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SHORT COMMUNICATION

Fourier, Legendre and Jacobi

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The front page of the December 2009 issue of the Notices of the American Mathematical Society reproduced a caricature of Joseph Fourier (1768–1830) and Adrien Marie Legendre (1752–1833) coming from the library of Institut de France. It is part of a collection of caricatures of scientists and other people made by the painter Bailly around 1820. There are other portraits of Fourier but none other of Legendre. Actually there are other portraits of Legendre, but of another Legendre; the situation is well described in an article of Peter Duren in the same issue of the Notices [2].

Why Fourier and Legendre on the same picture? There were other prominent mathematicians at the French Academy of Sciences at that time, linked to either Fourier or Legendre: Laplace, Cauchy, Biot, Poisson, Lacroix, Prony, Ampère among others. There is no much in common between the main works of Legendre, Elements of geometry, Theory of numbers and Treatise of elliptic functions and Eulerian integrals, and those of Fourier, Analytical theory of heat and Theory of equations. There is a relation between Fourier series and Legendre polynomials, but the mere terms of Fourier series as well as Legendre polynomials appeared much later.

By the time of the caricature, Fourier should have been elected as permanent secretary (“secrétaire perpétuel”) of the Academy of Sciences (1822) and he had been a member since 1817. Legendre had been an adjoin member in 1783, then a full

member in 1795 (an IV of the French Republic), then president in (1805–1806) (an XIV) when the Royal Academy of Sciences was replaced by the first class of Institut de France. Legendre (spelled Le Gendre at that time) was a well established mathematician when Fourier was still a very young man, and, though Fourier was very active in many respects, his recognition as scientist came rather late in his life. Fourier died when Legendre was still alive. I shall try to point out the importance of Legendre in Fourier’s life and reputation, before and after the time of the caricature. Carl Gustav Jacob Jacobi (1804–1851) will appear through his correspondence with Legendre, as initiator of new views on elliptic functions; he had been elected as correspondent in 1830, just before Fourier died, and associate member of the Academy in 1846.

From everything we know, Legendre was a very decent human being. He could have been reluctant in front of the new look on elliptic functions due to Jacobi, but on the contrary he was enthusiastic. He introduced Jacobi in the Academy. Their correspondence is an exchange between peers and friends, though the age difference is more than 50 years. We shall return to Legendre and Jacobi at the time of the death of Fourier, 1830.

Actually I shall select three times in which Legendre played a role in relation with Fourier. It is an opportunity to have a look on Fourier’s life and works.

The first episode takes place in 1787, when Fourier was 19 years old, two years before the beginning of the French Revolution. Joseph Fourier was born in the town of Auxerre in a rather poor family. His grandparents were peasants, his father was a taylor and had 16 children, and both his father and mother died when he was very young. At the age of 10 he was an orphan, but noticed as a bright boy, he was taught French and Latin by the organist of the cathedral and entered the Military College of Auxerre. Military education was reserved previously to young members of the nobility but it was not the case any more in Auxerre. He was taught by

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Benedictine monks, graduated at the age of 14 and was appointed as a teacher in the same College when he was 16 1/2. He had learnt mathematics by himself, reading Clairault and Bezout, and got interested in a subject that he studied later again and again: how to solve algebraic equations? Later he extended the question: how to solve a system of inequations? It is what he called “analyse des équations déterminées” et “calcul des inégalités”. He always was interested in explicit procedures in order to get numerical solutions and, for example, he introduced the method of steepest descent on convex polyhedra in what we called later linear programming, as part of what he named “analyse indéterminée”. But that part of Fourier’s works remained ignored for a long time, and it was left out by Gaston Darboux when he published his “Œuvres” (actually, Selected Works) in 1888–1890; Darboux explained that Fourier gave an “exaggerated importance” to these things. Our point of view is different now because we praise algorithms and Fourier’s contributions in that field deserve to be known.

His first contribution was written when he was 17, rewritten, and sent to Legendre, who appreciated this first work of a newcomer. It was in 1787, and Fourier had to look for a position in life, with two possible ways: the Army or the Church. Fourier thought he was well prepared to enter the Artillery, the most scientific section of the Army. He applied and got the support of Legendre, who had a position by the Ministry of War. According to Arago, who read the obituary of Fourier in 1833, the minister replied to Legendre that “even if he was a second Newton, Fourier could not enter Artillery since he is not a noble”. Then Fourier had to choose the Church. It was not for long, as we shall see in a moment.

The second episode took place in 1815. In the meantime France had known the end of the reign of the Bourbon king Louis XVI, the Revolution, the war against the European kingdoms, the fall of Louis XVI and the first French Republic, the raise of Bonaparte with the campaigns in Italy and Egypt, the Napoléon empire, the defeat and exile of Napoléon and the restauration of a Bourbon kingdom with Louis XVIII, a brother of Louis XVI (the reign of Louis XVII, son of Louis XVI, was just a fiction), the “Hundred days” during which Napoléon came back and took power again, the defeat of Waterloo and the new exile of Napoléon and new Bourbon Restauration with Louis XVIII again.

