Journal of the Egyptian Mathematical Society (2011) 19, 45–51

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Introduction to some conjectures for spectral minimal partitions ☆

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Available online 18 November 2011

KEYWORDS

Minimal partitions;
Nodal sets;
Courant theorem

Abstract Given a bounded open set Ω in $\mathbb{R}^n$ (or in a Riemannian manifold) and a partition of Ω by k open sets D_j, we consider the quantity $\max_j \lambda(D_j)$ where $\lambda(D_j)$ is the ground state energy of the Dirichlet realization of the Laplacian in D_j. If we denote by $\mathfrak{L}_k(\Omega)$ the infimum over all the k-partitions of $\max_j \lambda(D_j)$, a minimal k-partition is then a partition which realizes the infimum. When $k = 2$, we find the two nodal domains of a second eigenfunction, but the analysis of higher k's is non trivial and quite interesting. In this paper, which is complementary of the survey [20], we consider the two-dimensional case and present the properties of minimal spectral partitions, illustrate the difficulties by considering simple cases like the disk, the rectangle or the sphere ($k = 3$). We will present also the main conjectures in this rather new subject.

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1. Introduction

We consider mainly the Dirichlet Laplacian in a bounded domain $\Omega \subset \mathbb{R}^2$. We would like to analyze the relations between the nodal domains of the eigenfunctions of this Laplacian and the partitions of Ω by k open sets D_i which are minimal in the sense that the maximum over the D_i's of the ground state energy[1] of the Dirichlet realization of the Laplacian $H(D_i)$ in D_i is minimal. In the case of a Riemannian compact manifold, the natural extension is to consider the Laplace Beltrami operator. We denote by $\lambda_j(\Omega)$ the increasing sequence of its eigenvalues and by u_j some associated orthonormal basis of eigenfunctions. The groundstate u_1 can be chosen to be strictly positive in Ω, but the other eigenfunctions u_k must have zerosets. For any $u \in C_0^0(\overline{\Omega})$, we define the zero set as

$$N(u) = \overline{\{x \in \Omega \,|u(x) = 0\}}, \tag{1}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u. The number of nodal domains of u is called $\mu(u)$. These $\mu(u)$ nodal domains define a k-partition of Ω, with $k = \mu(u)$.

☆ This work has started in collaboration with T. Hoffmann-Ostenhof and has been continued with the coauthors V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, S. Terracini, and G. Vial.

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Peer review under responsibility of Egyptian Mathematical Society.
doi:10.1016/j.joems.2011.09.003

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[1] The ground state energy is the smallest eigenvalue.

Let us start by recalling two very classical theorems in spectral theory. The first one is called the Courant nodal theorem.

Theorem 1.1 (*Courant*). *Let $k \geqslant 1$, λ_k be the k-th eigenvalue and $E(\lambda_k)$ the eigenspace of $H(\Omega)$ associated to λ_k. Then, $\forall u \in E(\lambda_k)\backslash\{0\}, \mu(u) \leqslant k$.*

If in dimension 1 the Sturm-Liouville theory says that we have always equality in the previous theorem (this is what we will call later a Courant-sharp situation), the second theorem due to Pleijel in 1956 says that this cannot be true when the dimension (here we consider the 2D-case) is larger than one.

Theorem 1.2 (*Pleijel*). *There exists k_0 such that if $k \geqslant k_0$, then*

$$\mu(u) < k, \forall u \in E(\lambda_k) \setminus \{0\}.$$

The proof involves notions which will play an important role for the partitions.

Proposition 1.3. *For any eigenvalue λ of $H(\Omega)$ corresponding to an eigenfunction u with k nodal domains we have*

$$\lambda \geqslant k\frac{\pi j^2}{|\Omega|}. \tag{2}$$

where $|\Omega|$ denotes the area of Ω and j is the smallest positive zero of the Bessel function J_0.

The proof is actually a side result of the proof by Pleijel of his theorem [33]. The main point is the Faber-Krahn Inequality:

$$\lambda(\omega) \geqslant \frac{\pi j^2}{|\omega|}. \tag{3}$$

If u is an eigenfunction of H attached to the eigenvalue λ with k nodal sets then we have for any of these nodal domains D_i:

$$|D_i|\lambda \geqslant \pi j^2. \tag{4}$$

Summing over i, we get (2).

