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Original Article

On the invalidity of semigroup property for the Mittag–Leffler function with two parameters

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Abstract It is shown that the following property

$$E_{\alpha,\beta}\big(a(s+t)^{\alpha\beta}\big) = E_{\alpha,\beta}(as^{\alpha\beta})E_{\alpha,\beta}(at^{\alpha\beta}),\ s,t \geq 0, a \in \mathbb{R}, \alpha, \beta > 0 \tag{1}$$

is true only when $\alpha = \beta = 1$, and $a = 0$, $\beta = 1$ or $\beta = 2$. Moreover, a new equality on $E_{\alpha,\beta}(at^{\alpha\beta})$ is developed, whose limit state as $\alpha \uparrow 1$ and $\beta > \alpha$ is just the above property (1) and if $\beta = 1$, then the result is the same as in [16]. Also, it is proved that this equality is the characteristic of the function $t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$. Finally, we showed that all results in [16] are special cases of our results when $\beta = 1$.

Mathematics Subject Classification: 34A12

1. Introduction

As a result of researchers' and scientists' increasing interest in pure as well as applied mathematics in non-conventional models, particularly those using fractional calculus, Mittag–Leffler functions have recently caught the interest of the scientific community. Focusing on the theory of the Mittag–Leffler functions, the present volume offers a self-contained, comprehensive treatment, ranging from rather elementary matters to the latest research results. In addition to the theory the authors devote some sections of the work to the applications, treating various situations and processes in viscoelasticity, physics, hydrodynamics, diffusion and wave phenomena, as well as stochastics. In particular the Mittag–Leffler functions allow us to describe phenomena in processes that progress or decay too slowly to be represented by classical functions like the exponential function and its successors. The two parameter Mittag–Leffler function is such a two-parameter function defined in the complex plane $\mathbb{C}$ by $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$, where $\alpha > 0$ is the parameter and Γ the Gamma function [1].

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It was originally introduced by Mittag–Leffler in 1902 [2]. Obviously, the exponential function e^z is a particular Mittag–Leffler function with the specified parameter $\alpha = \beta = 1$, or in other words, the Mittag–Leffler function is the parameterized exponential function. In recent years the Mittag–Leffler function has caused extensive interest among scientists, engineers and applied mathematicians, due to its role played in investigations of fractional differential equations (see, for example, [3–8]). A large of its properties have been proved (see, e.g., [9–15]). In this paper we show that the following property

$$E_{\alpha,\beta}\big(a(s+t)^{\alpha\beta}\big) = E_{\alpha,\beta}(as^{\alpha\beta})E_{\alpha,\beta}(at^{\alpha\beta}),\ s,t \geq 0,\quad a \in \mathbb{R} \quad \text{and} \quad \alpha,\beta > 0$$

is true only when $\alpha = \beta = 1$, and $a = 0$, $\beta = 1$ or $\beta = 2$. Moreover, a new equality on $E_{\alpha,\beta}(at^{\alpha\beta})$ is developed, whose limit state as $\alpha \uparrow 1$ and $\beta > \alpha$ is just the above property (1), if $\beta = 1$, then the equality is the same as in [16]. Finally, it is proved that this equality is the characteristic of the function $t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$. To this purpose, the following properties of Mittag–Leffler function and Caputo's fractional derivative are needed:

(P1) (cf. [17, formula (2.140)]) The Laplace transform of Caputo's derivative is given by

$$\widehat{D_t^{\alpha} f(t)}(\lambda) = \lambda^{\alpha}\hat{f}(t) - \sum_{k=0}^{n-1}\lambda^{\alpha-k-1}f^{(k)}(0), \quad \text{where} \quad n-1 < \alpha \leq n,$$

$\widehat{D_t^{\alpha} f(t)}(\lambda)$ and $\lambda^{\alpha}\hat{f}(t)$ denote the laplace transform of $D_t^{\alpha}f(t)$ and $f(t)$, respectively.

(P2) (cf. [12, p.287]) The Laplace transform of the Mittag–Leffler functions can be derived from the formula

$$\int_0^{\infty} e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}-a},\ Re\lambda > a^{\frac{1}{\alpha}},\ a > 0,$$

where $Re\lambda$ represents the real part of the complex number λ.

