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REVIEW PAPER

Computer geometry and encoding the information on a manifold

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Abstract In this review, we try to answer the following question why should one study differential geometry? First of all, differential geometry is a Jewel of Mathematics. It is a prerequisite for theoretical physics. Secondly, in recent years, new and important applications have been discovered. Surprisingly, the structures of differential geometry are ideally suited for coding theory, information geometry and imaging process, kinematics of Robotics and computer aided geometric design, optimization and so on The main goal of the review is to establish a bridge between the theoretical aspects of modern geometry and topology on the one hand and computer experimental geometry on the other. The flood of information through various computer networks such as the internet characterizes the world situation in which we live. Information words, often called virtual spaces and cyberspace, have been formed on computer networks. The complexity of information worlds has been increasing almost exponentially through the exponential growth of computer networks. Such nonlinearity in growth and in scope characterizes information words. In other words the characterization of nonlinearity is the key to understanding, utilizing and living with the flood of information. The characterization approach is by characteristic points such as peaks, pits, and passes, according to the Morse theory on the manifold. Another approach is by singularity signs such as folds, cusps bifurcation, nodes, butterfly and swallowtail. Atoms and molecules are the other fundamental characterization approach. Topology and geometry including differential topology, serve as the framework for the characterization.

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1. Introduction

If one consider information sciences, how can one geometrize this information? What is a geometric property compatible with the characterization of the problem? Riemannian geometry has an unexpected applications in geometric interpretation of some problems of mathematical statistics and computer geometry and encoding the information on a surface (manifold). The review consists mainly of two parts and their applications. Here, we give a brief account on statistical inference as an application to information geometry. Also, we present geometric modeling as an application to computer geometry (Garment wrinkles) through the concept of singularities and Morse function on manifold.

2. Information geometry [1–4,10,12,15,20]

Statistics is a science which studies method of inference, from observed data, concerning the probabilistic structure underlying such data. Statisticians should, however, be remarked that is not necessarily faithful (includes the true distributions or approximately). Thus, it should be very important to know the shape of a statistical model in the whole of probability distributions (geometry of statistical model). Information geometry originated from the study of the manifold of probability distributions. To illustrate this statement, let M be an n-dimensional parameter space, $\vartheta = (\vartheta^1, \vartheta^2, \ldots, \vartheta^n)$ be an arbitrary parameter vector in M, x be a random variable and $f = (X;\vartheta)$ be probability density function. Let (ϑ^i) and $(\vartheta^i + d\vartheta_i)$ be two neighboring points. The infinitesimal distance ds between ϑ^i and $(\vartheta^i + d\vartheta_i)$

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(\vartheta) d\vartheta^i d\vartheta^j \tag{2.1}$$

where $g_{ij}(\vartheta)$ is the Fisher's information matrix:

$$g_{ij}(\vartheta) = \int \frac{\partial}{\partial \vartheta^i} \log f(X;\vartheta) \frac{\partial}{\partial \vartheta^j} \log f(X;\vartheta) f(x;\theta) dX \tag{2.2}$$

C.R. Rao has proved that the Fisher's information matrix has the properties of Riemannian metric and the parameter space M becomes a Riemannian one with a metric tensor $g_{ij}(\vartheta)$.

Definition 2.1. The J-divergence between two extremely closed probability density functions $p = f(x;\theta)$, $q = f(x;\theta + d\theta)$ is given by

$$J(p,q) = \int (p-q) \log \frac{p}{q} dx \tag{2.3}$$

or equivalent (Taylor expansion)

$$j(p,q) = g_{ij}(\theta) d\theta^i d\theta^j \tag{2.4}$$

which means that the J-divergence between the functions $f(x;\theta)$ and $f(x;\theta + d\theta)$ locally coincide with the square of the geodesic distance between the points θ^i and $\theta^i + d\theta^i$.

Remark 2.1. Locally the J-divergence and the geodesic distance are closely connected by the relation $J(p,q) = dS^2$

Remark 2.2. Globally, Although, no relation between the divergence and geodesic distance has been found yet. Geometrically, the geodesic distance is invariant, but the J-divergence is not invariant. That is why the relation between them is non-linear and generally much more complicated one. This problem can be solved in special cases say for fixed parameter. We give a brief survey of some notions of the theory of statistical inference from the geometrical standpoint.

Definition 2.2. A statistical model is a family S of probability distributions P of random variable which is smoothly parameterized by a finite number of real parameters $\vartheta = (\vartheta^i)$: $S = \{P_\vartheta$: $\vartheta \in R^n$ where The statistical model S carries the structure of smooth Riemannian manifold, with respect to which $\vartheta = (\vartheta^i)$ play the role of coordinates of a point $P_{\vartheta \in S}$, and whose metric $\langle , \rangle$ is defined by the information matrix $(g_{ij}(\vartheta))$.

Remark 2.3. Here $\{\frac{\partial}{\partial \vartheta^i} \log f(x;\theta)\} = \{\partial_i\}$ is a basis of a vector space of random values, which identified with the tangent space $T_\vartheta S$ (set of Fisher score functions). Thus, we can define the operator of covariant differentiation $\nabla_{\partial_i}\partial_i$ and some of different connections. The statistical parameter spaces became a wide field for investigations, e.g., the geodesic distances (divergence), their curvatures, the relation between the geometric structures and statistical properties of these spaces is given through the Riemannian curvature tensor R^l_{ijk}, associated with information loss, which is defined as

$$\begin{aligned}
&R^l_{ijk} = \frac{\partial}{\partial u^k}\Gamma^i_{jl} - \frac{\partial}{\partial u^l}\Gamma^i_{jk} + \Gamma^i_{\lambda k}\Gamma^\lambda_{jl} - \Gamma^i_{\lambda l}\Gamma^\lambda_{jk}, R^l_{ijk} = g^{\lambda i} R_{\lambda jk\lambda},\\
&R_{ik} = g^j_i R^i_{jkl} = R^i_{jki}, R = g^{\mu\nu} R_{\mu\nu} (\text{Ricc tensor})\\
&\Gamma^i_{jl} = g^{ik}\Gamma_{kjl}, \Gamma_{kjl} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial u^k} + \frac{\partial g_{lk}}{\partial u^j} - \frac{\partial g_{kj}}{\partial u^l}\right),\\
&R^l_{ijk} = 0, \text{ for flat manifold}
\end{aligned} \tag{2.5}$$

The two parameters θ_i and θ_j are said to be orthogonal if the element $g_{ij(i \neq j)}$ of the FIM is zero. Orthogonal parameters are easy to deal with in the sense that their maximum likelihood estimates are independent and can be calculated separately.