Let us now recall how the proof of the Pleijel theorem is achieved. The Weyl theory says that

$$\lambda_n \sim \frac{4\pi n}{|\Omega|}, \tag{5}$$

as $n \to +\infty$. If n is large, using (5) and (2), and having in mind the value of $j \sim 2.404$, we see that u_n cannot have n nodal domains.

2. Minimal partitions

We first introduce for $k \in \mathbb{N}$ ($k \geqslant 1$), the notion of k-partition. We will call k-partition of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of mutually disjoint sets in Ω. We call it open if the D_i are open sets of Ω, connected if the D_i are connected. We denote by $\mathfrak{O}_k(\Omega)$ the set of open connected partitions of Ω. We now introduce the notion of spectral minimal partition sequence.

Definition 2.1. For any integer $k \geqslant 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k(\Omega)$, we introduce

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \tag{6}$$

Then we define

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k} \Lambda(\mathcal{D}), \tag{7}$$

and call $\mathcal{D} \in \mathfrak{O}_k$ a minimal k-partition if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.

If $k = 2$, it is rather well known (see [22] or [17]) that $\mathfrak{L}_2 = \lambda_2$ and that the associated minimal 2-partition is a nodal partition, i.e. a partition whose elements are the nodal domains of some eigenfunction corresponding to λ_2.

A partition $\mathcal{D} = \{D_i\}_{i=1}^k$ of Ω in $\mathfrak{O}_k$ is called strong if

$$\mathrm{Int}(\overline{\cup_i D_i}) \setminus \partial\Omega = \Omega. \tag{8}$$

Attached to a strong partition, we associate a closed set in $\overline{\Omega}$, which is called the boundary set of the partition:

$$N(\mathcal{D}) = \overline{\cup_i(\partial D_i \cap \Omega)}. \tag{9}$$

$N(\mathcal{D})$ plays the role of the nodal set (in the case of a nodal partition).

This leads us to introduce the set $\mathcal{R}(\Omega)$ of regular partitions (or nodal like) through the properties of its associated boundary set N, which should satisfy:

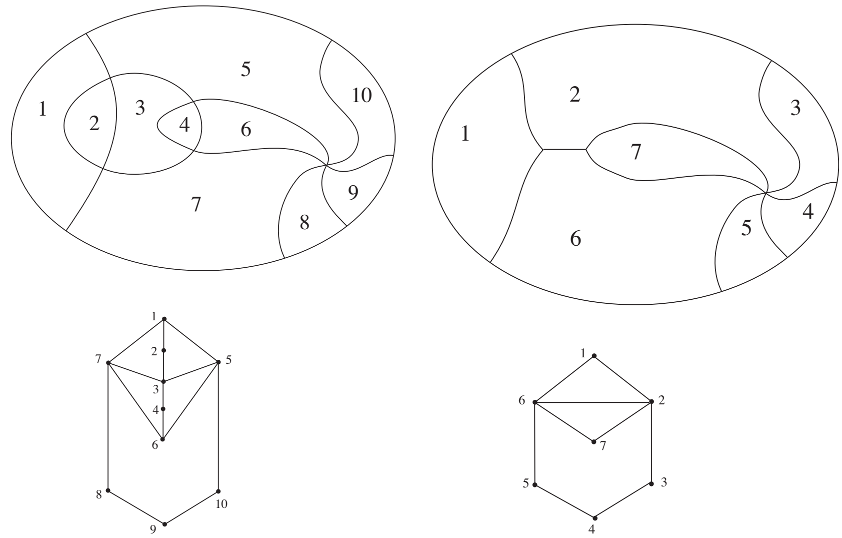
Definition 2.2.

(i) Except finitely many distinct $x_i \in \Omega \cap N$ in the neighborhood of which N is the union of $\nu_i = \nu(x_i)$ smooth curves ($\nu_i \geqslant 3$) with one end at x_i, N is locally diffeomorphic to a regular curve.
(ii) $\partial\Omega \cap N$ consists of a (possibly empty) finite set of points z_i. Moreover N is near z_i the union of ρ_i distinct smooth half-curves which hit z_i.
(iii) N has the equal angle meeting property.