2. A new equality characteristic of solution function

In this section, firstly we show the following general property

$$E_{\alpha,\beta}\big(a(s+t)^{\alpha\beta}\big) = E_{\alpha,\beta}(as^{\alpha\beta})E_{\alpha,\beta}(at^{\alpha\beta}),\ s,t \geq 0,\ a \in \mathbb{R} \quad \text{and} \quad \alpha,\beta > 0 \tag{1}$$

is false, to show that, it is sufficient to provide one counterexample. By using the fact $E_2(z^2) = (\frac{e^z+e^{-z}}{2}) = \cosh z$, valid for all z, then elementary calculation shows that (1) doesnot hold for the choice $\alpha = 2, \beta = 1, a = s = t = 1$ of parameters.This is because the difference of left hand side and right hand side of (1) comes out as $(\sinh 1)^2$. Hence (1) as a general property is false. Secondly we prove (1) is true only when $\alpha = \beta = 1$, and $a = 0$, $\beta = 1$ or $\beta = 2$, to do that, let

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}.$$

We want to find all $\alpha,\beta > 0$, $a \in \mathbb{R}$ for which

$$E_{\alpha,\beta}\big(a(s+t)^{\alpha\beta}\big) = E_{\alpha,\beta}(as^{\alpha\beta})E_{\alpha,\beta}(at^{\alpha\beta}),\ s,t \geq 0.$$

If we set $s = t = 0$ we can write

$$E_{\alpha,\beta}(0) = E_{\alpha,\beta}(0)E_{\alpha,\beta}(0).$$

Knowing that $E_{\alpha,\beta}(0) = \frac{1}{\Gamma(\beta)}$, yields $\Gamma(\beta) = 1$ which implies $\beta = 1$ or $\beta = 2$. So if $a = 0$ then (1) is true with $\beta = 1$ or $\beta = 2$ and this is the only condition, consequently we can say that for any $a \in$ R and $\alpha, \beta > 0$, the equality (1) may be true only when $\Gamma(\beta) = 1$. Therefore, let us assume that $a \neq 0$ and $\Gamma(\beta) = 1$. From (1), if we let $s = t$ then (1) becomes

$$E_{\alpha,\beta}(a(2s)^{\alpha\beta}) = E_{\alpha,\beta}(as^{\alpha\beta})E_{\alpha,\beta}(as^{\alpha\beta}) = \big(E_{\alpha,\beta}(as^{\alpha\beta})\big)^2.$$

Set $z = s^{\alpha\beta}$, then the above equation becomes

$$E_{\alpha,\beta}(az2^{\alpha\beta}) = (E_{\alpha,\beta}(az))^2 \quad \text{for all} \quad z \geq 0. \tag{2}$$

We differentiate (2) and set $z = 0$. This gives

$$\frac{a2^{\alpha\beta}}{\Gamma(\alpha+\beta)} = \frac{2a}{\Gamma(\beta)\Gamma(\alpha+\beta)}.$$

This implies $2^{\alpha\beta} = 2$. Therefore, $\alpha\beta = 1$. If $\beta = 1$ then $\alpha = 1$ and $E_{\alpha,\beta}(z) = e^z$, so (1) is true. Now assume that $\beta = 2$. Then $\alpha = \frac{1}{2}$. If we differentiate (2) twice with respect to z and set $z = 0$ we obtain

$$\frac{8a^2}{\Gamma(2\alpha+\beta)} = \frac{2a^2}{\Gamma(\alpha+\beta)^2} + \frac{4a^2}{\Gamma(\beta)\Gamma(2\alpha+\beta)}.$$

which gives (with $\beta = 2$ and $\alpha = \frac{1}{2}$)

$$4 = \frac{32}{9\pi} + 2,$$

which is not true. Therefore, (1) is true only when $\alpha = \beta = 1$, and $a = 0$, $\beta = 1$ or $\beta = 2$.

By definition of Caputo derivative it is clear that the Caputo's fractional derivative operator is nonlocal in the case of non-integer order α. The memory character of Caputo's derivative operator is perhaps the cause leading to the result that $E_{\alpha,\beta}(at^{\alpha\beta})$, as an eigenfunction of Caputo's derivative operator does not possess semigroup property that is non-memory. This seems to tell us that any equality relationship involving $E_{\alpha,\beta}(at^{\alpha\beta})$, $E_{\alpha,\beta}(as^{\alpha\beta})$ and $E_{\alpha,\beta}(a(s+t)^{\alpha\beta})$ should be of memory and hence be characterized with integrals. The equality relationship stated in the following theorem which is a generalization to Theorem 1 in [16] is a result of the above idea.

Theorem 2.1. *For every real a there holds that*

$$\int_0^t (t-\tau)^{\beta-\alpha-1}f(s+\tau)d\tau = \int_0^t (s+t-\tau)^{\beta-\alpha-1}f(\tau)d\tau + \frac{\alpha\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)}\int_0^s\int_0^t f(\tau_1)f(\tau_2)(t+s-\tau_1-\tau_2)^{-\alpha-1}d\tau_2 d\tau_1, \tag{3}$$

where $t, s \geq 0$ and $f(t) = t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$.