Remark 2.4. When dealing with research problem, it is very common for the researcher to invest some time searching for an orthogonal parameterization of densities involved in the problem.

Simple example: consider the set of all Gaussian distribution. Any Gaussian distributions is specified by the mean value μ and variance σ^2, so that it is regarded as a two dimensional manifold where (μ, σ^2) is a coordinate system. This model is called normal model. In these coordinates the Fisher's metric has the components:

$$g_{ij} = \frac{\delta_{ij}}{\sigma^2}, (i,j = 1,2) \tag{2.6}$$

The curvature $R = \frac{1}{2}$. Therefore S is the space of constant negative curvature (pseudo sphere) as shown in Fig. 1 (orthogonal parameters).

The normal model is special case (Sub manifold) of exponential family. What is the natural geometric structure to be introduced in such a manifold? Information geometry gives a definite answer to this problem, and an invariant geometrical structure can be introduced into the manifold consisting of a smooth family of general probability distribution. Only local properties of a statistical model are responsible for the asymptotic theory of statistical inference. Local properties are represented by the geometry of the tangent planes of the manifold. For studying higher order asymptotic properties of inference it is necessary to use the one-to-one affine correspondence between the tangent spaces (affine connection and its related geometric concepts) [10,12,15].

Remark 2.5. Information geometry is useful not only for applications to statistics, information theory, systems theory where probability distributions play a major role, but also

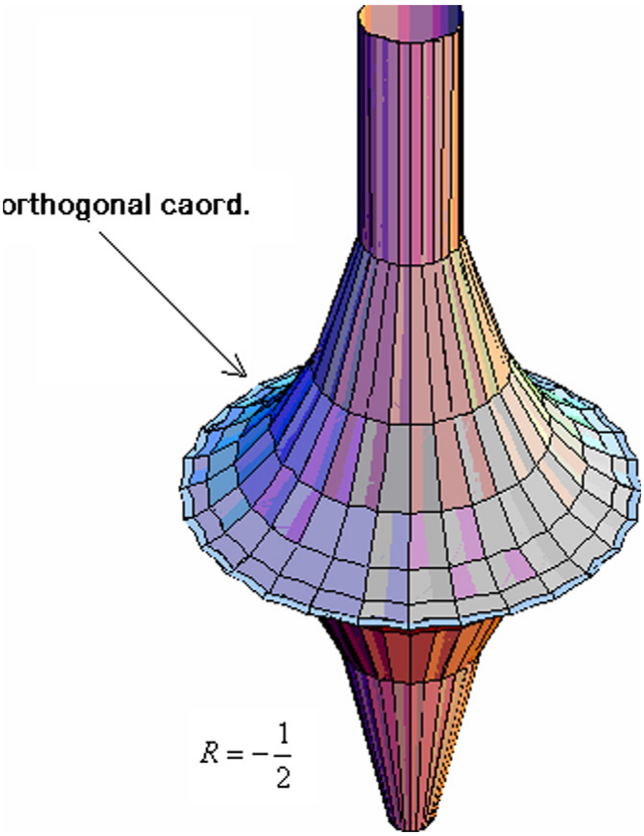

Figure 1 Pseudo sphere.

gives a new concept to differential geometry as we show in the following.

2.1. Application: statistical inference [1–4,20,21]

Because information geometry originated from the manifold of probabilities, it gives naturally a mathematical foundation of statistical inference. The basic structures on a curved exponential family $M = \{P(x,u)\}$, which is embedded in an exponential family (flat manifold), is defined as

$$q(x,\vartheta) = \exp\left\{\sum \vartheta^i x_{i-\psi(\vartheta)}\right\} \in S \tag{2.7}$$

$$p(x,u) = q(x,\vartheta(u)) = \exp\left\{\sum \vartheta^i(u) x_{i-\psi(\vartheta(u))}\right\} \in M \tag{2.8}$$

where $S = q(x,\vartheta)$, i.e, $M \subset S$ The manifold M is a smooth submanifold in S, where $u = (u^\alpha = (u^1,u^2,\ldots,u^m),\ m \prec n)$, is a coordinate system in M. The submanifold $M \subset S$ is called curved exponential family (not flat), and is represented by the parameterization

M: $\vartheta = \vartheta(u) = \theta(u^j)$, $j = 1,2,\ldots,n$ or
$\vartheta = \vartheta(u^1,u^2,\ldots,u^m)$, $j = 1,2,\ldots,n$

$$Rank\left(\frac{\partial \vartheta^i}{\partial u^i}\right) = m \tag{2.9}$$

the vector $\theta = \theta(u^i)$ is called the natural (canonical) parameter vector. If θ and u are two scalar parameterizations of an estimation problem such that $\theta = h(u)$ and h is a differentiable function, then $FIM(u) = FIM(\theta)(h'(u))^2$. In information geometry, this is seen as a change of coordinates on a Riemannian manifold and the intrinsic properties of curvature are unchanged under different parameterization. The probability distributions of M are parameterized by u in (2)–(8). Here the tangent vectors and the metric tensor are given as

$$\partial_i l(x,\theta) = x_i - \partial_i\psi(\theta), \partial_i\partial_j l(x,\theta) = -\partial_i\partial_j\psi(\theta) \tag{2.10}$$

$$g_{ij}(\theta) = \partial_i\partial_j\psi(\theta) = I_M(u) = B\begin{cases} 0, \psi(\theta) \text{ linear} \\ \neq 0, \psi(\theta) \text{ nonlinear}\end{cases} \tag{2.11}$$