The x_i are called the critical points and define the set $X(N)$. Similarly we denote by $Y(N)$ the set of the boundary points z_i. By equal angle meeting property, we mean that the half curves cross with equal angle at each critical point of N and also at the boundary together with the tangent to the boundary.

We say that D_i,D_j are neighbors or $D_i \sim D_j$, if $D_{ij} := \mathrm{Int}(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected. We associate with each $\mathcal{D}$ a graph $G(\mathcal{D})$ by associating to each D_i a vertex and to each pair $D_i \sim D_j$ an edge. We will say that the graph is bipartite if it can be colored by two colors (two neighbors having two different colors). We recall that the graph associated with a collection of nodal domains of an eigenfunction is always bipartite.

Next are two examples of partitions. The left figure corresponds to a regular strong bipartite partition with associated graph and the right figure corresponds to a regular strong bipartite partition with associated graph corresponds to a regular strong nonbipartite partition with associated graph.



3. Basic properties of minimal partitions

It has been proved by Conti–Terracini–Verzini [15–17] and Helffer–T. Hoffmann-Ostenhof–Terracini [25] that:

Theorem 3.1. *For any k, there exists a minimal regular k-partition. Moreover any minimal k-partition has a regular representative.*[2]

Other proofs of a somewhat weaker version of this statement have been given by Bucur–Buttazzo–Henrot [12], Caffarelli–Lin [14].

A natural question is whether a minimal partition of Ω is a nodal partition, i.e. the family of nodal domains of an eigenfunction of $H(\Omega)$. We have first the following converse theorem [22,25]:

Theorem 3.2. *If the graph of the minimal partition is bipartite this is a nodal partition.*

A natural question is now to determine how general is the previous situation. Surprisingly this only occurs in the so called Courant-sharp situation. We say that u is Courant-sharp if

$$u \in E(\lambda_k) \setminus \{0\} \quad \text{and} \quad \mu(u) = k.$$

For any integer $k \geqslant 1$, we denote by $L_k(\Omega)$ the smallest eigenvalue of $H(\Omega)$, whose eigenspace contains an eigenfunction with k nodal domains. We set $L_k(\Omega) = \infty$, if there are no eigenfunction with k nodal domains. In general, one can show, that

$$\lambda_k(\Omega) \leqslant \mathfrak{L}_k(\Omega) \leqslant L_k(\Omega). \tag{10}$$

The last result gives the full picture of the equality cases.

Theorem 3.3. *Suppose $\Omega \subset \mathbb{R}^2$ is regular. If $\mathfrak{L}_k(\Omega) = L_k(\Omega)$ or $\mathfrak{L}_k(\Omega) = \lambda_k(\Omega)$ then*

$$\lambda_k(\Omega) = \mathfrak{L}_k(\Omega) = L_k(\Omega).$$

In addition, one can find in $E(\lambda_k)$ a Courant-sharp eigenfunction.

[2] Modulo sets of capacity 0.

This answers a question in [13] (Section 7). Note that more recently this result has been extended to the 3D-case in [27].

Generalization: *p*-minimal *k*-partitions

More generally we can consider (see in [25]) for $p \in [1, +\infty]$

$$\Lambda^p(\mathcal{D}) = \left(\frac{1}{k}\sum_i \lambda(D_i)^p\right)^{\frac{1}{p}}, \tag{11}$$

and

$$\mathfrak{L}_{k,p}(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k} \Lambda^p(\mathcal{D}). \tag{12}$$

The case when $p = 1$ appears in probability [13] and harmonic analysis [5]. We write $\mathfrak{L}_{k,\infty}(\Omega) = \mathfrak{L}_k(\Omega)$ and recall the monotonicity property

$$\mathfrak{L}_{k,p}(\Omega) \leqslant \mathfrak{L}_{k,q}(\Omega) \text{ if } p \leqslant q. \tag{13}$$

The notion of p-minimal k-partition can be extended accordingly, by minimizing $\Lambda^p(\mathcal{D})$. Note that the inequalities can be strict: one can take a disjoint union of two disks (possibly related by a thin channel). A natural question is to determine if

$$\mathfrak{L}_{2,1}(\Omega) = \mathfrak{L}_{2,\infty}(\Omega)$$

This is indeed the case for the sphere [5]. We have proved recently [24] (in collaboration with T. Hoffmann-Ostenhof) that the inequality

$$\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega), \tag{14}$$

is ''generically'' satisfied. Moreover, we give in this article explicit examples (equilateral triangle) of convex domains for which (14) holds. This answers (by the negative) some question in [12].