Already in 1789 Fourier had taken advantage of a decision of the National Assembly in order to give up the vows he was supposed to take in order to become a priest. He was offered and glad to accept a position as “instituteur salarié par la nation”, a teacher paid by the Nation. Meanwhile he took part in the Revolution in an active and efficient way. When the National Convention decided to open the first Ecole normale (October 1794, Vendémiaire An III) Fourier was elected as one of the 1500 students. The mathematics teachers were the highly respected Lagrange (1736–1813) and the younger Monge (1746–1818) and Laplace (1749–1827); their lectures and stenographed discussions with the pupils were published by Jean Dhombres in 1992 [3] so that we can read an interesting discussion between Citizen Monge and Citizen Fourier about the definition of a circle and the foundation of geometry. Monge appreciated Fourier and was kind of a mentor for him during a few years. Monge was one of the founders of Ecole polytechnique and Fourier was hired immediately as a lecturer. When Bonaparte led the French expedition to Egypt, he orga-

nized with Monge a scientific expedition as well, with Fourier as a member. As a copy of Institut de France, the Institut d’Egypte was created, with Monge as president and Fourier as “secrétaire perpétuel” (permanent secretary). It was a very short perpetuity, but full of scientific activities of different kinds, including physics, history and geography as well as mathematics, and also administrative duties and diplomatic relations. Fourier in Egypt is the matter of an important chapter in the excellent book of Jean Dhombres and Jean-Bernard Robert on Fourier [4].

After Le Caire, Fourier came back to Paris and resumed his position at Ecole Polytechnique, now as a full professor. Shortly afterwards, in 1802, Bonaparte (on the way to become Napoléon) made him Préfet de l’Isère, prefect in Grenoble. It was an important position and could have been the end of his scientific activities, but it was not. He had to write hundreds of pages on Egypt and to assume his functions as prefect. He was in a complete isolation from a scientific point of view. However he elaborated in Grenoble his main mathematical work, the Analytical Theory of Heat.

His work included a careful investigation of the propagation of heat inside solid bodies, the heat equations (inside and at the boundary of the body) and a series of particular cases where he was able to give explicit solutions when initial values and boundary conditions were prescribed. The trigonometric series were introduced by him as a practical tool in order to get numerical values of the solutions in the simplest cases. This tool became an object of study and he arrived at the conclusion that every function could be expressed as the sum of a trigonometric series on a well prescribed interval. He had a series of examples; in particular he established that

$$\frac{1}{2}x = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots \quad (a)$$

when $-\pi < x < \pi$ (strict inequalities). More important, he associated the formulas that we now write as

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \int_0^{2\pi} f(x) e^{-inx} \frac{dx}{2\pi}$$

That led Riemann to declare that Fourier was the first who understood the true nature of trigonometric series, and to use the term of “Fourier series” as we do now, for a trigonometric series whose coefficients are given by these integral formulas [8].

Today the name of Fourier is mainly used in Fourier series, Fourier integrals, Fourier transforms, Fourier analysis, among physicists and engineers as well as mathematicians. But this part of Fourier’s works was not accepted at once. Here is what Riemann wrote:

“Als Fourier in einer seiner ersten Arbeiten über die Wärme, welche er der französische Akademie vorlegte (21 December 1807) zuerst den Satz aussprach, dass eine willkürlich (graphisch) gegebene Function sich durch eine trigonometrische Reihe ausdrücken lasse, war diese Behauptung dem greisen Lagrange so unerwartet, dass er ihr auf das Entschiedenste entgegtrat. Es soll sich hierüber noch ein Schriftstück im Archiv der Pariser Akademie befinden.”

When Fourier in one of his first works on heat, communicated to the French Academy on December 21, 1807, stated

that an arbitrary function (given in a graphic way) could be expressed by a trigonometric series, this statement was so unexpected to the old Lagrange that he opposed it in the strongest way. There should still be a written document about this in the Archives of the Parisian Academie.

I looked for this document and told the story in [7]. What I found in the huge collection of handwritten papers of Lagrange kept in the Library of Institut de France is rather strange. It is two pages long, each page on a different sheet. At the back of the second page is written

“Papier relatif au mémoire de Fourier, deux feuilles”

and it is signed De Prony, Le Gendre, Poisson, Lacroix. But the second page is not of Lagrange’s handwriting. It is Fourier’s, and just a refutation of what is written on the first page. This first page was written by Lagrange and was intended to prove that the formula (a) above written in the mémoire of Fourier, was false. Lagrange made a series of transformations and ended with a contradiction. Fourier explained that the contradiction was due to an illegitimate extension of (a) out of the domain $-\pi < x < \pi$ and he wrote as a conclusion that an equation of this type cannot be used without specifying the limits between which the values of the variable have to be considered. As far as I could see, that is the only page in the whole collection that was not written by Lagrange. How and why was it joined to the page of Lagrange and endorsed by the committee? Here is a partial answer.

