Journal of the Egyptian Mathematical Society (2011) 19, 71–77

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Fubini theorem for multiparameter stable process

Mohamed Erraoui, Youssef Ouknine *,[1]

Faculté des Sciences Semlalia, Département de Mathématiques, Université Cadi Ayyad, BP 2390, Marrakech, Morocco

Available online 8 November 2011

KEYWORDS

Integration by parts;
Fubini's theorem;
Stable processes;
Stable sheet;
Lévy functionals

Abstract We prove stochastic Fubini theorem for general stable measure which will be used to develop some identities in law for functionals of one and two-parameter stable processes. This result is subsequently used to establish the integration by parts formula for stable sheet.

1. Introduction

Stochastic Fubini theorem for double Wiener integrals was first proved by Donati-Martin and Yor [4] and then developed further by Yor and other researchers. See [3] and the references therein. Subsequently, this theorem was applied to establish some identities in law for some quadratic functionals of Brownian motion. Among these identities in law there is one similar to the integration by parts formula which allowed some interesting extensions of the famous Ciesielski-Taylor identity. A simple explanation of the Ciesielski-Taylor identity is presented in the paper [10]. In view of this, it is natural to ask if it is possible to developed this for other processes. Because of their generality, Lévy processes, in particular stable processes, have been the object of intense research activity in recent years (see e.g. [1,2] and [9]). In this regard it would be of interest to have a stochastic Fubini theorem for such processes. The first adequate extension of Stochastic Fubini theorem to symmetric stable process and the related results was established by Donati-Martin et al. [5].

Generalization of some well-known results for stochastic processes indexed by a single parameter to those indexed by two parameters has attracted considerable interest recently. In general, processes parametrized by two parameters can provide more flexibility in their applications in modelling physical phenomena. Of particular interest, for which several generalizations have been established, are the Brownian sheet and bivariate Brownian bridge. For example, as a consequences of Stochastic Fubini theorem for general Gaussian measures, the authors in [3] have obtained some identities in law, integration by parts formula and the law of a double stochastic integral for such processes. In the same context the authors in [7] have established new identities in law for quadratic functionals of conditioned bivariate Gaussian processes. In particular, their results provide a two-parameter generalization of a celebrated identity in law, involving the path variance of a Brownian bridge, due to Watson [12]. We will see how this kind of identities can be naturally extended to stable processes.

In Section 2, as a first step we establish stochastic Fubini theorem for general Stable measure. This brings us, first, to an identity in law of functionals of one parameter time changed stable process. In fact we extent the well-known identity

* Corresponding author.
E-mail address: ouknine@ucam.ac.ma (Y. Ouknine).
[1] This work was supported by the Academy Hassan II of Sciences and Technology.

1110-256X © 2011 Egyptian Mathematical Society. Production and hosting by Elsevier B.V. Open access under CC BY-NC-ND license.

Peer review under responsibility of Egyptian Mathematical Society.
doi:10.1016/j.joems.2011.09.010

Production and hosting by Elsevier

in law involving quadratic functionals of the Brownian bridge (for more details see [11]), which corresponds to 2-stable process, to general α-stable process. As a second consequence we produce, using minor additional technicalities, the same results for the well-known symmetric α-stable sheet $\{X^\alpha(t_1,t_2), (t_1,t_2) \in [0,1]^2\}$, which may be described as follows:

Let $\mathcal{I}$ be the class of all sets in $[0,1]^2$ of the type $\times_{i=1}^{2}(s_i,t_i]$, $s_i,t_i \in [0,1]$. For a given a function $f:[0,1]^2 \to \mathbb{R}$, the increment $f(I)$ of f over the set $I \in \mathcal{I}$ is defined by

$$f\left(\times_{i=1}^{2}(s_i,t_i]\right) = f(t_1,t_2) - f(t_1,s_2) - f(s_1,t_2) + f(s_1,s_2).$$

For $\alpha \in (0,2]\backslash\{1\}$, X^α is a stochastic process taking values in $\mathbb{R}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that:

(i) For any $k \in \mathbb{N}$ and any choice of disjoint sets $I_j \in \mathcal{I}$, $j \in \{1,\ldots,k\}$ the increments $X(I_j)$ are independents.
(ii) For any $I \in \mathcal{I}$ and $u \in \mathbb{R}$

$$\mathbb{E}\exp(iuX(I)) = \exp(-\lambda(I)|u|^\alpha), \tag{1}$$

where $\lambda(I)$ is the Lebesgue measure of I.

It is well known that X^α belongs to the space $D([0,1]^2,\mathbb{R})$ of functions Z from $[0,1]^2$ into $\mathbb{R}$ vanishing at the boundary and satisfying

$$\lim_{(t_1,t_2)\leqslant(s_1,s_2),(s_1,s_2)\to(t_1,t_2)} Z(s_1,s_2) = Z(t_1,t_2),$$

where $\leqslant$ denotes the natural partial ordering in $[0,1]^2$.

It should be noted that our results for the stable sheet are actually a continuation of those established by Peccati and Yor [7] for the Brownian sheet.

Section 3 is devoted to the integration by parts formula established first in [11] for the Brownian motion. Since then several extensions have been made to various processes. Namely the first one, for the one parameter stable process, was given in [5] whereas the second one, for Brownian sheet, was made in [3]. We are going here to show this formula for stable sheet. Our proof is based on the main result of Section 2 and time reversal stochastic integral with respect to stable process.

Let us fix some notations to be used throughout the paper: $X \overset{d}{=} Y$ means that the random variables X and Y have the same distribution. T_γ is a one-sided stable random variable with exponent γ if $\mathbb{E}(\exp(-uT_\gamma)) = \exp(-u^\gamma)$, for $u \geqslant 0$.

2. Some identities in law between some Lévy functionals

The starting point of this study is Fubini theorem for Stable measures. Let $(A,\mathcal{A},\mu)$ and $(B,\mathcal{B},\nu)$ be two measurable spaces, with μ and ν denoting positive and σ-finite measures.

