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Original Article

On certain subclasses of analytic and bi-univalent functions

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Abstract In this paper, we introduced two interesting subclasses of the function class σ of analytic and bi-univalent functions in the open unit disk U. Estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these classes are determined. Certain special cases are also indicated.

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1. Introduction

Let A denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, $\mathbb{C}$ being, as usual, the set of complex numbers. We also denote by S the subclass of all functions in A which are univalent in U. Let S_s^* be the subclass of S consisting of functions of the form (1.1) satisfying

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in U. \tag{1.2}$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [1] (see also Robertson [2], Stankiewicz [3], Wu [4] and Owa et al. [5]). Das and Singh [6] introduced another class C_s namely, convex functions with respect to symmetric points and satisfying the condition

$$\mathrm{Re}\left(\frac{(zf'(z))'}{(f(z) - f(-z))'}\right) > 0, \quad z \in U. \tag{1.3}$$

If f and g are analytic functions in U, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function φ, which (by definition) is analytic in U with $\varphi(0) = 0$ and $|\varphi(z)| < 1$ for all $z \in U$, such that $f(z) = g(\varphi(z))$, $z \in U$.

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Furthermore, if the function g is univalent in U, then we have the following equivalence

$$f(z) \prec g(z)(z \in U) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

For each $f \in S$, the Koebe one-quarter theorem [7] ensures the image of U under f contains a disk of radius 1/4. Thus every univalent function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(\omega)) = \omega \quad (|\omega|) < r_0(f); \quad r_0(f) \geq \frac{1}{4}\Big).$$

In fact, the inverse function $g = f^{-1}$ is given by

$$g(\omega) = f^{-1}(\omega) = w - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^2 - 5a_2a_3 + a_4)\omega^4 + \cdots.$$

A function $f \in A$ is said to be bi-univalent in U if f and f^{-1} are univalent in U. Let σ denote the class of bi-univalent functions in U given by (1.1). The familiar Koebe function is not a member of σ because it maps the unit disk U univalently onto the entire complex plane minus a slit along the line $\frac{-1}{4}$ to $-\infty$. Hence the image domain does not contain the unit disk U.

In 1985 Branges [8] proved the celebrated Bieberbach Conjecture which states that, for each $f(z) \in S$ given by the Taylor–Maclaurin series expansion (1.1), the following coefficient inequality holds true:

$$|a_n| \leq n \quad (n \in N - \{1\}),$$

N being the set of positive integers. The class of analytic bi-univalent functions was first introduced and studied by Lewin [9], where it was proved that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [10] improved Lewin's result to $|a_2| \leq \sqrt{2}$. Brannan and Taha [11] and Taha [12] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. For further historical account of functions in the class σ, see the work by Srivastava et al. [13] (see also [11,14]). In fact, the above-cited recent pioneering work of Srivastava et al. [13] has apparently revived the study of analytic and bi-univalent functions in recent years; it was followed by such works as those by Frasin and Aouf [15], Xu et al. [16,17], Hayami and Owa [18], and others (see, for example, [19–35]).

In the present paper, certain subclasses of the bi-univalent function class σ were introduced, and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ were found.

2. Coefficient estimates

In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk U, satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(U)$ is symmetric with respect to the real axis. Such a function has a Taylor series of the form

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \ (B_1 > 0). \tag{2.1}$$

Suppose that $u(z)$ and $v(z)$ are analytic in the unit disk U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$, and suppose that

$$u(z) = b_1z + \sum_{n=2}^{\infty} b_nz^n, \quad v(z) = c_1z + \sum_{n=2}^{\infty} c_nz^n \quad (z \in U). \tag{2.2}$$

It is well known that (see Nehari [36, p. 172])

$$|b_1| \leq 1, \quad |b_2| \leq 1 - |b_1|^2, \quad |c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2. \tag{2.3}$$

By a simple calculation, we have

$$\phi(u(z)) = 1 + B_1b_1z + (B_1b_2 + B_2b_1^2)z^2 + \cdots \ (z \in U), \tag{2.4}$$

and

$$\phi(v(\omega)) = 1 + B_1c_1\omega + (B_1c_2 + B_2c_1^2)\omega^2 + \cdots \ (\omega \in U). \tag{2.5}$$