Pleijel's theorem revisited

Inequality (2) was giving $L_k(\Omega) \geqslant k\frac{\pi j^2}{\lambda}$. We can actually get the better inequality:

$$\mathfrak{L}_{k,1}(\Omega) \geqslant k\frac{\pi j^2}{\lambda}. \tag{15}$$

We have indeed $\mathfrak{L}_{k,1}(\Omega) \geqslant \frac{1}{k}\inf_{\mathcal{B}\in\mathcal{B}_k}\sum_i\left(\pi j^2/|D_i|\right)$, for any partition $\mathcal{D}$ of Ω. But observing that $\sum|D_i| \leqslant |\Omega|$, the previous lower bound implies:

$$\mathfrak{L}_{k,1}(\Omega) \geqslant \frac{\pi j^2}{k|\Omega|} \inf_{\sum \lambda_i \leqslant 1} \sum_i \frac{1}{\lambda_i} = k \frac{\pi j^2}{|\Omega|}$$

The infimum is indeed obtained for $\lambda_i = \frac{1}{k}$, for all i.

4. Examples of k-minimal partitions for special domains

Using Theorem 3.3, it is now easier to analyze the situation for the disk or for rectangles (at least in the irrational case), since we have just to check for which eigenvalues one can find associated Courant-sharp eigenfunctions.

The case $k = 3$.

In the case of the square, it is not too difficult to see that $\mathfrak{L}_3$ is strictly less than λ_3. We observe indeed that there is no eigenfunction corresponding to $\lambda_2 = \lambda_3$ with three nodal domains and $\lambda_4 > \lambda_3$. Restricting to the half-rectangle and assuming that there is a minimal partition which is symmetric with one of the symmetry axes of the square perpendicular to two opposite sides, one is reduced to analyze a family of Dirichlet–Neumann problems. Numerical computations performed by V. Bonnaillie-Noël and G. Vial [9] lead to a natural candidate for a symmetric minimal partition. see (Fig. 1). Here we describe some results [23] on the possible ‘‘topological’’ types of 3-partitions. Let Ω be simply-connected and consider a minimal 3-partition $\mathcal{D} = (D_1, D_2, D_3)$ associated to $\mathfrak{L}_3$ and suppose that it is not bipartite.

Let $X(\mathcal{D}) = X(N(\mathcal{D}))$ be the set of singular points of $N(\mathcal{D}) \cap \Omega$ and let $Y(\mathcal{D}) = N(\mathcal{D}) \cap \partial\Omega$. Then there are three cases depending on the different configurations for the critical set.

- one singular point inside and three points at the boundary;
- two singular points inside and no point at the boundary,
- two singular points and two points at the boundary.

Figure 1 Trace on the half-square of the candidate for the 3-partition of the square. The complete structure is obtained from the half square by symmetry with respect to the horizontal axis.

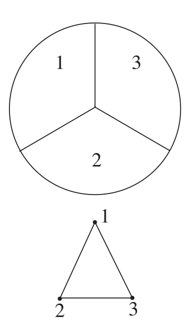


Figure 2 The Y-partition for the disk and corresponding graph.

The proof relies essentially on Euler formula (which is recalled below) together with the property that the associated graph should be a triangle.

Proposition 4.1. *Let U be an open set in $\mathbb{R}^2$ with piecewise $C^{1,+}$ boundary and let N a closed set such that $U \backslash N$ has k components and such that N satisfies the properties of Definition 2.2. Let b_0 be the number of components of ∂U and b_1 be the number of components of $N \cup \partial U$. Denote by $\nu(x_i)$ and $\rho(z_i)$ the numbers of arcs associated to the $x_i \in X(N)$, respectively, $z_i \in Y(N)$. Then*

$$k = b_1 - b_0 + \sum_{x_i \in X(N)} \left(\frac{\nu(x_i)}{2} - 1 \right) + \frac{1}{2} \sum_{z_i \in Y(N)} \rho(z_i) + 1. \qquad (16)$$

This leads (with some success) to analyze the minimal partition with some topological type. If in addition, we introduce some symmetries, this helps to guess some candidates for minimal partitions.