Let $\left\{X_\mu^\alpha(h) : h \in L^\alpha(A,\mathcal{A},\mu)\right\}$ and $\left\{X_\nu^\beta(k) : k \in L^\beta(B,\mathcal{B},\nu)\right\}$ be two independent stable symmetric processes, with $\alpha,\beta \in (0,2]\backslash\{1\}$, indexed respectively by functions in $L^\alpha(A,\mathcal{A},\mu)$ and $L^\beta(B,\mathcal{B},\nu)$, that is, for any $u \in \mathbb{R}$, $h \in L^\alpha(A,\mathcal{A},\mu)$ and $k \in L^\beta(B,\mathcal{B},\nu)$, we have

$$\mathbb{E}\left[\exp\left\{iuX_\mu^\alpha(h)\right\}\right] = \exp\left\{-\int_A |uh(a)|^\alpha \mu(da)\right\},$$

and

$$\mathbb{E}\left[\exp\left\{iuX_\nu^\beta(k)\right\}\right] = \exp\left\{-\int_B |uk(b)|^\beta \nu(db)\right\}.$$

Here we give some examples:

Let $\left\{X_t^\alpha, t \in [0,1]\right\}$ be a symmetric stable process with index α, that is a Lévy process such that for any $t \in [0,1]$ and $u \in \mathbb{R}$ its characteristic function is defined by

$$\mathbb{E}\left[\exp\left\{iuX_t^\alpha\right\}\right] = \exp(-t|u|^\alpha).$$

1. For $(A,\mathcal{A},\mu) = ([0,1],\mathcal{B}([0,1]),dt)$, then $X_\mu^\alpha(h)$ has a stochastic integral representation
$$X_\mu^\alpha(h) \overset{d}{=} \int_0^1 h(s)dX_s^\alpha.$$
2. For $(A,\mathcal{A},\mu) = ([0,1],\mathcal{B}([0,1]),\eta(dt))$, where η denotes a positive and σ-finite measure such that $\eta(\{0\}) = 0$, we have
$$X_\mu^\alpha(h) \overset{d}{=} \int_0^1 h(s)dX_{\eta[0,s]}^\alpha.$$
3. For $(A,\mathcal{A},\mu) = ([0,1]^2,\mathcal{B}([0,1]^2),dt\,ds)$, $X_\mu^\alpha(h)$ has the law representation as
$$X_\mu^\alpha(h) \overset{d}{=} \int_{[0,1]^2} h(t_1,t_2)dX^\alpha(t_1,t_2).$$

We now state a fundamental identity, which holds almost surely, on which our main result Theorem (1) is based on.

$$\int_A\left(\int_B \phi(a,b)X_\nu^\beta(db)\right)X_\mu^\alpha(da) = \int_B\left(\int_A \phi(a,b)X_\mu^\alpha(da)\right)X_\nu^\beta(db), \tag{2}$$

for any $\phi : A\times B \to \mathbb{R}$, $\mathcal{A}\otimes\mathcal{B}$-measurable function such that

$$\int_A \left|\int_B |\phi(a,b)|^\beta \nu(db)\right|^{\alpha/\beta} \mu(da) < +\infty,$$

and

$$\int_B \left|\int_A |\phi(a,b)|^\alpha \mu(da)\right|^{\beta/\alpha} \nu(db) < +\infty.$$

The main result in this section, which is fundamental for the rest of the development, is as follows:

Theorem 1. *Consider for $\alpha,\beta \in (0,2]\backslash\{1\}$ the random variables*

$$Y_{\beta,\alpha} = \int_A \left|\int_B \phi(a,b)X_\nu^\beta(db)\right|^\alpha \mu(da)$$

and

$$Y_{\alpha,\beta} = \int_B \left|\int_A \phi(a,b)X_\mu^\alpha(da)\right|^\beta \nu(db).$$

Then the following identity holds

$$\left(Y_{\beta,\alpha}\right)^{1/\gamma} T_\gamma \overset{d}{=} Y_{\alpha,\beta}, \tag{3}$$

where $\gamma = \alpha/\beta$ and T_γ is a one-sided stable random variable with exponent γ, which is assumed to be independent of $Y_{\beta,\alpha}$.

For $\alpha = \beta$ the identity in law (3) becomes

$$\int_A \left| \int_B \phi(a,b) X_\nu^\alpha(db) \right|^\alpha \mu(da) \stackrel{d}{=} \int_B \left| \int_A \phi(a,b) X_\mu^\alpha(da) \right|^\alpha \nu(db). \tag{4}$$

Proof 1. Taking the characteristic functions of both sides of (2), for any $u \in \mathbb{R}$, we obtain:

$$\mathbb{E}\left[\exp\left(-|u|^\alpha \int_A \left| \int_B \phi(a,b) X_\nu^\beta(db) \right|^\alpha \mu(da)\right)\right]$$
$$= \mathbb{E}\left[\exp\left(-|u|^\beta \int_B \left| \int_A \phi(a,b) X_\mu^\alpha(da) \right|^\beta \nu(db)\right)\right]. \tag{5}$$

Taking $r = |u|^\alpha$ as a new variable, the equality (5) becomes

$$\mathbb{E}\left[\exp\left(-r \int_A \left| \int_B \phi(a,b) X_\nu^\beta(db) \right|^\alpha \mu(da)\right)\right]$$
$$= \mathbb{E}\left[\exp\left(-r^{1/\gamma} \int_B \left| \int_A \phi(a,b) X_\mu^\alpha(da) \right|^\beta \nu(db)\right)\right].$$

On the other hand, it is easy to see that

$$\mathbb{E}\left[\exp\left(-r \int_A \left| \int_B \phi(a,b) X_\nu^\beta(db) \right|^\alpha \mu(da)\right)\right]$$
$$= \mathbb{E}\left[\exp\left(-\left\{ r^{1/\gamma} \left(\int_A \left| \int_B \phi(a,b) X_\nu^\beta(db) \right|^\alpha \mu(da) \right)^{1/\gamma} \right\}^\gamma\right)\right]$$
$$= \mathbb{E}\left[\exp\left(-r^{1/\gamma} \left(\int_A \left| \int_B \phi(a,b) X_\nu^\beta(db) \right|^\alpha \mu(da) \right)^{1/\gamma} T_\gamma\right)\right],$$

where T_γ is a one-sided stable random variable with exponent γ independent of $Y_{\beta,\alpha}$.