Definition 1. A function $f \in \sigma$ given by (1.1) is said to be in the class $S_\sigma(\alpha, \phi)(0 \leq \alpha \leq 1)$ if the following conditions are satisfied:

$$(1-\alpha)\frac{2zf'(z)}{f(z) - f(-z)} + \alpha\frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec \phi(z) \ (z \in U)$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega) - g(-\omega)} + \alpha\frac{2(\omega g'(\omega))'}{(g(\omega) - g(-\omega))'} \prec \phi(\omega), \ (\omega \in U),$$

where $g(\omega) := f^{-1}(\omega)$.

Theorem 1. *If $f(z)$ given by (1.1) be in the class $S_\sigma(\alpha, \phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{4(1+\alpha)^2B_1 + 2\left|(1+2\alpha)B_1^2 - 2(1+\alpha)^2B_2\right|}} \tag{2.6}$$

and

$$|a_3| \leq \begin{cases} \frac{B_1}{2(1+2\alpha)} & if\ B_1 \leq \frac{2(1+\alpha)^2}{1+2\alpha}, \\ \left(1 - \frac{2(1+\alpha)^2}{(1+2\alpha)B_1}\right) & if\ B_1 > \frac{2(1+\alpha)^2}{1+2\alpha} \\ \frac{B_1^3}{4(1+\alpha)^2B_1 + 2\left|(1+2\alpha)B_1^2 - 2(1+\alpha)^2B_2\right|} + \frac{B_1}{2(1+2\alpha)}. & \end{cases} \tag{2.7}$$

Proof. Let $f \in S_\sigma(\alpha, \phi)$. Then there are analytic functions u, v: $U \to U$ given by (2.2) such that

$$(1-\alpha)\frac{2zf'(z)}{f(z) - f(-z)} + \alpha\frac{2(zf'(z))'}{(f(z) - f(-z))'} = \phi(u(z)) \tag{2.8}$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega) - g(-\omega)} + \alpha\frac{2(\omega g'(\omega))'}{(g(\omega) - g(-\omega))'} = \phi(v(\omega)). \tag{2.9}$$

Since

$$(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)}+\alpha\frac{2(zf'(z))'}{(f(z)-f(-z))'}$$
$$=1+2(1+\alpha)a_2z+2(1+2\alpha)a_3z^2+\cdots,$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}+\alpha\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}$$
$$=1-2(1+\alpha)a_2\omega+2(1+2\alpha)\big(2a_2^2-a_3\big)\omega^2+\cdots,$$

it follows from (2.4), (2.5), (2.8) and (2.9) that

$$2(1+\alpha)a_2=B_1b_1, \tag{2.10}$$

$$2(1+2\alpha)a_3=B_1b_2+B_2b_1^2, \tag{2.11}$$

$$-2(1+\alpha)a_2=B_1c_1, \tag{2.12}$$

$$2(1+2\alpha)\big(2a_2^2-a_3\big)=B_1c_2+B_2c_1^2. \tag{2.13}$$

From (2.10) and (2.12), we get

$$b_1=-c_1, \tag{2.14}$$

$$8(1+\alpha)^2a_2^2=B_1^2\big(b_1^2+c_1^2\big). \tag{2.15}$$

By adding (2.13) to (2.11), further computations using (2.15) lead to

$$\big[4(1+2\alpha)B_1^2-8(1+\alpha)^2B_2\big]a_2^2=B_1^3(b_2+c_2), \tag{2.16}$$

(2.14), (2.16), together with (2.3), give that

$$2\big|(1+2\alpha)B_1^2-2(1+\alpha)^2B_2\big|\big|a_2^2\big|\le B_1^3\big(1-\big|b_1^2\big|\big). \tag{2.17}$$

From (2.10) and (2.17) we get

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{4(1+\alpha)^2B_1+2\big|(1+2\alpha)B_1^2-2(1+\alpha)^2B_2\big|}}. \tag{2.18}$$