This is not an exponential family unless $\vartheta(u)$ is linear. To illustrate this, let $x_1,x_2,\ldots,x_N$ be an N independent observations from an unknown probability distributions $p(x,u)$. From the observed sample, we need to estimate the true value u by $\hat{u} = \hat{u}(x_1,x_2,\ldots,x_N)$. In the framework of the exponential family S, we take the η-coordinate system which is given by (the expectation parameter)

$$\eta_i = \frac{\partial}{\partial \vartheta_i}\psi(\vartheta) = E_\vartheta[x_i] \tag{2.12}$$

Here $\psi(\vartheta)$ is the cumulate generating function. Then, the natural estimator $\hat{\eta}$ in S is given by replacing the expectation by the sample average

$$\hat{\eta} = \frac{1}{N}\sum x_i \tag{2.13}$$

This is the maximum likelihood estimator in S and the $\hat{\eta}$ summarizes all the Fusher information including in N observations. Let $\hat{p}$ be the point in S whose η-coordinates are given by $\hat{\eta}$. We call $\hat{p}$ the observed point. However, $\hat{p}$ does not, in general, belong to M so that it does not give an estimator û. Geometrically, an estimator is a projection

$$e : S \to M; \quad \hat{p} \to \hat{u} \quad \text{from } S \text{ to } M \tag{2.14}$$

where û is the parameter representing in M the closest distribution $p(x,\hat{u})$ in M to $p(x,\vartheta) \in S$ in the sense of divergence or geodesic (Fig. 2).

The manifold $e^{-1}(u)$ is called the ancillary manifold of u associated with the estimator û where $A(u)\{\eta \in S\colon \hat{u}(\eta) = u\}$. The estimator û takes the value u if the observed $\bar{x} \in A(u)$ and $dim\ A(u) = n - m$, where $A = \{A(u)\}$ is the ancillary family of M whose coordinate system is $V = (V^k)$, $K = m + 1,\ldots,n$ and the pair (u,v) for S (at least in a neighborhood of M). The pair (u,v) specifies a point in S such that it is included in the $A(u)$ attached to u and its position in $A(u)$ is given by V. When and only when the observed $\hat{p}$ belongs to $\hat{u}^{-1} = e^{-1}(u) = A(u)$, the estimated value is u (Fig. 2). For more details you can see Nassar et al. [24,25].

Remark 2.6. The importance of studying statistical structures as geometrical structures lies in the fact that geometric structures are invariant under coordinate transformations. For example, a family of probability distributions, such as Gaussian distributions, may be transformed by a change of variables into another family of distributions, such as log-normal distributions. However, the fact of it being an exponential family is not changed, since the later is a geometric property. The distance between two distributions in this family defined through Fisher metric will also be preserved.

3. Computer geometry [13,14,33,34]

This section considers the problem of encoding essential information about a compact surface in terms of a finite amount of

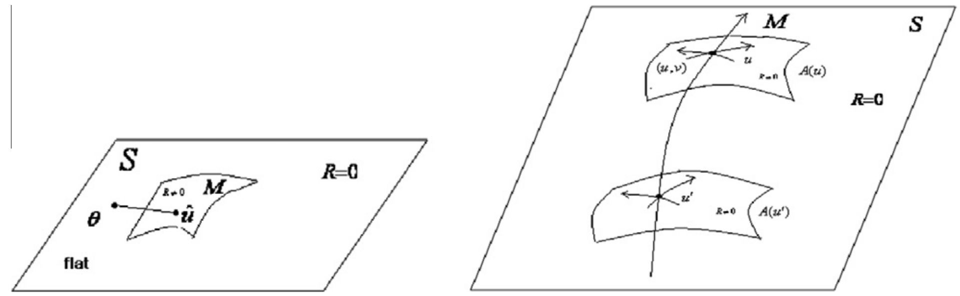

Figure 2 The ancillary manifold.

data that can stored and manipulated by computer (do not care about an exact geometric description of the surface, but all we want is the topological type of a surface, i.e., the genus and orientability are very important). In computer geometry and the graphic representation of a curvilinear coordinate system it is often to draw coordinate line (on the screen of a computer). In particular, the transformation to a curvilinear coordinate system is especially clear if the coordinate network is depicted. Thus the well-known coordinate transformations such as:

Polar coordinates on the plane, Cylindrical coordinate system R^3, Spherical coordinate system and Stereographic projection of the sphere S^2 onto the plane are very important.

3.1. Orientability and comfortability

Definition 3.1. The mapping f: $M \to N$ is called an immersion if the differential map df: $T_pM \to T_{f(p)}N$ is injective, that is, no nonzero vector maps to zero. If f is homeomorphism when f: $M \to f(M)$,we say that f is embedding, $f(M)$ is called submanifold of N, where M and N are a differentiable manifold.

Definition 3.2. A surface M is called orientable if transport of bases along any loop γ on M preserves orientation. Conversely, M is called nonorientable if there exists some loop γ on M such that transport along M reverses orientation. Mobius strip and sphere with cross caps are nonorientable as shown in Fig. 3 but spheres with handles are orientable.

Some time nonlinearity is useful for comfortability and application as shown in a simple example (may be for funny) in Fig. 4.

3.2. Height and distance functions

In computer geometry, there are two important classes, height and distance functions.

Definition 3.3. The orthogonal projection f_l: $R^3 \to R = l$ is called height function in R^3 with respect to l. For a smooth surface $M \subset R^3$ we have f_l: $M \subset R^3 \to R = l$ (height function of M). The level sets of f_l are the intersection of M with planes $\prod_\alpha, \alpha \in R$ orthogonal to l as shown in the Fig. 5. The level set may be complicated: isolated point, or curves or open subset of the planer (top level).