Hence, we have obtained

$$\mathbb{E}\left[\exp\left(-r^{1/\gamma} \int_A \left| \int_B \phi(a,b) X_\nu^\beta(db) \right|^\alpha \mu(da)\right)\right]$$
$$= \mathbb{E}\left[\exp\left(-r^{1/\gamma} \left(\int_B \left| \int_A \phi(a,b) X_\mu^\alpha(da) \right|^\beta \nu(db) \right)^{1/\gamma} T_\gamma\right)\right],$$

which is equivalent to (3). □

Remark 1. It should be noted that the case $\alpha = \beta = 2$, which corresponds to the Gaussian measures, has been considered in [3].

2.1. One parameter case

Let ϱ be a probability on $[0,1]$ and set $A_t = \varrho([0,t])$. It is well known that the variation of A corresponds to the total variation of ϱ. Define the right continuous inverse of A, namely for any $t \in [0,1]$,

$C_t = \inf\{s : A_s > t\}$.

It is easily seen that $A_{C_t} \geqslant t$ and $C_{A_t} \geqslant t$ for every t and $A_t = \inf\{s : C_s > t\}$.

Moreover we have a change variable formula stated as follows:

$$\int_0^t h(s)\varrho(ds) = \int_0^t h(s) dA_s = \int_0^{A_t} h(C_s) ds, \forall t \in [0,1]. \tag{6}$$

It should be noted that A and C play symmetric roles. The reader is referred to the book [8] for more details and some related results.

Proposition 1. *Let X^α be a symmetric stable process with index α. Then we have*

$$\int_0^1 \varrho(du) \left| \left\{ X_u^\alpha - \int_0^1 X_s^\alpha \varrho(ds) \right\} \right|^\alpha \stackrel{d}{=} \int_0^1 dC_u \left| X_u^\alpha - u X_1^\alpha \right|^\alpha, \tag{7}$$

and using the equality (6) we obtain

$$\int_0^1 \varrho(du) \left| \left\{ X_u^\alpha - \int_0^1 X_s^\alpha \varrho(ds) \right\} \right|^\alpha \stackrel{d}{=} \int_0^1 du \left| X_{A_u}^\alpha - A_u X_1^\alpha \right|^\alpha. \tag{8}$$

Proof 2. Let us consider the function $\phi : [0,1]^2 \to \mathbb{R}$ defined by

$$\phi(u,s) = \left[\mathbf{1}_{(s \leqslant u)} - (1-s)\right],$$

and set

$$(A, \mathcal{A}, \mu) = ([0,1], \mathcal{B}([0,1]), dC_t) \text{ and } (B, \mathcal{B}, \nu) = ([0,1], \mathcal{B}([0,1]), dt).$$

It is easily seen that

$$\int_0^1 dX_u^\alpha \phi(u,s) = \left(X_1^\alpha - X_s^\alpha\right) - (1-s)X_1^\alpha = -X_s^\alpha + sX_1^\alpha,$$

and

$$\int_0^1 dX_{C_s}^\alpha \phi(u,s) = X_{C_u}^\alpha - X_{C_0}^\alpha - \int_0^1 (1-s) dX_{C_s}^\alpha.$$

Now using integration by parts formula we obtain

$$\int_0^1 dX_{C_s}^\alpha \phi(u,s) = X_{C_u}^\alpha - \int_0^1 X_{C_{s^-}}^\alpha ds.$$

The set $\{s : C_s \neq C_{s^-}\}$ is countable and so far it is Lebesgue negligible. Owing to this fact we may replace $X_{C_{s^-}}^\alpha$ by $X_{C_s}^\alpha$ in the right-hand side. Thus we have

$$\int_0^1 dX_{C_s}^\alpha \phi(u,s) = X_{C_u}^\alpha - \int_0^1 X_{C_s}^\alpha ds.$$

Let $\widetilde{X}^\alpha$ be an independent copy of X^α. Applying the identity (4), with $X_\nu^\alpha = X_t^\alpha$ and $X_\mu^\alpha = \widetilde{X}_{C_t}^\alpha$, one has

$$\int_0^1 \left| X_{C_u}^\alpha - \int_0^1 X_{C_s}^\alpha ds \right|^\alpha du \stackrel{d}{=} \int_0^1 \left| X_s^\alpha - sX_1^\alpha \right|^\alpha dC_s.$$

Now the identity (7) follows simply from (6). This ends the proof. □

Remark 2. (i) Let $(B_s, s \leqslant 1)$ is a Brownian motion. It is well known, see [11], that the following identity holds

$$\int_0^1 \varrho(dt) \left(B_u - \int_0^1 \varrho(ds) B_s \right)^2 \stackrel{d}{=} \int_0^1 (\widetilde{B}_{\varrho[0,t]})^2 dt, \tag{9}$$

where $\left(\widetilde{B}_s, s \leqslant 1\right)$ is a standard Brownian bridge. It follows from (6) that

$$\int_0^1 \varrho(dt) \left(B_u - \int_0^1 \varrho(ds) B_s \right)^2 \stackrel{d}{=} \int_0^1 (\widetilde{B}_s)^2 dC_s,$$

which corresponds to the case $\alpha = 2$ in the identity (7) once we know that standard Brownian bridge has $(B_u - uB_1;\ u \leqslant 1)$ as a representation in law.

(ii) For $p > 0$, let $\varrho(du) = du\ p\ u^{p-1}$, we have

$A_t = t^p$ *and* $C_t = t^{1/p}$.

It follows from the identity (7) that

$$\int_0^1 du\, p\, u^{p-1}\left|\left\{B_u - \int_0^1 ps^{p-1}B_s ds\right\}\right|^2 \stackrel{d}{=} \int_0^1 du\left|B_{u^p} - u^p B_1\right|^2$$
$$\stackrel{d}{=} \int_0^1 du\left|\widetilde{B}_{u^p}\right|^2 = \frac{1}{p}\int_0^1 du\, u^{\frac{1}{p}-1}\left|\widetilde{B}_u\right|^2.$$

Note that for $p = 1$ we have

$$\int_0^1 du\left|\left\{B_u - \int_0^1 B_s ds\right\}\right|^2 \stackrel{d}{=} \int_0^1 du\left|\widetilde{B}_u\right|^2,$$

which was obtained in [4].

2.2. Two parameters case

Let $\mu = \nu$ be the Lebesgue measure on $[0,1]^2$, so that

$(A, \mathcal{A}, \mu) = (B, \mathcal{B}, \nu) = ([0,1]^2, \mathcal{B}([0,1]^2), dt_1 dt_2)$.