Lagrange died in 1813. In 1815, during the “hundred days”, the minister of Interior of Napoléon, the mathematician Lazare Carnot, got Lagrange’s papers and ordered the Institut to put them in order and to have them printed. They never were printed, but they were put in order by a committee consisting of Legendre, Prony (1753–1839), Lacroix (1765–1841) and Poisson (1781–1840). Legendre was the senior member. There was obviously a problem when the committee saw a paper of Fourier (actually, a draft) among those of Lagrange. Who took the decision to join this page to the erroneous page of Lagrange? Legendre very likely.

Fourier got aware of this decision, a kind of revenge towards the constant opposition of Lagrange to his treatment of trigonometric series. To young people he mentioned this “Schriftstück” as a proof of Lagrange’s opposition, not as a mistake that he corrected. Riemann knew the existence of the Schriftstück by Dirichlet, who was in Paris and had friendly relations with Fourier around 1820.

For me this episode expresses honesty and savoir-vivre of both Legendre and Fourier.

The third episode occurred just after the death of Fourier in 1830. The recognition of Fourier as an important scientist came late in his life, but then he was honoured in different manners. In 1816 he was elected in the Académie des sciences but refused by the king. In 1817 he was elected again, then accepted. He became secrétaire perpétuel in 1822, and published his main work, *La Théorie Analytique de la Chaleur* [5], the same year. He entered Académie française, Académie de médecine, Royal Society. After he died, Arago as a new secrétaire perpétuel wrote a beautiful obituary of Fourier, with a high consideration for all his accomplishments, with one significant exception: not a word on trigonometric series.

As I already mentioned, Legendre and Jacobi exchanged an extensive correspondence. Fourier died on May 16, 1830. On July 2, Jacobi wrote to Legendre the following appreciation

on Fourier [6]: “J’ai lu avec plaisir le rapport de M. Poisson sur mon ouvrage, et je vais pouvoir en être très content. . . Mais M. Poisson n’aurait pas dû reproduire dans son rapport une phrase peu adroite de feu M. Fourier, où ce dernier nous reproche, à Abel et à moi, de ne pas nous être occupés de préférence du mouvement de la chaleur. Il est vrai que M. Fourier avait l’opinion que le but principal des mathématiques était l’utilité publique et l’explication des phénomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c’est l’honneur de l’esprit humain et que, sous ce titre, une question de nombres vaut autant qu’une question de système du monde. Quoiqu’il en soit, on doit vivement regretter que M. Fourier n’ait pu achever son ouvrage sur les équations, et de tels hommes sont trop rares aujourd’hui, même en France, pour qu’il soit facile de les remplacer.” (I was pleased to read the report on my work done by M. Poisson, and I think I can be proud of it. . . However M. Poisson should not have reproduced an ill-timed appreciation of the late M. Fourier, reproaching Abel and me for not paying enough attention to the movement of heat. In truth, M. Fourier thought that public interest and explanation of natural phenomena were the main purpose of mathematics. But, as a philosopher, he should have known that the unique purpose of science is the honour of the human mind, and that, in this respect, a question about numbers is as valuable as a question about the universe. Anyway it is great pity that M. Fourier could not finish his work on equations, and there are so few such men today, even in France, that it is not easy to have them replaced.)

“L’honneur de l’esprit humain”, the honour of the human mind, became later a motto for pure mathematics. In particular, it is the title of a famous book of Jean Dieudonné [1].

“L’utilité publique et l’explication des phénomènes naturels” (the public interest and the explanation of natural phenomena) are truly the point of view of Fourier. It is expressed clearly many times in his works, in particular in his introduction to the *Analytical Theory of Heat*, “Discours préliminaire”. Actually it is not only a “purpose” of mathematics: the careful study of natural phenomena is source of mathematics: “L’étude approfondie de la nature est la source la plus féconde des découvertes mathématiques” ([5], Discours préliminaire).

Moreover, as I already mentioned, the methods introduced by Fourier are intended to give explicit numerical solutions to the problems under consideration. In the “Discours préliminaire” he cannot describe the method of trigonometric series as a tool in order to compute the temperature inside a solid body, but the program is expressed clearly; here is a rapid translation.

“The heat equation, as well as the equations concerning vibrating strings or motions of liquids, belongs to a quite recent brand of analysis [namely, PDE]. After establishing the equations one has to find the solutions, that is, go from a general expression to a particular solution subject to prescribed conditions. This investigation was difficult and needed a new kind of analysis, based on new theorems. The corresponding method leaves nothing vague in the solutions. It leads to final numerical applications, as any investigation should do in order to be useful”.

For a long time Joseph Fourier was underestimated in France. May be he was too much a physicist to be considered as a good mathematician. Nowadays his name is everywhere in the scientific literature, in the form of Fourier series, Fourier

integrals, Fast Fourier transforms, Fourier analysis, that is, the part of his work that was considered as most questionable in his time. And our point of view on Fourier's ideas changed drastically in a few years. Physics and mathematics are closer than ever. Informatics gave a new impulse on numerical methods. Fourier as a philosopher is better understood now that he was in the 19th century.

His relation with Legendre and Jacobi was just an occasion to reach this conclusion.

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