In the case of the disk, we have no proof that the minimal 3-partition is the ‘‘Mercedes star’’ or Y-partition see (Fig. 2). But if we assume that the minimal 3-partition is of the first type, then by going on the double covering of the punctured disk, one can show that it is indeed the Y-partition.

We emphasize that we have no proof that the candidates described for the disk or the square are minimal 3-partitions.

But if we assume that the minimal partition has one singular point and has the symmetry, then numerical computations lead to Fig. 1. Numerics suggest more: the center of the square is the critical point of the partition. Once this property is accepted, a double covering argument shows that this is the projection of a nodal partition on the covering. This point of view is explored numerically by Bonnaillie–Helffer [6] and theoretically by Noris–Terracini [31] and [32].

Note that there is an interesting alternative algorithmic approach [9] and [10].

One can also try to look for a minimal partition having the symmetry with respect to the diagonal. This leads to the same value of $\Lambda(\mathcal{D})$. So this strongly (Fig. 3) suggests that there is a continuous family of minimal 3-partitions of the square. This can be explained by a double covering argument [7], which is analogous to the argument of isospectrality of Jakobson–Levitin–Nadirashvili–Polterovich [29] and Levitin–Parnovski–Polterovich [30]. See also old papers by Bérard [2,3], Sunada [35] and the more recent paper by O. Parzanchevski and R. Band [34].

Minimal 5-partitions

Using the covering approach, we were able (with V. Bonnaillie) in [6] to produce the following candidate $\mathcal{D}_1$ for a minimal 5-partition of a specific topological type (Fig. 4).

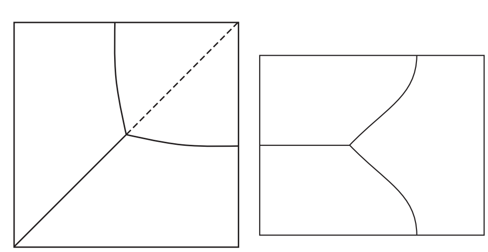


Figure 3 Two candidates for the square with different symmetries.

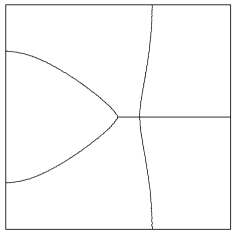


Figure 4 Candidate $\mathcal{D}_1$ for the 5-partition of the square.

It is interesting to compare with other possible topological types of minimal 5-partitions. They can be classified by using Euler formula (see (16)). Inspired by numerical computations in [18], one looks for a configuration which has the symmetries of the square and four critical points. We get two types of model that we can reduce to a Dirichlet–Neumann problem on a triangle corresponding to the eighth of the square. Moving the Neumann boundar y on one side like in [8] leads to two candidates $\mathcal{D}_2$ and $\mathcal{D}_3$. One has a lower energy $\Lambda(\mathcal{D})$ and one recovers the pictures in [18] (Fig. 5).

Note that in the case of the disk a similar analysis leads to a different answer. The partition of the disk by five halfrays with equal angle has a lower energy that the minimal 5-partition with four singular points (Fig. 6).

5. The problem for k large: the hexagonal conjecture

We learn of these conjectures from M. Van den Berg. They are also mentioned in Caffarelli–Lin [14]. The first one claims the existence of the limit.

Conjecture 5.1. *The limit of* $\mathfrak{L}_k(\Omega)/k$ *as* $k \to +\infty$ *exists.*

The second one says in particular that the limit is independent of Ω if Ω is a regular domain.

Conjecture 5.2

$$|\Omega| \lim_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \lambda_1(Hexa_1).$$

Of course the optimality of the regular hexagonal tiling appears in various contexts in Physics. It is easy to show the upper bound in the second conjecture and Faber-Krahn gives a weaker lower bound involving the first eigenvalue on the disk. Note that a stronger version of Conjecture 5.2 is that

$$|\Omega| \lim_{k\to+\infty} \frac{\mathfrak{L}_{k,1}(\Omega)}{k} = \lambda_1(Hexa_1). \tag{17}$$

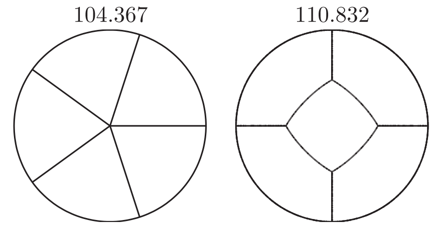


Figure 6 Two candidates for the 5-partition of the disk.