Proof. Let $0 < \alpha < 1$, $\beta > 1$, $a > 0$. Define

$$f(t) = t^{\beta-1}E_{\alpha,\beta}(at^{\alpha}) = \sum_{k=0}^{\infty}\frac{a^k t^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}. \qquad \square$$

Then its Caputo derivative satisfies

$$D_t^\alpha f(t) = af(t) + \frac{t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}.$$

Now

$$D_t^\alpha f(s+t) = af(s+t) + \frac{(s+t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} - \frac{1}{\Gamma(1-\alpha)} \times \int_0^s (s+t-\tau)^{-\alpha} \frac{df(\tau)}{d\tau} d\tau.$$

By integration by parts,

$$D_t^\alpha f(s+t) = af(s+t) + \frac{(s+t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} - \frac{t^{-\alpha} f(s)}{\Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)} \times \int_0^s (s+t-\tau)^{-\alpha-1} f(\tau) d\tau.$$

Applying the Laplace transform with respect to t, we get

$$\lambda^\alpha \hat{f}_s(\lambda) - \lambda^{\alpha-1} f(s) = a\hat{f}_s(\lambda) + \frac{\left\{(s+t)^{\beta-\alpha-1}\right\}^\wedge}{\Gamma(\beta-\alpha)} - \lambda^{\alpha-1} f(s) + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^s \left\{(s+t-\tau)^{-\alpha-1}\right\}^\wedge f(\tau) d\tau.$$

This simplifies to

$$(\lambda^\alpha - a)\hat{f}_s(\lambda) = \frac{\left\{(s+t)^{\beta-\alpha-1}\right\}^\wedge}{\Gamma(\beta-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)} \times \int_0^s \left\{(s+t-\tau)^{-\alpha-1}\right\}^\wedge f(\tau) d\tau.$$

We rewrite this as

$$\Gamma(\beta-\alpha)\lambda^{\alpha-\beta} \hat{f}_s(\lambda) = \left\{(s+t)^{\beta-\alpha-1}\right\}^\wedge \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - a} + \frac{\alpha\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)} \int_0^s \times \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - a} \left\{(s+t-\tau)^{-\alpha-1}\right\}^\wedge f(\tau) d\tau.$$

Now we use that

$$\hat{f}_s(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - a}.$$

Therefore, by applying inverse Laplace transform,

$$\int_0^t (t-\tau)^{\beta-\alpha-1} f(s+\tau) d\tau = \int_0^t (s+t-\tau)^{\beta-\alpha-1} f(\tau) d\tau + \frac{\alpha\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)} \times \int_0^s \int_0^t f(\tau_1) f(\tau_2)(t+s-\tau_1-\tau_2)^{-\alpha-1} d\tau_2 d\tau_1.$$

By analytic continuation one can see that this formula remains true when $0 < \alpha < 1$ and $\beta > \alpha$. If $\beta = 1$, the result is the same as in [16].

We now obtain our main result.

Theorem 2.2. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function which satisfies*

$|f(t)| \le Ce^{wt}$ for $t \ge 0, w \in \mathrm{R}$ and equality (3), then f must be equal $t^{\beta-1} E_{\alpha,\beta}(at^\alpha)$.

Proof. We use the two-dimensional Laplace transform for the function $f(s, t)$ of two variables $s, t \ge 0$ defined by

$$\hat{f}(\lambda, \mu) = \int_0^\infty \int_0^\infty e^{-\lambda s - \mu t} f(s, t) ds dt. \ \square$$

If

$$|f(s, t)| \le Ce^{w_1 s + w_2 t} \quad \text{for} \quad s, t \ge 0,$$

then $\hat{f}$ is defined for $\lambda > w_1$, $\mu > w_2$. We will use the following rules:

(1) The Laplace transform of $f(s, t) = g(s + t)$ is

$$\hat{f}(\lambda, \mu) = -\frac{\hat{g}(\mu) - \hat{g}(\lambda)}{\mu - \lambda},$$

where $\hat{g}$ is the one-dimensional Laplace transform of g. If $\lambda = \mu$ the right hand side has to be replaced by $-\frac{d}{d\lambda}\hat{g}(\lambda)$.