Definition 3.4. Let $q \in R^3$ be a fixed point, then the function d_q: $R^3 \to R, d_q = |q - p|^2$ is called distance function to (smooth function). For any embedded smooth surface $M \subset R^3$, we have d_q: $M \subset R^3 \to R$ as in Fig. 6.

Geometrical interpretation: for height function, we have foliated space R^3 into the union of parallel planes $\prod_\alpha$ and $\prod_\alpha \cap M$, for all α. If R^3 is foliated as the union of concentric spheres $S_o^2(r) \cup 0$ and $S^2(r) \cap M$. Thus concentric spheres are the level sets of d_q.

Main problem: you can reconstruct the original surface M if we know these parallel planes (medicine, geology).

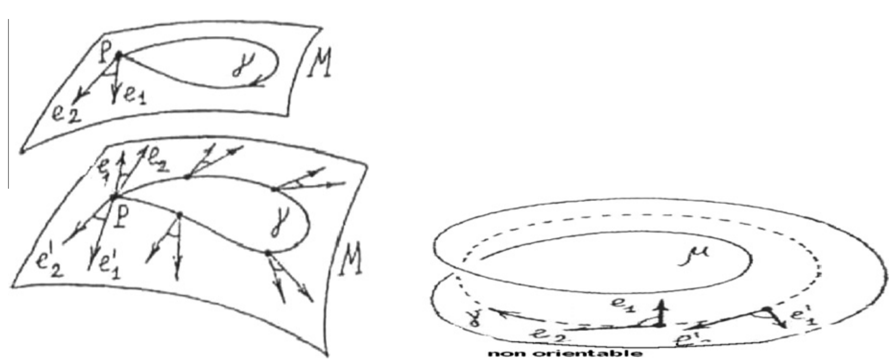

Figure 3 Orientable surfaces.

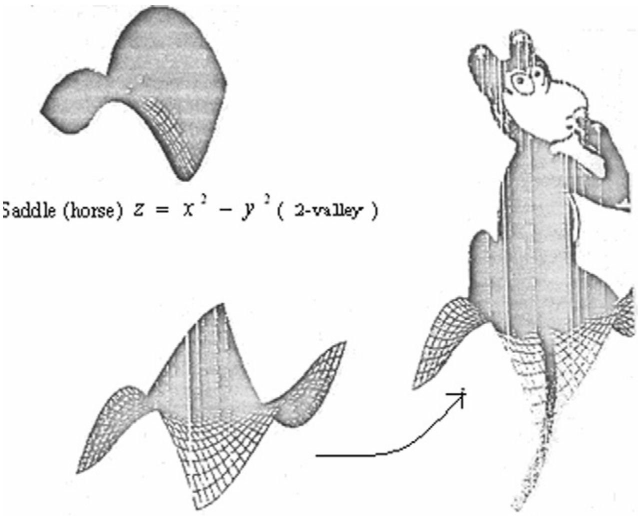

Figure 4 Monkey saddle $Z = X(X^2 - (\sqrt{3}y)^2)$ (3-valley-two legs and one tail).

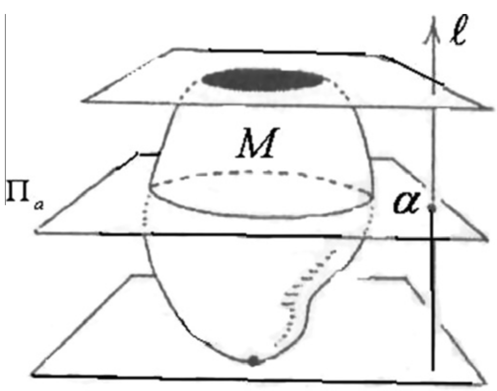

Figure 5 Hight function.

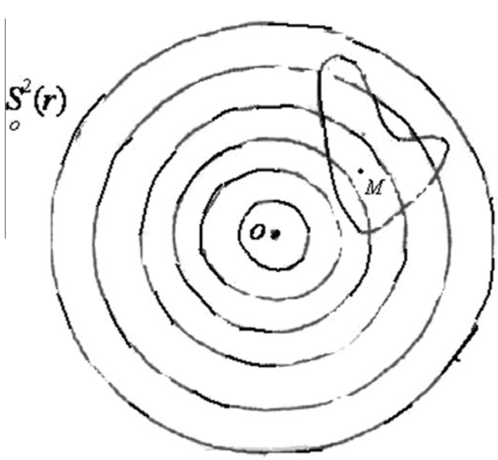

Figure 6 Distance function.

3.3. Height and distance functions

Morse theory describes the relationship between a function's critical points on a manifold (surface) and the homotopy type of the function's domain.

Definition 3.5. A critical point x of f: $M \to R$ is called **nondegenerate** if $d^2f = \sum \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i dx^j$ is no degenerate at that point, i.e., $Detd^2f \neq 0$ at x. The index of the quadratic form d^2f at x is the index of x. The nullity of x is the nullity of d^2f at x.

Definition 3.6. A smooth function on a smooth manifold is called Morse function if all its critical points are nondegenerate. For Morse function, we have the following results: (1) Morse functions exit on any smooth manifold. (2) Any smooth function on a smooth manifold can be approximated by a Morse function. (3) Non degenerate critical points are isolated (there cannot be a sequence of non degenerate critical points converging to a non degenerate critical point). (4) Morse function on a compact manifold has finitely many critical points and they are isolated. (5) If x_0 is a non degenerate critical point of a function on a manifold M, there is some open neighborhood of x_0 in M, such that$(f(x) = f(x_0) - \sum_{i=1}^{\lambda}(x^i)^2 + \sum_{j=\lambda+1}^{\lambda}(x^j)^2$. where λ is the index of the critical point and $\{x^j\}$ a set of local coordinate around x_0 (More's lemma).