Then X^α and X^β are two independent symmetric stable sheet. We consider now the random variables

$$Z^\alpha(s_1, s_2) = \int\int_{[0,1]^2} \phi(s_1, s_2, t_1, t_2) X^\alpha(dt_1, dt_2),$$

and

$$Z^\beta(t_1, t_2) = \int\int_{[0,1]^2} \phi(s_1, s_2, t_1, t_2) X^\beta(ds_1, ds_2).$$

In this special case, the conclusion of Theorem 1 becomes

$$\left(\int\int_{[0,1]^2} |Z^\alpha(s_1,s_2)|^\beta ds_1 ds_2\right)^{1/\gamma} T_\gamma \stackrel{d}{=} \int\int_{[0,1]^2}\left|Z^\beta(t_1,t_2)\right|^\alpha dt_1 dt_2. \tag{10}$$

The consideration developed above can be applied to the following:

$\phi_1(s_1, s_2, t_1, t_2) = 1_{[0,s_1]}(t_1)1_{[0,s_2]}(t_2) - (1-t_1)(1-t_2)$,

$\phi_2(s_1, s_2, t_1, t_2) = 1_{[0,s_1]}(t_1)1_{[0,s_2]}(t_2) - (1-t_1)1_{[0,s_2]}(t_2) - (1-t_2)1_{[0,s_1]}(t_1) + (1-t_1)(1-t_2)$.

In this setting, we have

Proposition 2. *For* $\alpha \in (0,2]\backslash\{1\}$ *the following identities hold*

$$\int_0^1\int_0^1\left|X^\alpha(s_1,s_2) - \int_0^1\int_0^1 X^\alpha(t_1,t_2)dt_1dt_2\right|^\alpha ds_1ds_2 \stackrel{d}{=} \int_0^1\int_0^1 |X^\alpha(t_1,t_2) - t_1t_2X^\alpha(1,1)|^\alpha dt_1dt_2, \tag{11}$$

and

$$\int_0^1\int_0^1\left|X^\alpha(s_1,s_2) - \int_0^1 X^\alpha(s_1,t_2)dt_2 - \int_0^1 X^\alpha(t_1,s_2)dt_1 + \int_0^1\int_0^1 X^\alpha(t_1,t_2)dt_1dt_2\right|^\alpha ds_1ds_2$$
$$\stackrel{d}{=} \int_0^1\int_0^1 |X^\alpha(t_1,t_2) - t_1X^\alpha(1,t_2) - t_2X^\alpha(t_1,1) + t_1t_2X^\alpha(1,1)|^\alpha dt_1dt_2 \tag{12}$$

Proof 3. Applying (10) to ϕ_1 and ϕ_2 with $\alpha = \beta$, we get

$$\int_0^1\int_0^1\left|X^\alpha(s_1,s_2) - \int_0^1\int_0^1(1-t_1)(1-t_2)X^\alpha(dt_1,dt_2)\right|^\alpha ds_1ds_2$$
$$\stackrel{d}{=} \int_0^1\int_0^1\left|X^\alpha\left(\mathop{\times}_{i=1}^{2}(t_i,1]\right) - (1-t_1)(1-t_2)X^\alpha(1,1)\right|^\alpha dt_1dt_2$$

and

$$\int_0^1\int_0^1 |X^\alpha(s_1,s_2) - \int_0^1\int_0^1(1-t_1)1_{[0,s_2]}(t_2)X^\alpha(dt_1,dt_2) - \int_0^1\int_0^1(1-t_2)1_{[0,s_1]}(t_1)X^\alpha(dt_1,dt_2) + \int_0^1\int_0^1(1-t_1)(1-t_2)X^\alpha(dt_1,dt_2)\Big|^\alpha ds_1ds_2$$
$$\stackrel{d}{=} \int_0^1\int_0^1\left|X^\alpha\left(\mathop{\times}_{i=1}^{2}(t_i,1]\right) - (1-t_1)X^\alpha(1,(t_2,1])\right.$$
$$\left. -(1-t_2)X^\alpha((t_1,1],1) + (1-t_1)(1-t_2)X^\alpha(1,1)\right|^\alpha dt_1dt_2.$$

It is readily checked that the following identities hold:

$$\int_0^1\int_0^1(1-t_1)(1-t_2)X^\alpha(dt_1,dt_2) = \int_0^1\int_0^1 X^\alpha(t_1,t_2)dt_1dt_2$$
$$\int_0^1\int_0^1(1-t_1)1_{[0,s_2]}(t_2)X^\alpha(dt_1,dt_2) = \int_0^1 X^\alpha(t_1,s_2)dt_1$$
$$\int_0^1\int_0^1(1-t_2)1_{[0,s_1]}(t_1)X^\alpha(dt_1,dt_2) = \int_0^1 X^\alpha(s_1,t_2)dt_2.$$

Hence, the above identities in law become:

$$\int_0^1\int_0^1\left|X^\alpha(s_1,s_2) - \int_0^1\int_0^1 X^\alpha(t_1,t_2)dt_1dt_2\right|^\alpha ds_1ds_2$$
$$\stackrel{d}{=} \int_0^1\int_0^1\left|X^\alpha\left(\mathop{\times}_{i=1}^{2}(t_i,1]\right) - (1-t_1)(1-t_2)X^\alpha(1,1)\right|^\alpha dt_1dt_2$$

and

$$\int_0^1\int_0^1\left|X^\alpha(s_1,s_2) - \int_0^1 X^\alpha(t_1,s_2)dt_1 - \int_0^1 X^\alpha(s_1,t_2)dt_2 + \int_0^1\int_0^1 X^\alpha(t_1,t_2)dt_1dt_2\right|^\alpha ds_1ds_2$$
$$\stackrel{d}{=} \int_0^1\int_0^1\left|X^\alpha\left(\mathop{\times}_{i=1}^{2}(t_i,1]\right) - (1-t_1)X^\alpha(1,(t_2,1])\right.$$
$$\left. -(1-t_2)X^\alpha((t_1,1],1) + (1-t_1)(1-t_2)X^\alpha(1,1)\right|^\alpha dt_1dt_2.$$

Now from (1) the following distributional identity between processes

$$\left\{X^\alpha\left(\mathop{\times}_{i=1}^{2}(t_i,1]\right), (t_1,t_2) \in [0,1]^2\right\}$$
$$\stackrel{d}{=} \left\{X^\alpha(1-t_1, 1-t_2), (t_1,t_2) \in [0,1]^2\right\},$$

hold which leads us, with the change variable $(r_1,r_2) = (1-t_1, 1-t_2)$, to the identities (11) and (12). □

Remark 3. It should be noted that the identities (11) and (12) are the extension of the identity (7), with $\varrho(du) = du$, to the two parameters case.

