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REVIEW PAPER

On generalized order statistics and maximal correlation as a measure of dependence

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Abstract In the first part of this review article some recent developments of maximal correlation coefficient, introduced by Gebelein (1941) [7], and its applications in various areas of statistics are discussed. The second part is devoted to find the distributions providing the maximal correlation coefficient between generalized order statistics (gos) and dual generalized order statistics (dgos), which are introduced by Kamps (1995) [8] and Burkschat et al. (2003) [4], respectively. Finally, in the third part, general theorems are presented, which give simple non-parametric criterion for the asymptotic independence between the different elements of gos, as well as dgos.

1. Introduction

When we talk about the dependence relations between random variables (rvs) it is convenient to distinguish between three interrelated kinds of such relations. The first is the linear relation, which is the simplest one. The most widely known measure of the linear relation is Pearson's product-moment correlation coefficient, which is invariant under location and scale changes. Clearly, the invariant property means that this measure interests only in the existence of the linear relation rather than its shape. The second kind of the dependence relation is the association, or concordance. Informally, a pair of rvs are concordant if "large" values of one tend to be associated with "large" values of the other, and "small" values of one with "small" values of the other. The most widely known scale-invariant measures of association are the population versions of Kendall's tau and Spearman's rho. The third kind of the dependence relation is the general one, which means the existence of a Borel-measurable function between the two rvs. The maximal coefficient of correlation, besides being a convenient measure of the general dependence relation between rvs, it plays a critical role in various areas of statistics including correspondence analysis, optimal transformation for regression, and the theory of Markov processes, see [14].

Rényi [12] gives a set of seven postulates which a measure of dependence $\mu(X, Y)$ between two rvs X and Y should satisfy.

(1) $\mu(X, Y)$ is defined for any pair of rvs X and Y, neither of them being constant with probability 1.
(2) $\mu(X, Y) = \mu(Y, X)$.
(3) $0 \leqslant \mu(X, Y) \leqslant 1$.
(4) $\mu(X, Y) = 0$, if and only if X and Y are independent.
(5) $\mu(X, Y) = 1$, if there is a strict dependence between X and Y, i.e., either $Y = \phi(X)$ or $X = \psi(Y)$, where ϕ and ψ are Borel-measurable functions.

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(6) If the Borel-measurable functions ϕ and ψ map the real axis in a one-to-one way onto itself, then $\mu(\phi(X), \psi(Y)) = \mu(X, Y)$.

(7) If the joint df of X and Y is normal, then $\mu(X, Y) = |\mathrm{corr}(X, Y)|$, where $\mathrm{corr}(X, Y)$ is the ordinary correlation coefficient of X and Y, which is defined by

$$\mathrm{corr}(X,\ Y) = \frac{\mathrm{cov}(X,\ Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}}.$$

In 1959, Rényi considered six dependence measures. Of these measures, only the maximal correlation satisfies all seven postulates.

Definition 1. The maximum correlation coefficient (or maximum correlation for short) between the two rvs X and Y is introduced in Gebelein [7] is defined by

$$R(X,\ Y) = \sup_{\phi,\psi} \mathrm{corr}(\phi(X),\ \psi(Y)),$$

where the supremum being over all Borel-measurable functions ϕ and ψ, for which $\mathrm{var}(\phi(X)) > 0$ and $\mathrm{var}(\psi(Y)) > 0$.

1.1. Some recent results regarding the maximal correlation in general

The maximal correlation enjoys the following remarkable properties:

Property 1. If (X, Y) is bivariate normal random vector, then

$$R(X,\ Y) = |\mathrm{corr}(X,\ Y)|.$$

The proof of this property is given by Lancaster [9], in which series expansion involving Hermite polynomial was used.

Property 2. More recently, Dembo et al. [6] have shown that the maximal correlation between two partial sums of independent and identically distributed rvs is also their usual correlation. Specially, if non-degenerate rvs $X_1, X_2, \ldots, X_n$ are independent and identically distributed, then

$$R(S_m,\ S_n) = \mathrm{corr}(S_m,\ S_n) = \sqrt{\frac{m}{n}},$$

where $m \leqslant n$ are integers and $S_k = \sum_{j=1}^{k} X_j,\ k = 1, 2, \ldots, \mathrm{n}$. With a few exceptions such as the above explicit analytical for R are usually unavailable. The generalization of Property 1.2 is given recently by Yaming [14].