3.3.1. Morse theory of surfaces(level sets)

Consider Morse function on a surface M. Then $f^{-1}(a)$ is called the level curve of f corresponding to the value a. The subset M_0 of M bounded above by a is defined as (Fig. 7).

$M_a = \{x: f(x) \preceq a\} = f^{-1}([-\infty,a])$

If a is a regular value of f, then $f^{-1}(a)$ is a smooth submanifold, say f_a of M with boundary $f^{-1}(a)$ (the level set).

From Fig. 7 one can see the following: (1) $f^{-1}([a,b]) \subset M^2$ has no critical point of f (Fig. 7(b)). (2) M_a and M_b are diffeomorphic as shifts along the trajectories of the vector field $-\nabla f$ (Fig. 7(c)).

Definition 3.7. Morse functions that have several critical point at the same level is called complicated. A complicated Morse function can be changed by small perturbations into a simple Morse function which has many application, as in Fig. 8.

Morse theorem(dynamical system): Morse functions on smooth manifold M are everywhere dense in the space of all smooth functions on M. Equivalently: Any smooth function on M can be converted into a Morse function as a result of a perturbation as slight as desired, i.e., the perturbation splits degenerate critical point into a certain number of non degenerate singularities as in the Fig. 9 (Rotating l or M by arbitrary small amount).

Really we can simplify the level sets by rotating the object M or the line l in the space. Consider, the level set of the graph of consists f_l: $R^2 \to R, f_l(x,y) = \text{Real }(x + i\,y)^3$, the level set of the graph of f_l consists of 3 lines through the origin and nearby level sets Real $z^3 = 0$ (tilt the line l).

If we tilt the line l slightly, one complicated critical point disintegrates into several simple non degenerate saddles (Fig. 10). For the application, the following three problems

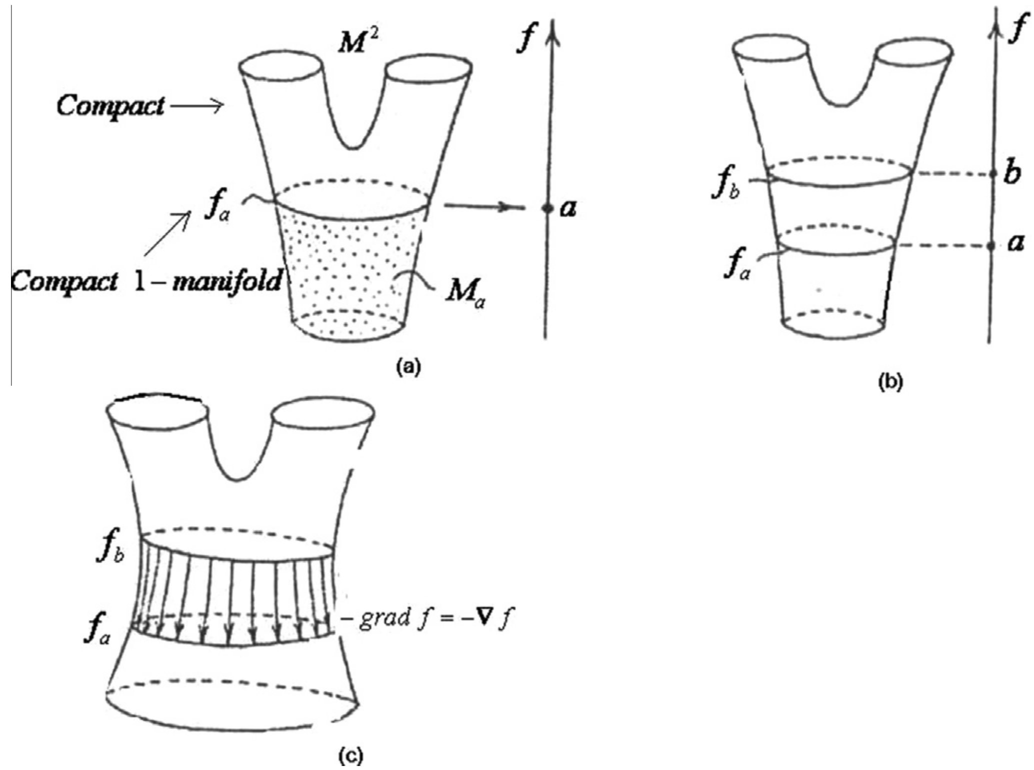

Figure 7 Level sets.

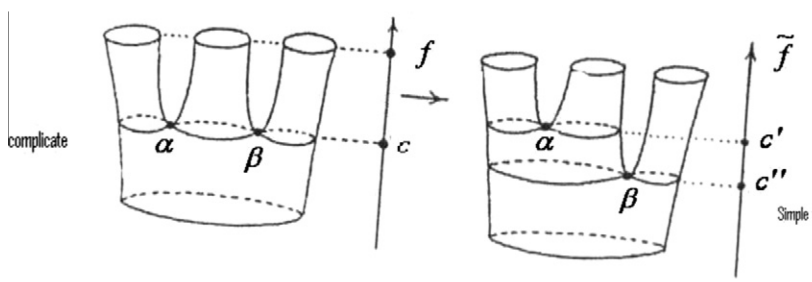

Figure 8 Complicated Morse function.