But we have at the moment no idea of any approach for proving this in our context. We have explored in [8] numerically why this conjecture looks reasonable, by controlling that many consequences of this conjecture are numerically correct. Other recent numerical computations devoted to $\lim_{k\to+\infty} \frac{1}{k}\mathfrak{L}_{k,1}(\Omega)$ and to the asymptotic structure of the minimal partitions by Bourdin–Bucur–Oudet [11] are very enlighting.

6. The problem on the sphere and the Bishop conjecture

Let us mention one interesting conjecture on $\mathbb{S}^2$ and a quite recent theorem. We parametrize $\mathbb{S}^2$ by the spherical coordinates $(\theta,\phi) \in [0,\pi]\times[-\pi,\pi]$ with $\theta = 0$ corresponding to the north pole, $\theta = \frac{\pi}{2}$ corresponding to the equator and $\theta = \pi$ corresponding to the south pole. There is a particular partition of $\mathbb{S}^2$ corresponding to cutting $\mathbb{S}^2$ by the half-hyperplanes $\phi = 0, \frac{2\pi}{3}, -\frac{2\pi}{3}$. We call this partition the Y-partition. The conjecture due to Bishop–Friedland–Hayman [19,5] is:

Conjecture 6.1. The Y-partition gives a minimal 3-partition for $\mathbb{S}^2$ when minimizing $\frac{1}{3}\sum_{j=1}^{3}\lambda(D_j)$ over all the 3-partitions of $\mathbb{S}^2$.

Actually one can have the same conjecture for $\max_j\lambda(D_j)$. This version of the conjecture is actually a consequence of the first conjecture (because all the groundstate energies are equal for the Y partition) but could be easier to prove. This is indeed the case and was proven by Helffer–T. Hoffmann-Ostenhof–Terracini in [26].

Theorem 6.2. *The Y-partition gives a minimal 3-partition for S^2 when minimizing $\max_j\lambda(D_j)$ over all the 3-partitions of S^2.*

The techniques developed in the previous parts give some insight on the second conjecture which has some similarity with the Mercedes star conjecture. A specific role is played by the proof that in the boundary of the 3-partition we have two antipodal points. This involves Lyusternik–Shnirelman's theorem.

We have seen that for the disk the minimal 4-partition for $\max_j\lambda(D_j)$ consists simply in the complement in the disk of

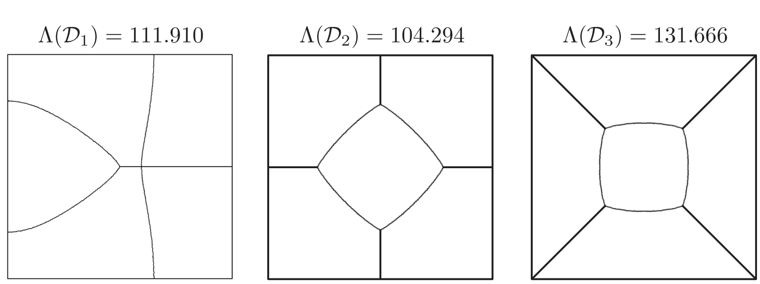


Figure 5 Three candidates for the 5-partition of the square.

the two perpendicular axes. One could think that a minimal 4-partition of $\mathbb{S}^2$ could be what is obtained by cutting $\mathbb{S}^2$, either by the two planes $\phi = 0$ and $\theta = \frac{\pi}{2}$ or by the two planes $\phi = 0$ and $\phi = \frac{\pi}{2}$. This is actually excluded: a minimal 4-partition on $\mathbb{S}^2$ cannot be a nodal partition. This is proven in [26] by observing that the multiplicity of the second eigenvalue is 3, and hence any eigenfunction in the spectral space attached to $\lambda_2 = \lambda_3 = \lambda_4$ has only two nodal domains (hence cannot be Courant sharp).