Property 3. Under the assumptions of Property 1.2, we have

$$R(S_m,\ S_n - S_\ell)\left(= R\left(\sum_{j=1}^{m} X_j,\ \sum_{j=\ell+1}^{n} X_j\right)\right) = \mathrm{corr}(S_m,\ S_n - S_\ell) = \frac{m-\ell}{\sqrt{m(n-\ell)}},$$

where integers ℓ, m, n satisfy $1 \leqslant \ell + 1 \leqslant m \leqslant n$.

2. The generalized order statistics and the maximal correlation

Kamps [8] introduced the concept of the generalized order statistics (gos) as a unification of several models of ascendingly ordered rvs. It is known that ordinary order statistics, upper record values, sequential order statistics and progressive type II censored order statistics are special cases of gos.

Definition 2. The uniform gos $U^*(r) \equiv U(r, n, k, \tilde{m}),\ r = 1, 2, \ldots, n$, are defined by their density function

$$f^{U^*(1),\ldots,U^*(n)}(u_1, \ldots, u_n) = \left(\prod_{j=1}^{n} \gamma_j\right)\left(\prod_{j=1}^{n-1} (1-u_j)^{\gamma_j - \gamma_{j+1} - 1}\right)(1-u_n)^{\gamma_n - 1}$$

on the cone $\{(u_1, \ldots, u_n) : 0 \leqslant u_1 \leqslant \ldots \leqslant u_n < 1\} \subset \Re^n$, with parameters $\gamma_1, \ldots, \gamma_n > 0$. The parameters $\gamma_1, \ldots, \gamma_n$ are defined by $\gamma_n = k > 0$ and $\gamma_r = k + n - r + M_r, r = 1, 2, \ldots, n-1$, where $M_r = \sum_{j=r}^{n-1} m_j$ and $\tilde{m} = (m_1 m_2 \cdots m_{n-1}) \in \Re^{n-1}$. Generalized order statistics based on some df F are defined via the quantile transformation $X^*(r) = F^{-1}(U^*(r)), r = 1, 2, \ldots, n$.

Particular choices of the parameters $\gamma_1, \ldots, \gamma_n$ lead to different models, e.g.,

(i) m-Generalized order statistics (m-gos): In this model we have $m_1 = m_2 = \ldots = m_{n-1} = m$ and $\gamma_n = k$. Thus, $\gamma_r = k + (n-r)(m+1), r = 1, \ldots, n-1$. Many important practical models of m-gos are included such as order statistics, order statistics with non-integer sample size, upper record values, sequential order statistics.

(ii) Ordinary order statistics: In this model we have $\gamma_n = 1$ and $\gamma_r = n - r + 1,\ r = 1, \ldots, n-1$, i.e., $k = 1$, $m_i = 0, i = 1, \ldots, n-1$. This model can be defined classically, if we arrange the rvs $X_1, X_2, \ldots, X_n$ in order of magnitude, and then written as $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$. We call $X_{r:n}$ the rth order statistic. The subject of order statistics deals with the properties and applications of these ordered rvs and of functions involving them.

(iii) Record values: In this model we have $m_i = -1$, $i = 1, \ldots, n-1$ and $k = 1$. Classically, we can define this model by supposing $Z_n = \max(X_1, X_2, \ldots, X_n)$, for $n \geqslant 1$. Then, we say that X_j is an upper record value of $\{X_n, n \geqslant 1\}$, if $Z_j > Z_{j-1}, j > 1$. By definition, X_1 is an upper record value. Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them, e.g., Olympic records or world records in sports.

(iv) Sequential order statistics: In this model we have $\gamma_n = \alpha_n,\ \gamma_r = (n - r + 1)\alpha_r,\ r = 1, \ldots, n-1$. Classically, this model is defined, if we let the rth order statistic be observed as the rth failure in some life-length test. Then we can define the sequential order statistics as the model in which the life-length distribution of the remaining components in the system may change after each failure of the components.

Burkschat et al. [4] have introduced the concept of dual generalized order statistics (dgos) to enable a common approach to descendingly ordered rvs like reversed order statistics and lower record values.