Next, from (2.13) and (2.11), we have

$$4(1+2\alpha)a_3=4(1+2\alpha)a_2^2+B_1(b_2-c_2).$$

From (2.3), (2.10), (2.14) and (2.18), it follows that

$$|a_3|\le a_2^2+\frac{B_1}{2(1+2\alpha)}\big(1-\big|b_1^2\big|\big)$$
$$=\left(1-\frac{2(1+\alpha)^2}{(1+2\alpha)B_1}\right)a_2^2+\frac{B_1}{2(1+2\alpha)}$$
$$\le\begin{cases}\frac{B_1}{2(1+2\alpha)} & if\ B_1\le\frac{2(1+\alpha)^2}{1+2\alpha},\\ \left(1-\frac{2(1+\alpha)^2}{(1+2\alpha)B_1}\right) & if\ B_1>\frac{2(1+\alpha)^2}{1+2\alpha}\\ \quad\frac{B_1^3}{4(1+\alpha)^2B_1+2|(1+2\alpha)B_1^2-2(1+\alpha)^2B_2|}+\frac{B_1}{2(1+2\alpha)}. & \end{cases}$$
□

If we set

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^\gamma=1+2\gamma z+2\gamma^2z^2+\cdots(0<\gamma\le1,z\in U)$$

in Definition 1 of the bi-univalent function class $S_\sigma(\alpha,\phi)$, we obtain a new class $S_\sigma(\alpha,\gamma)$ given by Definition 2 below.

Definition 2. For $0\le\alpha\le1$ and $0<\gamma\le1$, a function $f\in\sigma$ given by (1.1) is said to be in the class $S_\sigma(\alpha,\gamma)$ if the following conditions are satisfied:

$$(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)}+\alpha\frac{2(zf'(z))'}{(f(z)-f(-z))'}\prec\left(\frac{1+z}{1-z}\right)^\gamma\ (z\in U)$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}+\alpha\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}\prec\left(\frac{1+\omega}{1-\omega}\right)^\gamma\ (\omega\in U),$$

where $g(\omega):=f^{-1}(\omega)$.

Using the parameter setting of Definition 2 in Theorem 1, we get the following corollary.

Corollary 1. *For $0\le\alpha\le1$ and $0<\gamma\le1$, let the function $f\in S_\sigma(\alpha,\gamma)$ be of the form (1.1). Then*

$$|a_2|\le\frac{\gamma}{\sqrt{(1+\alpha)^2+\alpha^2\gamma}}$$

and

$$|a_3|\le\frac{\gamma}{1+2\alpha}.$$

If we set

$$\phi(z)=\frac{1+(1-2\upsilon)z}{1-z}=1+2(1-\upsilon)z+2(1-\gamma)z^2$$
$$+\cdots(0\le\upsilon<1,z\in U)$$

in Definition 1 of the bi-univalent function class $S_\sigma(\alpha,\phi)$, we obtain a new class $S_\sigma^\upsilon(\alpha)$ given by Definition 3 below.

Definition 3. For $0\le\alpha\le1$ and $0\le\upsilon<1$, a function $f\in\sigma$ given by (1.1) is said to be in the class $S_\sigma^\upsilon(\alpha)$, if the following conditions hold true:

$$(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)}+\alpha\frac{2(zf'(z))'}{(f(z)-f(-z))'}$$
$$\prec\frac{1+(1-2\upsilon)z}{1-z}\ (z\in U)$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}+\alpha\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}$$
$$\prec\frac{1+(1-2\upsilon)\omega}{1-z}\ (\omega\in U),$$

where $g(\omega):=f^{-1}(\omega)$.

Using the parameter setting of Definition 3 in the Theorem 1, we get the following corollary.

Corollary 2. *For $0\le\alpha\le1$ and $0\le\upsilon<1$, let the function $f\in S_\sigma^\upsilon(\alpha)$, be given by (1.1). Then*

$$|a_2|\le\frac{1-\upsilon}{\sqrt{(1+\alpha)^2+\big|(1+2\alpha)(1-\upsilon)-(1+\alpha)^2\big|}}$$

and

$$|a_3| \le \frac{1-\upsilon}{1+2\alpha}.$$

Definition 4. A function $f \in \sigma$ given by (1.1) is said to be in the class $C_\sigma(\alpha, \phi)(0 \le \alpha \le 1)$ if the following conditions are satisfied:

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)^{1-\alpha} \prec \phi(z)\ (z \in U)$$

and

$$\left(\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}\right)^{\alpha}\left(\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}\right)^{1-\alpha} \prec \phi(\omega),\ (\omega \in U),$$

where $g(\omega) := f^{-1}(\omega)$.