We close this section by a simple extension of the identities in the above proposition. Precisely, using the same techniques as before we obtain another variant of the identities in law (11) and (12) with $\varrho_1(dt_1)\varrho_2(dt_2)$ instead of dt_1dt_2 where ϱ_1 and ϱ_2 are two probability measures on $[0,1]$. Let $A_t^i = \varrho_i([0,t])$ and $C_t^i = \inf\{s : A_s > t\}$ for $i = 1, 2$ and $t \in [0,1]$. We set $(A, \mathcal{A}, \mu) = ([0,1]^2, \mathcal{B}([0,1]^2), dt_1dt_2)$ and $(B, \mathcal{B}, \nu) = ([0,1]^2, \mathcal{B}([0,1]^2), dC^1_{t_1}dC^2_{t_2})$. With these notations, we may state :

Proposition 3. *For the symmetric stable sheet X^α and the probability measures ϱ_1 and ϱ_2 we have*

$$\int_0^1\int_0^1\left|X^\alpha(s_1,s_2)-\int_0^1\int_0^1 X^\alpha(t_1,t_2)\varrho_1(dt_1)\varrho_2(dt_2)\right|^\alpha\varrho_1(ds_1)\varrho_2(ds_2)$$
$$\stackrel{d}{=}\int_0^1\int_0^1|X^\alpha(t_1,t_2)-t_1t_2X^\alpha(1,1)|^\alpha\varrho_1(dt_1)\varrho_2(dt_2), \qquad (13)$$

and

$$\int_0^1\int_0^1\left|X^\alpha(s_1,s_2)-\int_0^1 X^\alpha(s_1,t_2)\varrho_2(dt_2)\right.$$
$$\left.-\int_0^1 X^\alpha(t_1,s_2)\varrho_1(dt_1)+\int_0^1\int_0^1 X^\alpha(t_1,t_2)\varrho_1(dt_1)\varrho_2(dt_2)\right|^\alpha\varrho_1(ds_1)\varrho_2(ds_2).$$
$$\stackrel{d}{=}\int_0^1\int_0^1|X^\alpha(t_1,t_2)-t_1X^\alpha(1,t_2)-t_2X^\alpha(t_1,1)+t_1t_2X^\alpha(1,1)|^\alpha$$
$$\varrho_1(dt_1)\varrho_2(dt_2). \qquad (14)$$

The above proposition allows us to establish some extensions of certain identities in law developed by Peccati and Yor in [7] for the Brownian sheet.

Let $p,q > 0$ and $\{W = W(t_1,t_2):(t_1,t_2) \in [0,1]^2\}$ is a standard Brownian sheet on $[0,1]$ vanishing on the axes. We associate to W the following processes:

- $\{B^{(W)} = B^{(W)}(t_1,t_2):(t_1,t_2) \in [0,1]^2\}$ is the canonical bivariate Brownian bridge associated to W, i.e.

  $B^{(W)}(t_1,t_2) = W(t_1,t_2) - t_1t_2W(1,1)$;
- $\left\{B_0^{(W)} = B_0^{(W)}(t_1,t_2) : (t_1,t_2) \in [0,1]^2\right\}$ is the canonical bivariate tied down Brownian bridge associated to W, i.e.

  $B_0^{(W)}(t_1,t_2) = W(t_1,t_2) - t_1W(1,t_2) - t_2W(t_1,1) + t_1t_2W(1,1).$

Then for $\varrho_1(du) = du\, p\, u^{p-1}$ and $\varrho_2(du) = du\, q\, u^{q-1}$ the identity identities (13)–(14) become:

$$\int_0^1\int_0^1\left|W(s_1,s_2)-pq\int_0^1\int_0^1 W(t_1,t_2)t_1^{p-1}t_2^{q-1}dt_1dt_2\right|^2 s_1^{p-1}s_2^{q-1}ds_1ds_2$$
$$\stackrel{d}{=};\int_0^1\int_0^1\left|B^{(W)}(t_1,t_2)\right|^2t_1^{p-1}t_2^{q-1}dt_1dt_2,$$

and

$$\int_0^1\int_0^1\left|W(s_1,s_2)-q\int_0^1 W(s_1,t_2)t_2^{q-1}dt_2-p\int_0^1 W(t_1,s_2)t_1^{p-1}dt_1\right.$$
$$\left.+pq\int_0^1\int_0^1 W(t_1,t_2)t_1^{p-1}t_2^{q-1}dt_1dt_2\right|^\alpha s_1^{p-1}s_2^{q-1}ds_1ds_2$$
$$\stackrel{d}{=}\int_0^1\int_0^1\left|B_0^{(W)}(t_1,t_2)\right|^2t_1^{p-1}t_2^{q-1}dt_1dt_2.$$

3. Application: integration by parts formula

It is well known that Theorem 1 has several applications. Namely, an identity which resembles to integration by parts formula. Here, we give two examples yielding such identity. The first one deals with one parameter stable process and it was previously given in [5] but our method is quite different. Whereas the second one is new and consider the two parameters case. In the sequel we write δ_x for the unit mass at point x.

3.1. One parameter case

Let $f, g : [0,1] \to \mathbb{R}^+$ be two continuous functions with f decreasing and g increasing. Let us now choose $A = B = [0,1]$ and define the measure μ and ν by

$$\mu(ds) = -df(s) + f(1)\delta_1(ds) \quad \text{and} \quad \nu(dt) = g(0)\delta_0(dt) + dg(t).$$

Here we are mainly concerned with the definition of time reversal stochastic integral with respect to Lévy process. First we make the following notation: Let Z be a process with càdlàg paths defined on $[0,1]$. $\overline{Z} = \{\overline{Z}_t;\ t \in [0,1]\}$ will always denote the associated time reversed process of the process Z given by:

$$\overline{Z}_t = \begin{cases} 0 & \text{if} \quad t = 0,\\ Z_{(1-t)^-} - Z_{1^-} & \text{if} \quad 0 < t < 1,\\ Z_0 - Z_{1^-} & \text{if} \quad t = 1,\end{cases}$$

where Z_{u^-} denotes the left limit at u, $0 < u \leqslant 1$.