Definition 3. The uniform dgos $U_d^*(r) \equiv U_d(r, n, k, \tilde{m}),\ r = 1, 2, \ldots, n$, are defined by their density function

$$f^{U_d^*(1),\ldots,U_d^*(n)}(u_1,\ldots,u_n)=\left(\prod_{j=1}^{n}\gamma_j\right)\left(\prod_{j=1}^{n-1}u_j^{\gamma_j-\gamma_{j+1}-1}\right)u_n^{\gamma_n-1}$$

on the cone $\{(u_1,\ldots,u_n): 1\geqslant u_1\geqslant\ldots\geqslant u_n>0\}\subset\Re^n$. The quantile transformation $X_d^*(r)=F_d^{-1}(U_d^*(r)),\ r=1,2,\ldots,n$, yields dgos based on arbitrary df F_d.

Cramer [5] and Burkschat et al. [4] have shown that the gos and dgos, respectively, can be defined by the product of the independent power function distributed rvs.

Definition 4. Let B_j, $1\leqslant j\leqslant n$, be independent rvs with respective Beta distribution $Beta(\gamma_j,1)$, i.e., B_j follows a power function distribution with exponent γ_j. Then $X^*(r)$ and $X_d^*(r)$ can be defined, respectively by

$$X^*(r)\ \sim\ F^{-1}\left(1-\prod_{j=1}^{r}B_j\right),\ r=1,2,\ldots,n,$$

and

$$X_d^*(r)\ \sim\ F_d^{-1}\left(\prod_{j=1}^{r}B_j\right),\ r=1,2,\ldots,n.$$

From the above definition, we can easily see that the two relations $X^*(1)\leqslant X^*(2)\leqslant\ldots\leqslant X^*(n)$ and $X_d^*(1)\geqslant X_d^*(2)\geqslant\ldots\geqslant X_d^*(n)$ hold almost surely.

2.1. The maximal correlation for gos and dgos

In the classical ordinary order statistics, Terrel [11] proved that, when the sample size $n=2$, and if the order statistics $X_{1:2}$ and $X_{2:2}$ have finite variances, then their correlation coefficient satisfies the inequality

$$\mathrm{corr}(X_{1:2},X_{2:2})\leqslant\frac{1}{2},$$

with equality if and only if F is uniformly distributed (this relation means that $R(X_{1:2},X_{2:2})=\frac{1}{2}=\mathrm{corr}(F(X_{1:2}),F(X_{2:2}))$). Latter, Szekely and Móri [10] have shown that

$$\mathrm{corr}(X_{r:n},X_{s:n})\leqslant\sqrt{\frac{r(n-s+1)}{s(n-r+1)}},\ 1\leqslant r<s\leqslant n,$$

here it is supposed that $\mathrm{var}(X_{r:n})$ and $\mathrm{var}(X_{s:n})$ are finite and the sample size can be arbitrary.

An interesting alternative proof of last relation is given by Rohatgi and Szekeli [13]. Recently, Barakat [1] by using the method of Rohatgi and Szekeli [13], the above result is extended to a wide subclass of gos and dgos. Namely, for any $1\leqslant r<s\leqslant n$ we consider the gos $X^*(r),X^*(s)$ and the dgos $X_d^*(r),\ X_d^*(s)$, for which $m_1=m_2=\cdots m_{s-1}=m$. The three exhaustive and distinct cases $m+1>0$, $m+1=0$ and $m+1<0$ are considered. Clearly the m-gos and m-dgos, where $m_1=m_2=\cdots=m_{n-1}=m$ are special cases of these subclasses.

Theorem 1. ([1]). *Let $X^*(1)\leqslant X^*(2)\leqslant\cdots\leqslant X^*(n)$ and $X_d^*(n)\leqslant X_d^*(n-1)\leqslant\cdots\leqslant X_d^*(1)$ be gos and dgos based on arbitrary continuous df's F and F_d, respectively, such that for any $1\leqslant r<s\leqslant n$ we have $m_1=m_2=\cdots=m_{s-1}=m$. Furthermore, let $X^*(r)$, $X^*(s)$,$X_d^*(r)$ and $X_d^*(s)$ have finite variances. Then*

