le \alpha a(s)$$

for all $t \ge s \ge 0$, and if for some $t_1 \ge 0$ there are positive constants k and K such that

$$k \le a(t) \le \alpha K$$

for all $t \ge t_1$ and a constant $\lambda \in (0,1)$ such that

$$\int_s^t |b(u,s)|du \le \lambda \alpha a(s)$$

$t \ge s \ge t_1$, then all solutions of (2) are uniformly continuous on $[0,\infty)$ and belong to $L^1[0,\infty)$. Moreover, the zero solution of (2) is globally asymptotically stable.

Proof. By Lemma 4, the zero solution of (2) is stable. For any $\varphi \in [0, t_0]$, consider the corresponding solution $x(t) = x(t, t_0, \varphi)$. By (16), we have

$$|x(t)| \le V(t_0)$$

for all $t \ge t_0$.

This, together with $\int_0^t |b(t,s)|ds \le \alpha a(t)$ and $k \le a(t) \le \alpha K$, applied to Eq. (2) gives

$$\begin{aligned} |x'(t)| &\le a(t) h_1(x(t))|x(t)| + \int_0^{t_0} |b(t,s)||g_1(\varphi(s))||\varphi(s)|ds \\ &\quad + \int_{t_0}^t |b(t,s)||g_1(x(s))||x(s)|ds \\ &\le \alpha a(t)|x(t)| + \int_0^{t_0} |b(t,s)||\varphi(s)|ds + \int_{t_0}^t |b(t,s)||x(s)|ds \\ &\le 2KV(t_0) + \alpha a(t_0)|\varphi|_{t_0} \end{aligned}$$

for all $t \ge \tau$, where as before $\tau = \max\{t_0, t_1\}$. In short, $x'(t)$ is bounded on $[\tau, \infty)$. Consequently, by the uniform continuity argument in the proof of Theorem 1, $x(t)$ is uniformly continuous on $[0,\infty)$. Also, by Lemma 1, $x(t) \in L^1[0,\infty)$. Therefore, by Barbălat's Lemma, it follows that $x(t) \to 0$ as $t \to \infty$. The proof of Theorem 2 is completed. □

Example 2. Consider the non-linear Volterra integro-differential equation of the form

$$\begin{aligned} x'(t) &= -\left(k + \frac{1+\beta}{1+t}\right)\left(x(t) + \frac{x(t)}{1+x^2(t)}\right) \\ &\quad + \int_0^t \frac{\cos s}{(1+t)^2}\left(\frac{x(s)}{2} + \frac{x^3(s)}{1+2x^2(s)}\right)ds \end{aligned}$$

for $t \ge 0$, where k and β are any positive constants.

It is obvious that the assumption

$$k \le a(t) \le \alpha K$$

holds, where $a(t) = k + \frac{1+\beta}{1+t}$ and $a(t)$ is bounded by positive constants. Besides, the assumption

$$\int_0^t |b(t,s)|ds \le \alpha a(t)$$

holds since

$$\int_0^t |b(t,s)|ds = \int_0^t \frac{|\cos s|}{(1+t)^2}ds \le \frac{t}{(1+t)^2} < \frac{2}{1+t} = \alpha a(t)$$

for all $t \ge 0$. Finally, we have

$$\begin{aligned} \int_s^t |b(u,s)|du &\le \int_s^t \frac{1}{(1+u)^2}du < \frac{1}{1+s} < \frac{1}{1+\beta}\left(k + \frac{1+\beta}{1+s}\right) \\ &= \frac{1}{1+\beta}a(s) < \frac{2}{1+\beta}a(s) \end{aligned}$$

for all $t \ge s \ge 0$. Thus, all the assumptions of Theorem 2 hold. Hence, we can conclude that all solutions of the equation given are uniformly continuous on $[0,\infty)$ and belong to $L^1[0,\infty)$. Moreover, the zero solution of the equation given is globally asymptotically stable.

3. Conclusion

A kind of non-linear Volterra integro-differential equations has been considered. The stability/global asymptotic stability/uniformly continuity of the solutions on $[0, \infty)$, boundedness of the first order derivative of solutions and absolutely Riemann integrability of the solutions on $[0, \infty)$ have been discussed by using the Lyapunov's second approach. The obtained results extend and improve some recent results in the literature from linear case to the non-linear case. Examples are also given to illustrate the importance of our results. The results of this paper are also new and complement previously known results.

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