Note that the function $t \mapsto f(1-t)$ is increasing. It is well known that a semimartingale remains a semimartingale under time changes. Thus the process $\{Y_t := X^\alpha(f(1-t));\ t \in [0,1]\}$ is a semimartingale since Lévy processes are semimartingales. Thus the process

$$\overline{Y}_t = \begin{cases} 0 & \text{if} \quad t = 0,\\ X^\alpha(f(t)) - X^\alpha(f(0)) & \text{if} \quad 0 < t < 1,\\ X^\alpha(f(1)) - X^\alpha(f(0)) & \text{if} \quad t = 1,\end{cases}$$

is again a semimartingale.

The stochastic integral of $\varphi \in L^\alpha(A, \mathcal{A}, -df)$ with respect to the process $\{X^\alpha(f(t)),\ t \in [0,1]\}$ is defined by time reversal as follows:

$$\int_0^t \varphi(s)dX^\alpha(f(s)) := \overline{\int_0^t \varphi(1-s)dX^\alpha(f(1-s))}, \qquad t \in [0,1]. \qquad (15)$$

Consequently, for $u \in \mathbb{R}$ and $t \in [0,1]$, we have

$$\mathbb{E}\left[\exp\left\{iu\int_0^t \varphi(s)dX^\alpha(f(s))\right\}\right]$$
$$= \mathbb{E}\left[\exp\left\{iu\int_{1-t}^1 \varphi(1-s)dX^\alpha(f(1-s))\right\}\right]$$
$$= \exp\left\{-\int_{1-t}^1 |u\varphi(1-s)|^\alpha df(1-s)\right\}$$
$$= \exp\left\{\int_0^t |u\varphi(a)|^\alpha df(a)\right\}.$$

For more details on time reversal stochastic integrals the reader is referred to [6].

Let X^α and $\widetilde{X}^\alpha$ are two α-stable symmetric processes and set

$$X^\alpha_\mu(h) = h(1)X^\alpha(f(1)) - \int_0^1 h(s)dX^\alpha(f(s)),\ \ h \in L^\alpha(A, \mathcal{A}, \mu),$$

where the stochastic integral is defined by (15) and

$$X_\nu^\alpha(k) = k(0)\widetilde{X}^\alpha(g(0)) + \int_0^1 k(t)d\widetilde{X}^\alpha(g(t)),\ k \in L^\alpha(B,\mathcal{B},\nu),$$

where the stochastic integral is defined in the usual sense.

Since f is decreasing (resp. g is increasing) and using the fact that the increments of X are independents it follows that $X^\alpha(f(1))$ and $\int_0^1 h(s)dX^\alpha(f(s))$ (resp. $X^\alpha(g(0))$ and $\int_0^1 h(s)dX^\alpha(g(s))$) are independent. So, for all $u \in \mathbb{R}$, we have

$$\mathbb{E}\Big[\exp\Big\{iuX_\mu^\alpha(h)\Big\}\Big] = \exp\Big\{-\int_0^1 |uh(s)|^\alpha \mu(ds)\Big\},$$

and

$$\mathbb{E}\big[\exp\{iuX_\nu^\beta(k)\}\big] = \exp\Big\{-\int_0^1 |uk(t)|^\beta \nu(dt)\Big\}.$$

A simple calculation, on one hand yields

$$\int_0^1 \mathbf{1}_{[s\geqslant t]}X_\nu^\alpha(dt) = X_\nu^\alpha(1_{[0,s]}) = \widetilde{X}^\alpha(g(s)),$$

and

$$\int_0^1 \left|\int_0^1 \mathbf{1}_{[s\geqslant t]}X_\nu^\alpha(dt)\right|^\alpha \mu(ds) = \int_0^1 -df(s)\left|\widetilde{X}^\alpha(g(s))\right|^\alpha + f(1)\left|\widetilde{X}^\alpha(g(1))\right|^\alpha.$$

On the other hand

$$\int_0^1 \mathbf{1}_{[s\geqslant t]}X_\mu^\alpha(ds) = X_\mu^\alpha(1_{[t,1]}) = X^\alpha(f(t)),$$

and

$$\int_0^1 \left|\int_0^1 \mathbf{1}_{[s\geqslant t]}X_\mu^\alpha(ds)\right|^\alpha \nu(dt) = \int_0^1 dg(t)|X^\alpha(f(t))|^\alpha + g(0)|X^\alpha(f(0))|^\alpha.$$

In order to obtain the desired integration by parts formula, we shall use (4) of Theorem 1 with $\phi(s,t) = \mathbf{1}_{[s\geqslant t]}$, which enable us to get the following

Proposition 4. *For any symmetric stable process X^α we have*

$$\int_0^1 -df(s)|X^\alpha(g(s))|^\alpha + f(1)|X^\alpha(g(1))|^\alpha \stackrel{d}{=} \int_0^1 dg(t)|X^\alpha(f(t))|^\alpha + g(0)|X^\alpha(f(0))|^\alpha$$

We recall that the above integration by parts formula was shown by Donati-Martin et al. in [5] using the discrete version of the Fubini-type identity in law (3).

3.2. Two parameters case

Let $f_1, g_1 : [0,1] \to \mathbb{R}^+$ (resp. $f_2, g_2 : [0,1] \to \mathbb{R}^+$) be two continuous functions, with f_1 (resp. f_2) decreasing, and g_1 (resp. g_2) increasing. Let us now choose $A = B = [0,1]\times[0,1]$ and define the measure μ and ν by:

$$\mu(ds_1, ds_2) = \{-df_1(s_1) + \delta_1(ds_1)f_1(1)\}\{-df_2(s_2) + \delta_1(ds_2)f_2(1)\}$$
$$\nu(dt_1, dt_2) = \{dg_1(t_1) + \delta_0(dt_1)g_1(0)\}\{dg_2(t_2) + \delta_0(dt_2)g_2(0)\}.$$