$$R(X^*(r),X^*(s))=R(X_d^*(r),X_d^*(s))=\rho_n(r,s,m,\tilde{\gamma}_s)=\sqrt{\frac{A_s(m)}{A_r(m)}\times\frac{B_r(m)-A_r(m)}{B_s(m)-A_s(m)}},$$

where

$$A_l(m)=\begin{cases}\prod_{j=1}^{l}\frac{\beta_j}{1+\beta_j}, & m\neq-1,\\ 1, & m=-1,\end{cases}$$

$$B_l(m)=\begin{cases}\prod_{j=1}^{l}\frac{\beta_j+1}{\beta_j+2}, & m\neq-1,\\ 1+\sum_{j=1}^{l}\frac{1}{\gamma_j^2}, & m=-1,\end{cases}$$

$\beta_j=\frac{\gamma_j}{m+1}$, with $\beta_1=\frac{\gamma_1}{m+1}<-2$, if $m+1<0$, and $\tilde{\gamma}_s=(\gamma_1\gamma_2\cdots\gamma_s)$. Moreover, the correlation coefficient, in the gos and the dgos cases, coincides with the maximal correlation only if

$$F(x)=\begin{cases}F_1(x)=1-(1-x)^{\frac{1}{m+1}},\ 0<x<1, & \text{if } m+1>0,\\ F_2(x)=1-e^{-x},\ 0<x<\infty, & \text{if } m=-1,\\ F_3(x)=1-x^{\frac{1}{m+1}},\ 1<x<\infty, & \text{if } m+1<0,\end{cases}$$

and

$$F_d(x)=\begin{cases}F_{1d}(x)=x^{\frac{1}{m+1}},\ 0<x<1, & \text{if } m+1>0,\\ F_{2d}(x)=e^{x},\ -\infty<x<0, & \text{if } m=-1,\\ F_{3d}(x)=(1-x)^{\frac{1}{m+1}},\ -\infty<x<0, & \text{if } m+1<0.\end{cases}$$

Corollary 1. *Let $\mu_i^{r/s}(x_s)=\mathrm{E}(X_i^*(r)/X_i^*(s)=x_s)$ be the regression curve of $X_i^*(r)$ given $X_i^*(s)$, where $X_i^*(r)$ and $X_i^*(s)$ are the gos based on the df $F_i(x)$, $i=1,2,3$. Similarly, let $\mu_{id}^{r/s}(x_s)=\mathrm{E}(X_{id}^*(r)/X_{id}^*(s)=x_s)$ be the regression curve of $X_{id}^*(r)$ given $X_{id}^*(s)$, where $X_{id}^*(r)$ and $X_{id}^*(s)$ are the dgos based on the df $F_{id}(x)$, $i=1,2,3$. Then, we have the following relations:*

1. $\mu_{1d}^{r/s}(x_s)-\mu_1^{r/s}(x_s)=1-\frac{r}{s}$, *for all $0<x_s<1, m+1>0$.*
2. $\mu_1^{s/r}(x_r)-\mu_{1d}^{s/r}(x_r)=\frac{(s-r)(m+1)}{\gamma_s+(s-r)(m+1)}$, *for all $0<x_r<1$, $m+1>0$.*
3. $\mu_2^{r/s}(x_s)+\mu_{2d}^{r/s}(-x_s)=0$, *for all $0<x_s<\infty$, $m=-1$.*
4. $\mu_2^{s/r}(x_r)+\mu_{2d}^{s/r}(-x_r)=0$, *for all $0<x_r<\infty$, $m=-1$.*
5. $\mu_3^{r/s}(x_s)+\mu_{3d}^{r/s}(-x_s)=1-\frac{r}{s}$, *for all $1<x_s<\infty$, $m+1<0$.*
6. $\mu_3^{s/r}(x_r)+\mu_{3d}^{s/r}(-x_r)=\frac{(s-r)(m+1)}{\gamma_s+(s-r)(m+1)}$, *for all $1<x_r<\infty$, $m+1<0$.*

The following two results are direct consequences of Theorem 1.

Corollary 2. *For any $r<s$, we have*

$$\rho_n(r,s,m,\tilde{\gamma}_s)\leqslant\min\left(\sqrt{\frac{A_s(m)}{A_r(m)}},\ \sqrt{\frac{Br(m)-A_r(m)}{B_s(m)-A_s(m)}}\right).$$

Moreover, the asymptotic independence between the gos $X_{r:n}^*$ and $X_{s:n}^*$ occurs if, and only if, at least one of the relations $\frac{A_s(m)}{A_r(m)}\to0$ and $\frac{B_r(m)-A_r(m)}{B_s(m)-A_s(m)}\to0$ holds.