(2) If

$$h(s, t) = \int_0^s \int_0^t f(\tau_1, \tau_2) g(s - \tau_1, t - \tau_2) d\tau_1 d\tau_2,$$

then

$$\hat{h}(\lambda, \mu) = \hat{f}(\lambda, \mu)\hat{g}(\lambda, \mu).$$

(3) If

$$h(s, t) = \int_0^t f(s, t - \tau) g(\tau) d\tau,$$

then

$$\hat{h}(\lambda, \mu) = \hat{f}(\lambda, \mu)\hat{g}(\mu).$$

Now let $f : [0, \infty) \to \mathbb{R}$ be a continuous function which satisfies an estimate of the form

$$|f(t)| \le Ce^{wt} \quad \text{for} \quad t \ge 0, w \in \mathrm{R}. \tag{4}$$

Then the Laplace transform $\hat{f}(\lambda)$ is defined for $\lambda > w$. Let $\alpha \in (0, 1)$ and $\beta > \alpha$. Suppose that the following equation holds

$$f_1(s, t) - f_2(s, t) - f_3(s, t) = f_4(s, t) \quad \text{for} \quad s, t \ge 0, \tag{5}$$

where

$$f_1(s, t) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^{s+t} (s+t-\tau)^{\beta-\alpha-1} f(\tau) d\tau$$

$$f_2(s, t) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^t (s+t-\tau)^{\beta-\alpha-1} f(\tau) d\tau,$$

$$f_3(s, t) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^s (s+t-\tau)^{\beta-\alpha-1} f(\tau) d\tau,$$

$$f_4(s, t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^s \int_0^t f(\tau_1) f(\tau_2)(t+s-\tau_1-\tau_2)^{-\alpha-1} d\tau_2 d\tau_1.$$

By rule (1),

$$\hat{f}_1(\lambda, \mu) = \frac{\lambda^{\alpha-\beta}\hat{f}(\lambda) - \mu^{\alpha-\beta}\hat{f}(\mu)}{\mu - \lambda}.$$

By rules (1) and (3),

$$\hat{f}_2(\lambda,\mu) = \frac{\lambda^{\alpha-\beta}-\mu^{\alpha-\beta}}{\mu-\lambda}\hat{f}(\mu),$$
$$\hat{f}_3(\lambda,\mu) = \frac{\mu^{\alpha-\beta}-\lambda^{\alpha-\beta}}{\lambda-\mu}\hat{f}(\lambda).$$

By rules (1) and (2),

$$\hat{f}_4(\lambda,\mu) = -\frac{\lambda^{\alpha}-\mu^{\alpha}}{\mu-\lambda}\hat{f}(\lambda)\hat{f}(\mu).$$

It follows from (5) that

$$\hat{f}_1(\lambda,\mu)-\hat{f}_2(\lambda,\mu)-\hat{f}_3(\lambda,\mu)=\hat{f}_4(\lambda,\mu) \quad \text{for} \quad \lambda,\mu > w.$$

If we substitute our formula for $\hat{f}_j$, $j=1,2,3,4$, in (5), we obtain after simplification

$$\mu^{\alpha-\beta}\hat{f}(\lambda)-\lambda^{\alpha-\beta}\hat{f}(\mu)=(\mu^{\alpha}-\lambda^{\alpha})\hat{f}(\lambda)\hat{f}(\mu). \tag{6}$$

If $f(t)=0$ for all $t\geq 0$ then (5) holds. Otherwise, there is $\mu > w$ such that $\hat{f}(\mu) \neq 0$. Then (6) yields by solving for $\hat{f}(\lambda)$

$$\hat{f}(\lambda)=\frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}-a},$$

where

$$a=\mu^{\alpha}-\frac{\mu^{\alpha-\beta}}{\hat{f}(\mu)}.$$

We know that the Laplace transform of the function

$$h(t)=t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$$

is

$$\hat{h}(\lambda)=\frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}-a}.$$

Therefore, it follows that $f=h$. This shows that a part from the trivial solution $f(t)=0$ the only solution of (5) is $f(t)=h(t)$.

Remark 2.1. Theorem 2.2, shows that every characteristic function f satisfies an estimate of the form $|f(t)|\leq Ce^{wt}$ for $t\geq 0, w\in \mathrm{R}$ and also satisfies equality (3) must be equal $t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$.

3. Conclusion

Recently the authors have found in some publications that the following property.

$$E_{\alpha,\beta}(a(s+t)^{\alpha\beta})=E_{\alpha,\beta}(as^{\alpha\beta})E_{\alpha,\beta}(at^{\alpha\beta}),\ s,t \geq 0,\ a\in \mathrm{R},\ \alpha,\beta>0(1).$$

Of Mittag–Leffler function with two parameters is taken for granted and used to derive other properties. Also in this note it is proved that the above property is unavailable unless $\alpha=\beta=1$, and $a=0$, $\beta=1$ or $\beta=2$. Moreover, a new equality on $E_{\alpha,\beta}(at^{\alpha\beta})$ is developed, whose limit state as $\alpha\uparrow 1$ and $\beta>\alpha$ is just the above property (1). Also it is shown that if $\beta=1$, then the equality is the same as in [16]. Finally, it is proved that this equality is the characteristic of the function $t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$.

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