Theorem 2. *Let $f \in \sigma$ given by (1.1) be in the class $C_\sigma(\alpha, \phi)$. Then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{4(2-\alpha)^2B_1 + 2\left|[\alpha^2+3(1-\alpha)]B_1^2 - 2(2-\alpha)^2B_2\right|}}, \tag{2.19}$$

and

$$|a_3| \le \begin{cases} \dfrac{B_1}{2[\alpha^2+3(1-\alpha)]} & if\ |B_2| \le B_1, \\ \dfrac{2(2-\alpha)^2B_1|B_2| + B_1|[\alpha^2+3(1-\alpha)]B_1^2 - 2(2-\alpha)^2B_2|}{2[\alpha^2+3(1-\alpha)][2(2-\alpha)^2B_1 + |[\alpha^2+3(1-\alpha)]B_1^2 - 2(2-\alpha)^2B_2|]} & if\ |B_2| > B_1. \end{cases} \tag{2.20}$$

Proof. Let $f \in C_\sigma(\alpha, \phi)$. Then there are analytic functions u, v: $U \to U$ given by (2.2) such that

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)^{1-\alpha} = \phi(u(z)) \tag{2.21}$$

and

$$\left(\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}\right)^{\alpha}\left(\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}\right)^{1-\alpha} = \phi(v(\omega)). \tag{2.22}$$

Since

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)^{1-\alpha} = 1 + 2(2-\alpha)a_2z + 2\left[(3-2\alpha)a_3 - \alpha(1-\alpha)a_2^2\right]z^2 + \cdots,$$

and

$$\left(\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}\right)^{\alpha}\left(\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}\right)^{1-\alpha} = 1 - 2(2-\alpha)a_2\omega + 2\left[(3-2\alpha)\left(2a_2^2-a_3\right) - \alpha(1-\alpha)a_2^2\right]\omega^2 + \cdots,$$

it follows from (2.4), (2.5), (2.21) and (2.22) that

$$2(2-\alpha)a_2 = B_1b_1, \tag{2.23}$$

$$2\left[(3-2\alpha)a_3 - \alpha(1-\alpha)a_2^2\right] = B_1b_2 + B_2b_1^2, \tag{2.24}$$

$$-2(2-\alpha)a_2 = B_1c_1, \tag{2.25}$$

$$2\left[(3-2\alpha)\left(2a_2^2-a_3\right) - \alpha(1-\alpha)a_2^2\right] = B_1c_2 + B_2c_1^2. \tag{2.26}$$

From (2.23) and (2.25), we get

$$b_1 = -c_1, \tag{2.27}$$

and

$$8(2-\alpha)^2a_2^2 = B_1^2\left(b_1^2+c_1^2\right). \tag{2.28}$$

By adding (2.24) to (2.26), further computations using (2.28) lead to

$$4\left[(\alpha^2+3(1-\alpha))B_1^2 - 2(2-\alpha)^2B_2\right]a_2^2 = B_1^3(b_2+c_2), \tag{2.29}$$

(2.27), (2.29), together with (2.3), give that

$$2\left|[\alpha^2+3(1-\alpha)]B_1^2 - 2(2-\alpha)^2B_2\right|\left|a_2^2\right| \le B_1^3\left(1-\left|b_1^2\right|\right). \tag{2.30}$$

From (2.23) and (2.30) we get

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{4(2-\alpha)^2B_1 + 2\left|[\alpha^2+3(1-\alpha)]B_1^2 - 2(2-\alpha)^2B_2\right|}}, \tag{2.31}$$

Next, from (2.24) and (2.26), we have

$$4(3-2\alpha)[\alpha^2+3(1-\alpha)]a_3 = [2(3-2\alpha) - \alpha(1-\alpha)]B_1b_2 + \alpha(1-\alpha)B_1c_2 + 2(3-2\alpha)B_2b_1^2.$$