Corollary 3. *For any $r \neq s$ and any n, we have $X^*_{r:n}$ and $X^*_{s:n}$ are independent if, and only if, $X^*_{d;r:n}$ and $X^*_{d;s:n}$ are independent.*

3. Discussion and applications

In this section, we discuss the maximal correlation and its applications for some important models of ascendingly ordered rvs.

3.1. The ordinary order statistics model

For this model, we have

$$\rho_n(r, s, m, \tilde{\gamma}_s) = \sqrt{\frac{r(n-s+1)}{s(n-r+1)}}.$$

Therefore, any two order statistics $X_{r:n}$ and $X_{s:n}$, with non-decreasing ranks r and s, with respect to n, are asymptotically independent if, and only if, $\min\left(\frac{r}{s}, \frac{n-s+1}{n-r+1}\right) \to 0$.

We can note that, as the sample size increases to infinity, any order statistic $X_{r:n}$ can belong to one and only one of the following types:

- Lower extreme, where r is fixed $\left(\frac{r}{n} \to 0\right)$.
- Lower intermediate, where $\min(r,n) \to \infty$ and $\frac{r}{n} \to 0$.
- Central, where $\min(r,n) \to \infty$ and $\frac{r}{n} \to \lambda \in (0, 1)$.
- Upper intermediate, where $r = n - r' + 1$, $r' \to \infty$ and $\frac{r}{n} \to 1$.
- Upper extreme, where $r = n - r' + 1$, r' is fixed and $\frac{r}{n} \to 1$.

By using Theorem 1, we can state the following theorem which is concerning with the asymptotic interrelation between the above types of order statistics.

Theorem 2. *Let $\underline{X}^E_{r:n}$, $\overline{X}^E_{r:n}$, $\underline{X}^I_{r:n}$, $\overline{X}^I_{r:n}$ and $X^C_{r:n}$ be the rth lower extreme, upper extreme, lower intermediate, upper intermediate and central order statistics, respectively. Then*

(1) *the rvs $\underline{X}^E_{r_1:n}$, $\overline{X}^E_{r_2:n}$, $\underline{X}^I_{r_3:n}$, $\overline{X}^I_{r_4:n}$, and $X^C_{r_5:n}$ are asymptotically mutually independent, for all $1 \leqslant r_1, r_2, \ldots, r_5 \leqslant n$,*
(2) *the components of each pair $(\underline{X}^E_{r:n}, \underline{X}^E_{s:n})$, $(X^C_{r:n}, X^C_{s:n})$ and $(\overline{X}^E_{r:n}, \overline{X}^E_{s:n})$ are asymptotically dependent, for all $1 \leqslant r < s \leqslant n$,*
(3) *the components of each pair $(\underline{X}^I_{r:n}, \underline{X}^I_{s:n})$ and $(\overline{X}^I_{r:n}, \overline{X}^I_{s:n})$ are asymptotically independent if, and only if, $\min\left(\frac{r}{s}, \frac{n-s+1}{n-r+1}\right) \to 0$.*

The following consequence studies the rate of convergence to asymptotic independence between different order statistics. Although, the proof of this consequence is simple, but to the best of the author knowledge this result is new.

Theorem 3. *Keeping the notations of Theorem 2, we get*

(I) *The convergence to the asymptotic independence of the couple $(\underline{X}^E_{r_1:n}, \overline{X}^E_{r_5:n})$ is faster than the couple $(\underline{X}^I_{r_2:n}, \overline{X}^I_{r_4:n})$.*
(II) *The convergence to the asymptotic independence of the couple $(\underline{X}^E_{r_1:n}, X^C_{r_3:n})$ is faster than the couples $(\underline{X}^E_{r_1:n}, \underline{X}^I_{r_2:n})$ and $(\underline{X}^I_{r_2:n}, X^C_{r_3:n})$.*
(III) *The convergence to the asymptotic independence of the couple $(\underline{X}^I_{r_2:n}, X^C_{r_3:n})$ is faster than the couple $(\underline{X}^E_{r_1:n}, \underline{X}^I_{r_2:n})$, if and only if $r_2 = \circ(\sqrt{n})$, as $n \to \infty$.*