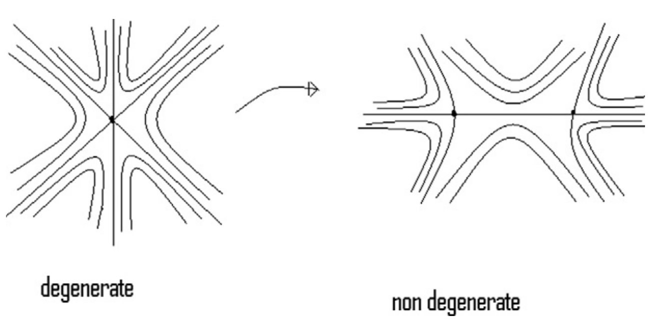

Figure 9 Node generate singularities.

are considered: (i) How to code a Morse function on a smooth manifold (ii) How to code a surface using a Morse function defined on it (iii) How to reconstruct a surface if we know a coding of it. Therefore we present the following abstractization:

3.3.2. Reeb graph

The structure of a continuous function can sometimes be made explicit by plotting the evolution of the components of the level set. This leads to the concept of the Reeb graph of the function. In graph theory, a Reeb graph of a scalar function describes the connectivity of its level sets. Reeb graphs and their loop-free version, called contour trees, have a wide variety of applications including computer aided geometric design, topology-based shape matching, surface computation and parameterization.

Definition 3.8. The Reeb graph of f: $M \to R$ on a compact manifold M is defined as the quotient space of M by the equivalence relation defined by $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$ and x_1, x_2 are in the same connected component of $f^{-1}(f(x_1))$. For example, the Reeb graph of the height function of a surface embedded in is the quotient space of the surface under the equivalence relation $(x_1,y_1,z) \sim (x_1,y_1,z)$ if these two points lie in the same connected component of the horizontal cross

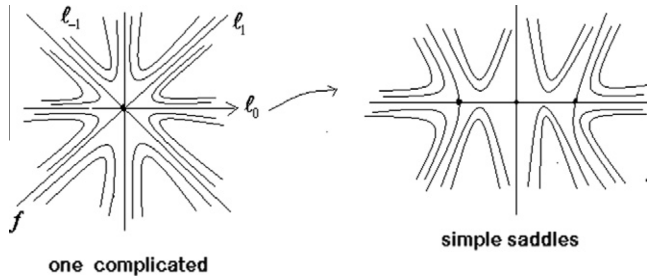

Figure 10 Node generate saddles.

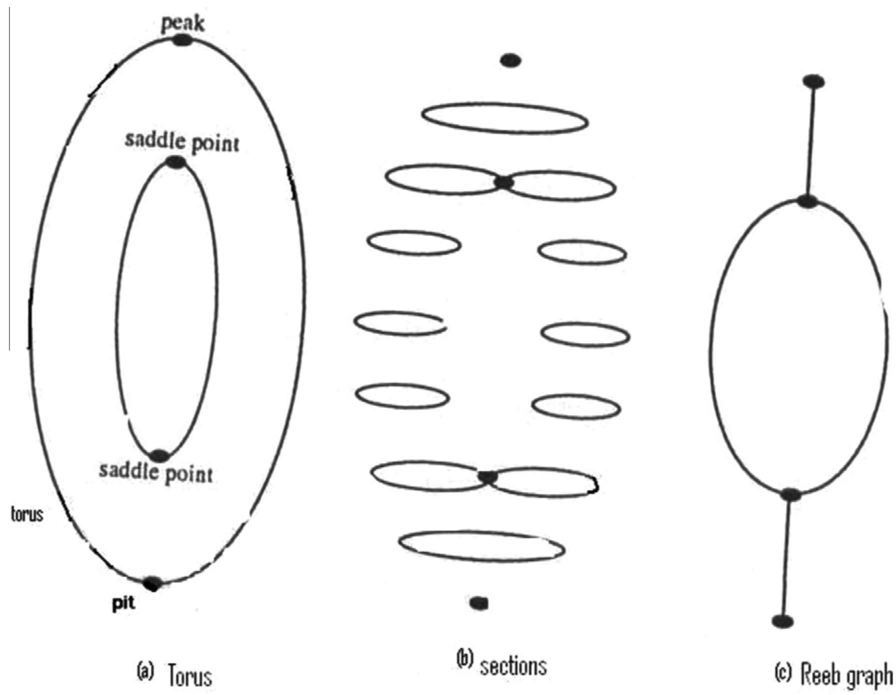

Figure 11 Reeb graph.

section of the surface at height z. Each connected contour on each horizontal plane is represented by a point in the Reeb graph. For the tours and the critical points of its height function as shown in Fig. 11(a), cross section in Fig. 11(b) and the Reeb graph of the height function on the tours as in Fig. 11(c).

3.3.3. Practical visualization

For practical visualization, the Reep graph consists of two types of elements: arcs or edges, coming from the connected components of the part of the surface that is situated strictly between two critical points and nodes or vertices coming from critical points. To distinguish different types of critical points, we label the nodes of the graph. For simple Morse function on an orientable manifold, we need just two labels nodes that represent a maximum or minimum labeled A, they are leaves of the graph, that is, only one edge goes out of them, pointing up in the case of minimum, or down in the case of maximum as shown in Fig. 12(a and b), respectively.

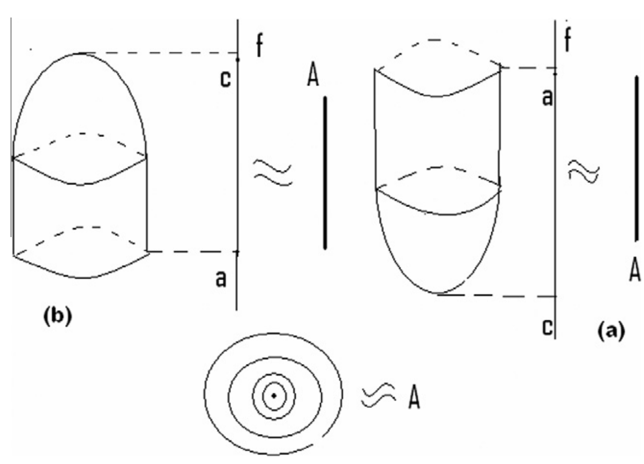

Figure 12 Label nodes.