Then, in view of (2.3), we have

$$2[\alpha^2+3(1-\alpha)]|a_3| \le B_1 + [|B_2| - B_1]\left|b_1^2\right|.$$

Notice that

$$\left|b_1^2\right| = \frac{4(2-\alpha)^2}{B_1^2}\left|a_2^2\right| \le \frac{2(2-\alpha)^2B_1}{2(2-\alpha)^2B_1 + \left|[\alpha^2+3(1-\alpha)]B_1^2 - 2(2-\alpha)^2B_2\right|},$$

we get

$$|a_3| \le \begin{cases} \dfrac{B_1}{2[\alpha^2+3(1-\alpha)]} & \text{if } |B_2| \le B_1, \\ \dfrac{2(2-\alpha)^2B_1|B_2| + B_1|[\alpha^2+3(1-\alpha)]B_1^2 - 2(2-\alpha)^2B_2|}{2[\alpha^2+3(1-\alpha)][2(2-\alpha)^2B_1 + |[\alpha^2+3(1-\alpha)]B_1^2 - 2(2-\alpha)^2B_2|]} & \text{if } |B_2| > B_1. \end{cases}$$

□

If we set

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2z^2 + \cdots\ (0 < \gamma \le 1, z \in U)$$

in Definition 4 of the bi-univalent function class $C_\sigma(\alpha, \phi)$, we obtain a new class $C_\sigma(\alpha, \gamma)$ given by Definition 5 below.

Definition 5. For $0 \le \alpha \le 1$ and $0 < \gamma \le 1$, a function $f \in \sigma$ given by (1.1) is said to be in the class $C_\sigma(\alpha, \gamma)$ if the following subordinations hold:

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)^{1-\alpha} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}(z \in U)$$

and

$$\left(\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}\right)^{\alpha}\left(\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}\right)^{1-\alpha} \prec \left(\frac{1+\omega}{1-\omega}\right)^{\gamma} (\omega \in U),$$

where $g(\omega) := f^{-1}(\omega)$.

Using the parameter setting of Definition 5 in Theorem 2, we get the following corollary.

Corollary 3. *For $0 \le \alpha \le 1$ and $0 < \gamma \le 1$, let the function $f \in C_\sigma(\alpha, \gamma)$ be of the form (1.1). Then*

$$|a_2| \le \frac{\gamma}{\sqrt{(2-\alpha)^2+\gamma(1-\alpha)}}$$

and

$$|a_3| \le \frac{\gamma}{\alpha^2+3(1-\alpha)}.$$

If we set

$$\phi(z) = \frac{1+(1-2\upsilon)z}{1-z} = 1+2(1-\upsilon)z + 2(1-\gamma)z^2+\cdots (0 \le \upsilon < 1, z \in U)$$

in Definition 4 of the bi-univalent function class $C_\sigma(\alpha, \phi)$, we obtain a new class $C_\sigma^\upsilon(\alpha)$ given by Definition 6 below.

Definition 6. For $0 \le \alpha \le 1$ and $0 \le \upsilon < 1$, a function $f \in \sigma$ given by (1.1) is said to be in the class $C_\sigma^\upsilon(\alpha)$, if the following conditions are satisfied:

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)^{1-\alpha} \prec \frac{1+(1-2\upsilon)z}{1-z} (z \in U)$$

and

$$\left(\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}\right)^{\alpha}\left(\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}\right)^{1-\alpha} \prec \frac{1+(1-2\upsilon)\omega}{1-z} (\omega \in U),$$

where $g(\omega) := f^{-1}(\omega)$.

Using the parameter setting of Definition 6 in Theorem 2, we get the following corollary.

Corollary 4. *For $0 \le \alpha \le 1, 0 \le \upsilon < 1$, let the function $f \in C_\sigma^\upsilon(\alpha)$ be of the form (1.1). Then*

$$|a_2| \le \frac{1-\upsilon}{\sqrt{(2-\alpha)^2+|[\alpha^2+3(1-\alpha)](1-\upsilon)-(2-\alpha)^2|}}$$

and

$$|a_3| \le \frac{1-\upsilon}{\alpha^2+3(1-\alpha)}.$$

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