Exmaple 1 (The determination of a suitable type of a given order statistic). As an interesting application of Theorem 3, we consider a requirement of a certain statistical problem which stipulates the asymptotic independence between two order statistics $X_{10:100}$ and $X_{50:100}$. For performing a goodness of fit test to identify the suitable limit distribution type of each of the statistics $X_{10:100}$ and $X_{50:100}$, we first have to choose their types (extreme or intermediate or central type). The type of the order statistic $X_{50:100}$ can reasonably regarded as a central type, i.e., $X_{50:100} = X^C_{r_3:n}$. However, on one hand side, the type of the order statistic $X_{10:100}$ may be regarded as extreme type, i.e., $X_{10:100} = \underline{X}^E_{r_1:n}$, $r_1 = 10$, $n = 100$, but on the other side, it may be regarded as lower intermediate type $X_{10:100} = \underline{X}^I_{r_2:n}$, $r_2 = [\sqrt{100}]$, $n = 100$. In view of our requirement, Theorem 3, part (II), enables us to decide that the choice of extreme type for the order statistic $X_{10:100}$ is better than the choice of lower intermediate type.

3.2. The record values model

For this model, it is easy to see that

$$\rho_n(r, s, m, \tilde{\gamma}_s) = \sqrt{\left(\frac{3}{4}\right)^{s-r} \times \left(\frac{1-\left(\frac{3}{4}\right)^r}{1-\left(\frac{3}{4}\right)^s}\right)}. \qquad (1)$$

Therefore, the asymptotic dependence between any two upper records R^*_r and R^*_s, as well as any two lower records L_s and L_r, depends on the asymptotic behavior of the difference $s - r$. Since, the relation $\min\left(\frac{r}{s}, \frac{n-s+1}{n-r+1}\right) \to 0$ implies the relation $s - r \to \infty$, we get the following interesting fact: *The asymptotic independence between any two order statistics $X_{r:n}$ and $X_{s:n}$ implies the asymptotic independence between the upper records* R^*_r and R^*_s, *as well as the lower records L_s and L_r.*

3.3. Type II right censored samples

Let the censoring scheme be $R_1 = R_2 = \ldots = R_{M-1} = 0$, $R_M = n - M$. Therefore, we get $\beta_j = \gamma_j = 2n + M - j + 1$. Thus, if $r < s$, we get, after simple calculations,

$$\rho_n(r, s, m, \tilde{\gamma}_s) = \sqrt{\frac{r(2n-M-s+1)}{s(2n-M-r+1)}}.$$

Therefore, if M is constant with respect to n then $X^*(r)$ and $X^*(s)$ as well as $X^*_d(r)$ and $X^*_d(s)$ are dependent for all r, s and n. On the other hand, by assuming that $M = M(n) \to \infty$ and $\frac{M}{n} \to 0$, as $n \to \infty$, we can easily deduce that Theorem 2, which is concerned with the asymptotic dependence between ordinary order statistics, will hold for this model.

An open problem: In [2] it is proved that $\rho_n(r, s, 0, \tilde{\gamma}_s) = \sqrt{\frac{A_s(0)}{A_r(0)} \times \frac{B_r(0)-A_r(0)}{B_s(0)-A_s(0)}}$ is the correlation coefficient between any two uniform gos $U^*(r)$ and $U^*(s)$ or any two uniform dgos $U^*_d(r)$ and $U^*_d(s)$, where no any restriction is imposed on the parameters $k, m_1, m_2, \ldots, m_{n-1}$. Moreover, in the same paper it is proved that the measure $\sigma^*_{r,s:n} = 12\rho_n(r, s, 0, \tilde{\gamma}_s)$ provides a non-parametric criterion of asymptotically independence between the elements of gos and between the elements of dgos in general setting (where no any restriction is imposed on the parameters $k, m_1, m_2, \ldots, m_{n-1}$). This measure is based on a non-parametric criterion of independence derived from a generalized version of the Schweizer–Wolff non-parametric measure of dependence (see [3]). The derivation of the maximal correlation of gos and dgos in this general setting is still unsolved problem.

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