For the connected component of a saddle, we have a vertex (node) with three edges coming out: two pointing up and one

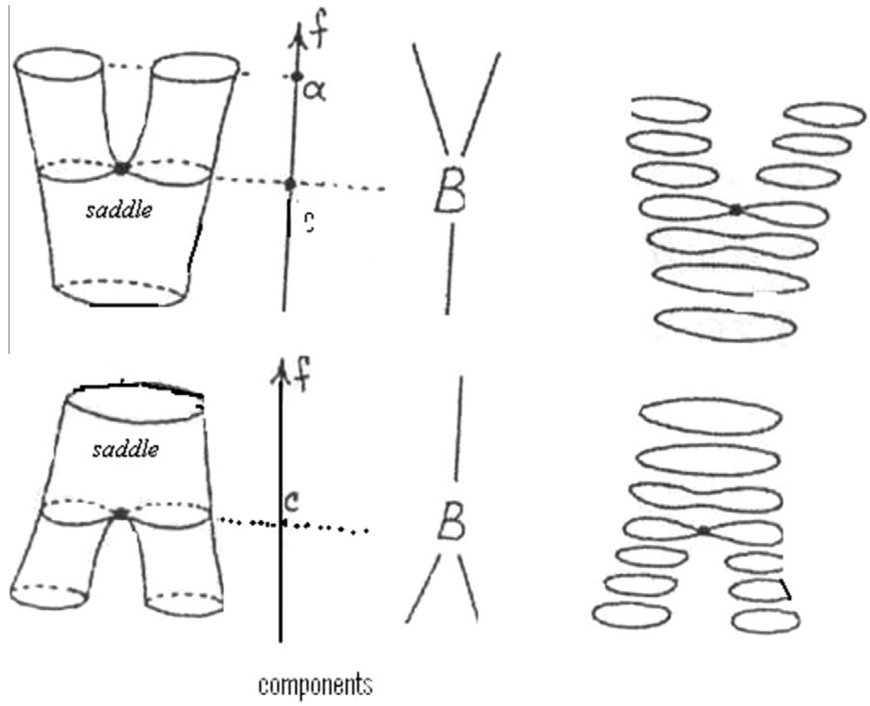

Figure 13 Orientalbe saddle transformation.

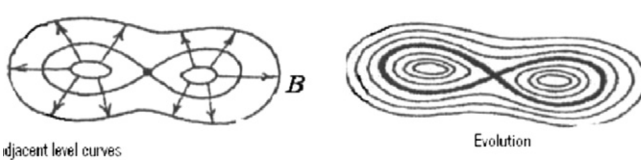

Figure 14 Evolution curves.

down or vice versa. We label such vertices; they represent an orientable saddle transformation (Fig. 13). The connected component of an orientable saddle surface is topologically a figure eight. The evolution of the level curves is given through Morse's lemma as the following: on opposite sides of the critical point, the function has the same sign. On one side the level set evolves into a single circle, and on the other into two circles as shown in Fig. 14.

The labeled nodes of the Reeb graph is denoted by atoms and the labeled graph itself a molecule, see the two examples in Fig. 15.

Remark 3.1. In the case of simple Morse function, we draw the molecule in such a way that the direction of increasing of f is upwards.

Remark 3.2. The molecule of a Morse function is not necessarily a planar graph, it may not be possible to arrange it in the plane R^2 without edges crossing, i.e., it possible to arrange it in the space R^3 (graph theory).

3.3.4. The molecule as a code

If we have a molecule to any simple Morse function on an orientable compact surface, whether we can reconstruct the surface and the function from the molecule?

For give the answer, we consider the set $\{(M_i,f_i)\}$ where M_i is closed orientable surface and f_i is a Morse function on M_i. The two pairs (M_i,f_i) and (M_j,f_j) are fiber equivalent if there exists a diffeomorphism ξ: $M_i \to M_j$ which maps level lines of f_i to those of f_j.

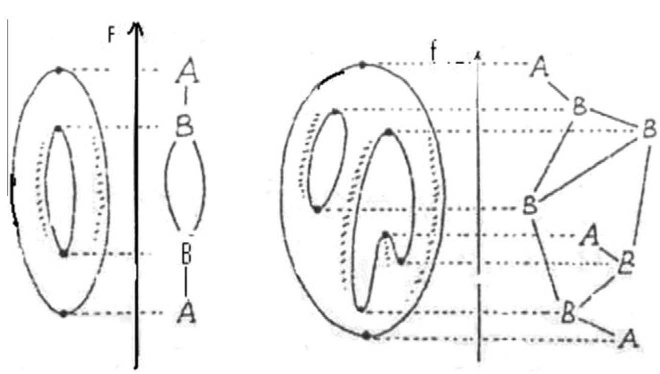

Figure 15 Atoms and molecule.

Remark 3.3. The molecule of a simple Morse function f can be considered as a cod that allows to define the equivalence class of (M,f) uniquely.

Remark 3.4. The pairs (M,f) and $(M,-f)$ are equivalent (the functions $f,-f$ have the same level lines). The only difference between the corresponding molecules is the orientations of their edges ($f,-f$ have opposite orientations).

Definition 3.9. Two molecules W_i and W_j with oriented edges are isomorphic if there is a homeomorphism between them for which preserves its atom labels and either preserves the orientation on the edges or change it on all the edges simultaneously.

Thus we have

Lemma 3.1. *The pairs (M,f) and $(M,-f)$ are fiber equivalent iff W_i and W_j are isomorphic.*

Lemma 3.2. *In particular if $W_i \approx W_j$ then M_i and M_j are diffeomorphic.*

Definition 3.10. Two embeddings $M_1, M_2 \rightarrow R^3$ are isotopic (equivalent) if one can be smoothly transformed into the other without self-intersections arising.