For a process Z in $D\Big([0,1]^2, \mathbb{R}\Big)$, we denote by $\overline{Z}$ the associated time reversed process of the process Z given by:

$$\overline{Z}(t_1,t_2) = \begin{cases} 0 & \text{if } t_1t_2 = 0, \\ Z((1-t_1)^-,(1-t_2)^-) - Z(1^-,(1-t_2)^-) \\ -Z((1-t_1)^-,1^-) + Z(1^-,1^-) & \text{if } 0 < t_1,t_2 < 1, \\ Z(0,(1-t_2)^-) - Z(1^-,(1-t_2)^-) \\ -Z(0,1^-) + Z(1^-,1^-) & \text{if } t_1 = 1, t_2 < 1, \\ Z((1-t_1)^-,0) - Z(1^-,0) \\ -Z((1-t_1)^-,1^-) + Z(1^-,1^-) & \text{if } t_1 < 1, t_2 = 1 \\ Z(0,0) - Z(1^-,0) \\ -Z(0,1^-) + Z(1^-,1^-) & \text{if } t_1 = t_2 = 1 \end{cases}$$

where $Z(u^-,v^-)$ denotes $\lim_{(s_1,s_2)\leqslant(u,v),(s_1,s_2)\to(u,v)} Z(s_1,s_2)$ and $\leqslant$denotes the natural partial ordering in $[0,1]^2$.

Let $\Big\{X_i^\alpha(s_1,s_2) : (s_1,s_2) \in [0,1]^2\Big\}$, $i = 1, 2$ be a pair of independent stable sheet. The stochastic integral of $\psi \in L^\alpha(A,\mathcal{A}, df_1df_2)$ with respect to the process $\{X_1^\alpha(f_1(s_1), f_2(s_2)), (s_1,s_2) \in [0,1]^2\}$ is defined by time reversal as follows:

$$\int_0^{t_1}\int_0^{t_2} \psi(s_1,s_2)dX_1^\alpha(f_1(s_1),f_2(s_2)) = \overline{\int_0^{t_1}\int_0^{t_2} \psi(1-s_1,1-s_2)dX_1^\alpha(f_1(1-s_1),f_2(1-s_2))}, \quad (16)$$

for $(t_1,t_2) \in [0,1]^2$. Moreover, for $u \in \mathbb{R}$ and $t_1,t_2 \in [0,1]$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{iu\int_0^{t_1}\int_0^{t_2} \psi(s_1,s_2)dX_1^\alpha(f_1(s_1),f_2(s_2))\Big\}\Big] \\
&= \mathbb{E}\Big[\exp\Big\{iu\int_{1-t_1}^1\int_{1-t_2}^1 \psi(1-s_1,1-s_2)dX_1^\alpha(f_1(1-s_1),f_2(1-s_2))\Big\}\Big] \\
&= \exp\Big\{-\int_{1-t_1}^1\int_{1-t_2}^1 |u\psi(1-s_1,1-s_2)|^\alpha df_1(1-s_1)df_2(1-s_2)\Big\} \\
&= \exp\Big\{-\int_0^{t_1}\int_0^{t_2} |u\psi(a,b)|^\alpha df_1(a)df_2(b)\Big\}.
\end{aligned}$$

Now, for $h \in L^\alpha(A,\mathcal{A},\mu)$ and $k \in L^\alpha(A,\mathcal{A},\mu)$, we define

$$\begin{aligned}
X_\mu^\alpha(h) = &\int_0^1\int_0^1 h(s_1,s_2)dX_1^\alpha(f_1(s_1),f_2(s_2)) \\
&\times \int_0^1 h(1,s_2)d_{s_2}X_1^\alpha(f_1(1),f_2(s_2)) \\
&+ \int_0^1 h(s_1,1)d_{s_1}X_1^\alpha(f_1(s_1),f_2(1)) + h(1,1)X_1^\alpha(f_1(1),f_2(1)),
\end{aligned}$$

where the first stochastic integral is defined by (16) and d_u means stochastic integration with respect to the variable u defined by (15) and

$$\begin{aligned}
X_\nu^\alpha(k) = &\int_0^1\int_0^1 k(t_1,t_2)dX_2^\alpha(g_1(t_1),g_2(t_2)) \\
&+ \int_0^1 k(0,t_2)d_{t_2}X_2^\alpha(g_1(0),g_2(t_2)) \\
&+ \int_0^1 k(t_1,0)d_{t_1}X_2^\alpha(g_1(t_1),g_2(0)) \\
&+ k(0,0)X_2^\alpha(g_1(0),g_2(0)).
\end{aligned}$$

Since the increments of each process X_1^α and X_2^α are independents it follows, for all $u \in \mathbb{R}$, $h \in L^\alpha(A,\mathcal{A},\mu)$ and $k \in L^\beta(B,\mathcal{B},\nu)$, that

$$\mathbb{E}\Big[\exp\Big\{iuX^{\alpha}_{\mu}(h)\Big\}\Big] = \exp\Big\{-\int_0^1\int_0^1 |uh(s_1,s_2)|^{\alpha}\mu(ds_1,ds_2)\Big\}$$

and

$$\mathbb{E}\big[\exp\{iuX^{\beta}_{\nu}(k)\}\big] = \exp\Big\{-\int_0^1\int_0^1 |uk(t_1,t_2)|^{\beta}\nu(dt_1,dt_2)\Big\}.$$

Now using a simple calculation we get

$$\int_0^1\int_0^1 \mathbf{1}_{\{s_1\geqslant t_1\}}\mathbf{1}_{\{s_2\geqslant t_2\}}X^{\alpha}_{\nu}(dt_1,dt_2) = X^{\alpha}_{\nu}(1_{[0,s_1]\times[0,s_2]}) = X^{\alpha}_2(g_1(s_1),g_2(s_2)),$$

and

$$\int_0^1\int_0^1 \mu(ds_1,ds_2)\left|\int_0^1\int_0^1 \mathbf{1}_{\{s_1\geqslant t_1\}}\mathbf{1}_{\{s_2\geqslant t_2\}}X^{\alpha}_{\nu}(dt_1,dt_2)\right|^{\alpha}$$
$$= \int_0^1\int_0^1 df_1(s_1)df_2(s_2)\big|X^{\alpha}_2(g_1(s_1),g_2(s_2))\big|^{\alpha}$$
$$-f_1(1)\int_0^1 df_2(s_2)\big|X^{\alpha}_2(g_1(1),g_2(s_2))\big|^{\alpha}$$
$$-f_2(1)\int_0^1 df_1(s_1)\big|X^{\alpha}_2(g_1(s_1),g_2(1))\big|^{\alpha}$$
$$+f_1(1)f_2(1)\big|X^{\alpha}_2(g_1(1),g_2(1))\big|^{\alpha}.$$