As already mentioned in [19], there is at least a natural candidate which is the spherical regular tetrahedron. Numerical computations,[3] give, for the corresponding 4-partition $\mathcal{D}_4^{Tetra}$

$$\Lambda(\mathcal{D}_4^{Tetra}) \sim 5.13. \tag{18}$$

Hence we obtain that

$$\frac{15}{4} < \mathfrak{L}_4(\mathbb{S}^2) \leqslant \Lambda(\mathcal{D}_4^{Tetra}) < 6 = L_4(\mathbb{S}^2). \tag{19}$$

Concerning the large k behavior, it is natural to conjecture that.

Conjecture 6.3. "Hexagonal" conjecture on $\mathbb{S}^2$

$$\lim_{k\to+\infty} \frac{\mathfrak{L}_k(\mathbb{S}^2)}{k} = \lim_{k\to+\infty} \frac{\mathfrak{L}_{k,1}(\mathbb{S}^2)}{k} = \frac{1}{\mathrm{Area}(\mathbb{S}^2)} \lambda(Hexa^1). \tag{20}$$

The first equality in the conjecture corresponds to the idea, which is well illustrated in the recent paper by Bourdin–Bucur–Oudet [11] that, asymptotically as $k \to +\infty$, a minimal k-partition for Λ^p will correspond to D_j's such that the $\lambda(D_j)$ are equal.

This hexagonal conjecture is probably true for any compact surface M (replace in (20) $\mathbb{S}^2$ by M). The guess is that this hexagonal conjecture is a "local result" where the curvature should not play a role.

7. An Aharonov-Bohm approach

7.1. The Aharonov-Bohm operator

Let us recall some definitions and results about the Aharonov-Bohm Hamiltonian (for short $\mathbf{AB}X$-Hamiltonian) with a singularity at X introduced in [7,21] and motivated by the work of Berger–Rubinstein [4]. We denote by $X = (x_0, y_0)$ the coordinates of the pole and consider the magnetic potential with renormalized flux at $X \frac{\Phi}{2\pi} = 1/2$:

$$\mathbf{A}^X(x,y) = (A_1^X(x,y), A_2^X(x,y)) = \frac{1}{2}\Big(-\frac{y-y_0}{r^2}, \frac{x-x_0}{r^2}\Big). \tag{21}$$

We know that the magnetic field vanishes identically in $\dot{\Omega}_X$. The $\mathbf{AB}X$-Hamiltonian is defined by considering the Friedrichs extension starting from $C_0^\infty(\dot{\Omega}_X)$ and the associated differential operator is

$$-\Delta_{\mathbf{A}^X} := (D_x - A_1^X)^2 + (D_y - A_2^X)^2 \text{ with } D_x = -i\partial_x \text{ and } D_y = -i\partial_y. \tag{22}$$

Let K_X be the antilinear operator

$$K_X = e^{i\theta_X}\Gamma,$$

with $(x-x_0) + i(y-y_0) = \sqrt{|x-x_0|^2 + |y-y_0|^2}\, e^{i\theta_X}$, and where Γ is the complex conjugation operator $\Gamma\, u = \bar{u}$. We say that a function u is K_X-real, if it satisfies $K_X u = u$. Then the operator $-\Delta_{\mathbf{A}^X}$ is preserving the K_X-real functions and we can consider a basis of K_X-real eigenfunctions. Hence we only analyze the restriction of the $\mathbf{AB}X$-Hamiltonian to the K_X-real space $L^2_{K_X}$ where

$$L^2_{K_X}(\dot{\Omega}_X) = \{u \in L^2(\dot{\Omega}_X), K_X u = u\}.$$

It was shown that the nodal set of such a K_X real eigenfunction has the same structure as the nodal set of an eigenfunction of the Laplacian except that an odd number of half-lines should met at X.