Lemma 3.3. *Two diffeomorphic surfaces embedded in R^3 with Morse height functions, the corresponding molecules may be different even if the embeddings are isotopic. Conversely, the molecules can be isomorphic even if the embeddings are not isotopic. For example: consider Fig. 16. where two non isotopic embedding of the tours T^2 in R^3 and the corresponding height functions. Their molecules are isomorphic. The first embedding is isotopic to the standard embedding, but the second is knotted which cannot be isotopically deformed into the standard embedding. As an application to computer geometry, we give the following.*

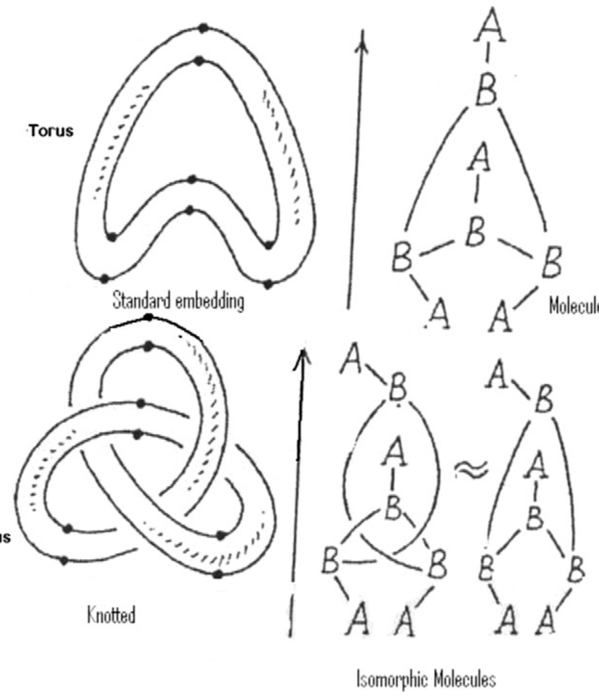

Figure 16 Isomorphic molecules.

4. Geometric modeling

There is an important application to differential geometry which is called a geometric modeling starting from the simple forms (curves, surfaces, solids), you can represent and assemble these forms into more complex objects. Beginning with the topology of models and proceeding with the theory and application of graph-based, Boolean, Boundary, and space partitioning models. From the geometry, there are several methods to describe the geometric model for a given shape. These methods include computer-aided geometric design, solid modeling, algebraic geometry and computational geometry. The geometric modeling techniques are used in numerically controlled machining. In other words, solid modeling involves the representation of the shapes of 3-dimensional objects. Many types of there representation are possible [6,8,16,18,19,35–37]. Say for example: (1) in the boundary representation method, an object is represented by means of the vertices, edges and faces of its boundary. Faces need not be planes, but can be taken from a family of functions, large enough to approximate the surface of the object being modeled to within the desired precision through Splines and Bezier surfaces (computational geometry). (2) In constructive solid geometry, by contrast, an object is represented as the result of set operations (union, intersection, set difference) on primitives such as cubes, spheres and cylinders. (3) Capture the topology of a surface or set of surfaces. The method codes a surface by describing the topological evolution of the level sets of a Morse height function defined on it as we see previously.

Remark 4.1. Geometric modeling, using differential and analytic geometry, linear algebra, tensors, topology, set theory and numerical computation methods to capture the complex description of an object.

4.1. Garment wrinkles modeling (distance function singularities) [5,9,13,14,33,34]

Wrinkles are formed because of excess material covering a body. Let us try to glue a hole in a plane using a patch that's bigger than the hole (the solution: the patch that has the minimal Dirichlet integral, which can be considered as an analog of elastic energy in linear elasticity theory). It turns out that the graph of the solution of the considered variational problem, with area constraints, has singularities that can roughly be considered as wrinkles.

4.2. Mathematical formation [7,11,17]

Let $\Omega \subset R^2$ be a bounded domain as a hole, with area $|\Omega|$. Let the patch is given by the graph of a function f: $\Omega \rightarrow R$, with area S and $S \succeq |\Omega|$. Consider the set of functions

$$F = \{f : f(x) = 0 \text{ on } \partial\Omega \int_{\Omega} \sqrt{1 + |\nabla f|^2} dx_1 dx_2 = S\}$$

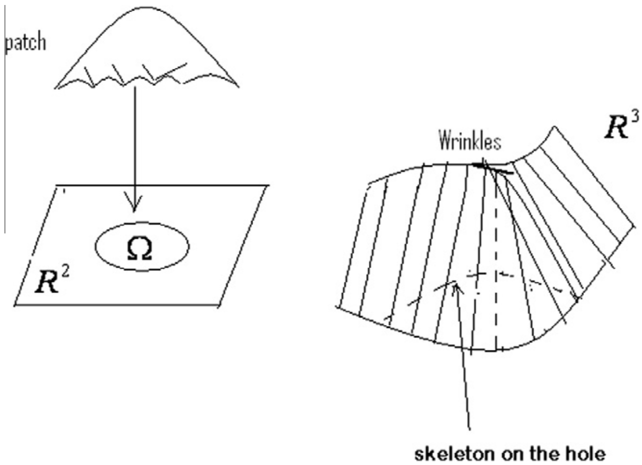

Figure 17 Skeleton and singularities.

The optimal patch is the solution of the minimization problem

$$\widehat{J} = Inf_{f\in F}J(f), J(f) = \int_{\Omega} |\nabla f|^2 dx_1 dx_2$$

Using Lagrange multipliers, you can find the solution as a ruled surface of type developable. Geometrically the problem can be described through the distance function from $\partial\Omega$ on the plane R^2. The graph of the solution and the graph of distance function have the same singularities. Then project these singularities onto the hole plane give the skeleton of the hole as in Fig. 17.

5. Conclusion [22–32]

Conclusion of this review is that the mathematics which were for funny and jewels (Projective geometry and numbers theory) in the past, now it's a very important for the applications. Surprising, the structures of classical projective geometry are ideally suited for modern communications (coding theory, cryptography). It is hoped that the present review ignites cooperation between mathematical scientists and pure mathematicians. Team work is requested for geometric modeling. Several direction of differential geometry with the other branches of mathematics are introduced.

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