We also obtain

$$\int_0^1\int_0^1 \mathbf{1}_{\{s_1\geqslant t_1\}}\mathbf{1}_{\{s_2\geqslant t_2\}}X^{\alpha}_{\mu}(ds_1,ds_2) = X^{\alpha}_{\mu}\big(1_{[t_1,1]\times[t_2,1]}\big)$$
$$= X^{\alpha}_1(f_1(t_1),f_2(t_2)),$$

and

$$\int_0^1\int_0^1 \nu(dt_1,dt_2)\left|\int_0^1\int_0^1 \mathbf{1}_{\{s_1\geqslant t_1\}}\mathbf{1}_{\{s_2\geqslant t_2\}}X^{\alpha}_{\mu}(ds_1,ds_2)\right|^{\alpha}$$
$$= \int_0^1\int_0^1 dg_1(t_1)dg_2(t_2)\big|X^{\alpha}_1(f_1(t_1),f_2(t_2))\big|^{\alpha}$$
$$+g_1(0)\int_0^1 dg_2(t_2)\big|X^{\alpha}_1(f_1(0),f_2(t_2))\big|^{\alpha}$$
$$+g_2(0)\int_0^1 dg_1(t_1)\big|X^{\alpha}_1(f_1(t_1),f_2(0))\big|^{\alpha}$$
$$+g_1(0)g_2(0)\big|X^{\alpha}_1(f_1(0),f_2(0))\big|^{\alpha}.$$

Having all these preliminaries in mind and using once again (4) with $\phi(s_1,s_2,t_1,t_2) = \mathbf{1}_{\{s_1\geqslant t_1\}}\mathbf{1}_{\{s_2\geqslant t_2\}}$, we obtain the two parameters version of Proposition 4 as follows:

Proposition 5. *For every X^{α} stable sheet we have*

$$\int_0^1\int_0^1 df_1(s_1)df_2(s_2)|X^{\alpha}(g_1(s_1),g_2(s_2))|^{\alpha}$$
$$-f_1(1)\int_0^1 df_2(s_2)|X^{\alpha}(g_1(1),g_2(s_2))|^{\alpha}$$
$$-f_2(1)\int_0^1 df_1(s_1)|X^{\alpha}(g_1(s_1),g_2(1))|^{\alpha}$$
$$+f_1(1)f_2(1)|X^{\alpha}(g_1(1),g_2(1))|^{\alpha}$$
$$\stackrel{d}{=} \int_0^1\int_0^1 dg_1(t_1)dg_2(t_2)|X^{\alpha}(f_1(t_1),f_2(t_2))|^{\alpha}$$
$$+g_1(0)\int_0^1 dg_2(t_2)|X^{\alpha}(f_1(0),f_2(t_2))|^{\alpha}$$
$$+g_2(0)\int_0^1 dg_1(t_1)|X^{\alpha}(f_1(t_1),f_2(0))|^{\alpha}$$
$$+g_1(0)g_2(0)|X^{\alpha}(f_1(0),f_2(0))|^{\alpha}$$

It should be noted that the case $\alpha = 2$, that is X^{α} is a Brownian sheet, was covered in [3].

3.3. Particular cases

1. $g_1(0) = g_2(0) = f_1(1) = f_2(1) = 0$, we obtain

$$\int_0^1\int_0^1 df_1(s_1)df_2(s_2)|X^{\alpha}(g_1(s_1),g_2(s_2))|^{\alpha}$$
$$\stackrel{d}{=} \int_0^1\int_0^1 dg_1(t_1)dg_2(t_2)|X^{\alpha}(f_1(t_1),f_2(t_2))|^{\alpha}.$$

2. $g_1(t) = g_2(t) = t^2$ and $f_1(s) = f_2(s) = \log(1/s)$

$$\int_0^1\int_0^1 \left|\frac{1}{(s_1s_2)^{1/\alpha}}X^{\alpha}\big(s_1^2,s_2^2\big)\right|^{\alpha} ds_1ds_2$$
$$\stackrel{d}{=} 4\int_0^1\int_0^1 \Big|(t_1t_2)^{1/\alpha}X^{\alpha}(\log(1/t_1),\log(1/t_2))\Big|^{\alpha}dt_1dt_2.$$

References

[1] J. Bertoin, Lévy Process, Cambridge University Press, Cambridge, 1996.
[2] O.E. Barndorff-Nielsen, N. Shephard, Modelling by Lévy processes for financial econometrics, in: O.E. Barndorrf-Nielsen, T. Mikosch, S.I. Resnick (Eds.), Lévy Processes, Birkhäuser, Boston, MA, 2001, pp. 283–318.
[3] P. Deheuvels, G. Peccati, M. Yor, On quadratic functionals of the Brownian sheet and related processes, Stoch. Proc. Appl. 116 (3) (2006) 493–538.
[4] C. Donati-Martin, M. Yor, Fubini's theorem for double Wiener integrals and variance of the Brownian path, Annales de l'Institut H. Poincaré (1991) 181–200.
[5] C. Donati-Martin, S. Song, M. Yor, Symmetric stable processes, Fubini's theorem and some extension of the Ciesielski-Taylor identities in law, Stochastics and Stochastics Reports 50 (1994) 1–33.
[6] J. Jacod, P. Protter, Time reversal on Lévy processes, The Annals of Probability 16 (2) (1988) 620–641.
[7] G. Peccati, M. Yor, Identities in law between quadratic functionals of bivariate Gaussian processes, through Fubini theorems and symmetric projections. Approximation and probability, Banach Center Publ., 72, Polish Acad. Sci., Warsaw, 2006, pp. 235–250.
[8] D. Revuz, M. Yor, Continuous martingales and Brownian motion, third ed., Springer-Verlag, Berlin, 1999.
[9] K. Sato, Lévy processes and infinitely divisible distributions, Cambridge Press, Cambridge, 1999.
[10] M. Yor, Une explication du théorème de Ciesielski-Taylor, Annales de l'Institut H. Poincaré 27 (2) (1991) 201–213.
[11] M. Yor, Some aspects of Brownian motion, Part I, Some special functionals, Lectures in Mathematics ETH Zürich, Birkäuser Verlag, Basel, 1992.
[12] G.S. Watson, Goodness of fits tests on a circle, Biometrika 48 (1961) 109–114.