First we can extend our construction of an Aharonov-Bohm Hamiltonian in the case of a configuration with ℓdistinct points $X_1, \ldots, X_\ell$ (putting a (renormalized) flux $\frac{1}{2}$ at each of these points). We can just take as magnetic potential

$$\mathbf{A}^{\mathbf{X}} = \sum_{j=1}^{\ell} \mathbf{A}^{X_j},$$

where $\mathbf{X} = (X_1, \ldots, X_\ell)$. We can also construct (see [21]) the antilinear operator $K_{\mathbf{X}}$, where θ_X is replaced by a multi-valued-function $\phi_{\mathbf{X}}$ such that $d\phi_X = 2\mathbf{A}^{\mathbf{X}}$ and $e^{i\phi_{\mathbf{X}}}$ is univalued and C^∞. We can then consider the real subspace of the $K_{\mathbf{X}}$-real functions in $L^2_{K_{\mathbf{X}}}(\dot{\Omega}_{\mathbf{X}})$. It has been shown in [21] (see in addition [1]) that the $K_{\mathbf{X}}$-real eigenfunctions have a regular nodal set (like the eigenfunctions of the Dirichlet Laplacian) with the exception that at each singular point X_j $(j = 1, \ldots, \ell)$ an odd number of half-lines should meet. In the case of one singular point, this fact was observed by Berger–Rubinstein [4] for the first eigenfunction. We denote by $L_k(\dot{\Omega}_{\mathbf{X}})$ the lowest eigenvalue (if any) such that there exists a $K_{\mathbf{X}}$-real eigenfunction with k nodal domains [20,28,31,32].

7.2. Toward a magnetic characterization of a minimal partition

We now discuss the following conjecture presented in [6] (short version).

Conjecture 7.1. Let Ω be simply connected. Then

$$\mathfrak{L}_k(\Omega) = \inf_{\ell\in\mathbb{N}} \inf_{X_1,\ldots,X_\ell} L_k(\dot{\Omega}_{\mathbf{X}}).$$

Let us present a few examples illustrating the conjecture. When $k = 2$, there is no need to consider punctured Ω's. The infimum is obtained for $\ell = 0$. When $k = 3$, it is possible to show (see the second remark below) that it is enough, to minimize over $\ell = 0$, $\ell = 1$ and $\ell = 2$. In the case of the disk and the square, it is proven that the infimum cannot be for $\ell = 0$ and we conjecture that the infimum is for $\ell = 1$ and attained for the punctured domain at the center. For $k = 5$, it seems that the infimum is for $\ell = 4$ in the case of the square and for $\ell = 1$ in the case of the disk.

Let us explain very briefly why this conjecture is natural. Considering a minimal k-partition $\mathcal{D} = (D_1, \ldots, D_k)$, we know that it has a regular representative and we denote by $X^{odd}(\mathcal{D}) := (X_1, \ldots, X_\ell)$ the critical points of the partition corresponding to an odd number of meeting half-lines. Then we suspect that $\mathfrak{L}_k(\Omega) = \lambda_k(\dot{\Omega}_{\mathbf{X}})$ (Courant sharp situation). One point to observe is that we have proven in [25] the existence of a family u_i such that u_i is a groundstate of $H(D_i)$ and $u_i - u_j$

[3] Transmitted to us by M. Costabel.

is a second eigenfunction of $H(D_{ij})$ when $D_i \sim D_j$. The hope is to find a sequence $\epsilon_i(x)$ of $\mathbb{S}^1$-valued functions, where ϵ_i is a suitable[4] square root of $e^{i\phi_{\mathbf{X}}}$ in D_i, such that $\sum_i \epsilon_i(x)u_i(x)$ is an eigenfunction of the **ABX**-Hamiltonian associated with the eigenvalue $\mathfrak{L}_k$.

Conversely, any family of nodal domains of an Aharonov-Bohm operator on $\dot{\Omega}_{\mathbf{X}}$ corresponding to L_k gives a k-partition.

Remark 7.2

1. In the case when Ω is not simply connected, one should also add the possibility to create renormalized flux $\frac{1}{2}$ in some of the holes.
2. Euler's formula (16), implies that for a minimal k-partition $\mathcal{D}$ of a simply connected domain Ω the cardinal of $X^{odd}(\mathcal{D})$ satisfies

$$\#X^{odd}(\mathcal{D}) \leqslant 2k - 3. \quad (23)$$

Acknowledgements

We thank all our collaborators V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, S. Terracini, and G. Vial. We are also very grateful to the organizers of this Egypt France meeting in Cairo.

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[4] Note that by construction the D_i's never contain any pole.

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