Journal of the Egyptian Mathematical Society (2011) **19**, 88–90

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

SHORT COMMUNICATION

Ill-Posedness of sublinear minimization problems

S. Issa [a,*], **M. Jazar** [a], **A. El Hamidi** [b]

[a] *LaMA-Liban, Azm Research Center, EDST, Lebanese University, Tripoli, Lebanon*
[b] *LMA, Université de La Rochelle, France*

Available online 3 December 2011

KEYWORDS

Bounded variation;
Nonconvex regularization;
Chambolle's projection;
Texture

Abstract It is well known that minimization problems involving sublinear regularization terms are ill-posed, in Sobolev spaces. Extended results to spaces of bounded variation functions BV were recently showed in the special case of *bounded* regularization terms. In this note, a generalization to sublinear regularization is presented in BV spaces. Notice that our results are optimal in the sense that linear regularization leads to well-posed minimization problems in BV spaces.

1. Introduction

The aim of this note is the study of minimization problems on the space of functions of bounded variation of functionals involving sublinear terms of the total variation. These problems are motivated by applications in image restoration.

More precisely, we are interested by the ill-posedness of minimization problems of the form

$$\inf_{u\in BV(\Omega)} J(u), \tag{1}$$

where

$$J(u) := \frac{\lambda}{2}\|f-u\|_{L^2(\Omega)} + \int_\Omega \Phi(|Du|), \quad \lambda > 0,$$

where the function Φ is sublinear at infinity. The functional space $BV(\Omega)$ is the space of functions with bounded variation $BV(\Omega)$ [2].

The set Ω is a bounded domain of $\mathbb{R}^N$, $N \geqslant 2$, f is a given function in $BV(\Omega)$, which may represent an observed image (for $N = 2$). The first term in $J(u)$ measures the fidelity to the data while the second one is a nontrivial smoothing term involving the generalized gradient Du of the function u.

In what follows, we will assume the following hypotheses on the smooth function Φ:

$(H1)\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\Phi(0) = \Phi'(0) = 0$,

$(H2)\Phi$ is sublinear at infinity, i.e. $\lim_{s\to+\infty} \frac{\Phi(s)}{s} = 0$.

The condition (H1) implies that the function Φ is quadratic at the origin. In image restoration, this means that at locations where the variations of the intensity are weak (low gradients), we would like to encourage smoothing, the same in all direc-

* Corresponding author.
E-mail addresses: samar.issa@univ-lr.fr (S. Issa), mjazar@ul.edu.lb (M. Jazar), aelhamid@univ-lr.fr (A. El Hamidi).

Peer review under responsibility of Egyptian Mathematical Society.
doi:10.1016/j.joems.2011.09.004

Production and hosting by Elsevier

tions. Conversely, the condition (H2) means that the "cost" of edges is "low" and consequently, the corresponding regularizing term preserves edges.

It is clear that Φ can not be convex since the unique convex function satisfying the conditions (H1) and (H2) is the trivial function. This fact implies that there is no hope to recover the lower semicontinuity of J with respect to the weak ★ convergence of $\mathcal{M}(\Omega, \mathbb{R}^N)$, the space of all N-vector bounded measures. More precisely, in [7,4,8,6] functionals of the form

$$F(\lambda) := \int_X f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_X f^\infty\left(x, \frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s|, \qquad (2)$$

have been studied, where X is a locally compact space, *mu* is a given positive measure in $\mathcal{M}(X, \mathbb{R}^N)$, f^∞ is the recession function of f with respect to its second variable and $\lambda = (d\lambda/d\mu) \cdot \mu + \lambda^s$ is the Lebesgue–Nikodym decomposition of λ into absolutely continuous and singular parts with respect to *mu*. It is shown that for functionals of the form (2), the convexity of f is a necessary condition to guarantee the lower semicontinuity in the weak ★ convergence of $\mathcal{M}(X, \mathbb{R}^N)$. Moreover, every convex and weak ★ lower semicontinuous functional $F : \mathcal{M}(X, \mathbb{R}^N) \to [0, +\infty]$ is representable in the form (2) with a suitable convex function f, provided the additivity condition

$$F(\lambda_1 + \lambda_2) = F(\lambda_1) + F(\lambda_2), \text{ for every } \lambda_1, \lambda_2 \in \mathcal{M}(X, \mathbb{R}^N) \text{ with } \lambda_1 \perp \lambda_2,$$

is satisfied.

For the reader's convenience, we recall some background facts used here. Let us define

$$\mathcal{K}(\Omega, \mathbb{R}^N) := \{\varphi \in C(\Omega, \mathbb{R}^N) : \operatorname{supp}(\varphi) \subset \Omega\},$$

$$\mathcal{BC}(\Omega, \mathbb{R}^N) := \left\{\varphi \in C(\Omega, \mathbb{R}^N) : \|\varphi\|_\infty := \sup_{x\in\Omega} \sqrt{\sum_{i=1}^N \varphi_i(x)^2} < +\infty\right\},$$

where $\operatorname{supp}(\varphi)$ denotes the support of φ. The space $C_0(\Omega, \mathbb{R}^N)$ is the closure of $\mathcal{K}(\Omega, \mathbb{R}^N)$ in $\mathcal{BC}(\Omega, \mathbb{R}^N)$ with respect to the uniform norm. The $\mathbb{R}^N$-valued Borel measures $\mu \in \mathcal{M}(\Omega, \mathbb{R}^N)$ represent the dual of $C_0(\Omega, \mathbb{R}^N)$. The norm of *mu* is then

$$\|\mu\| := \sup\{\langle \mu, \varphi\rangle : \|\varphi\|_\infty \leqslant 1\}.$$

The variation $|\mu| \in \mathcal{M}(\Omega, \mathbb{R}^N)$ is defined by its values on open subsets of Ω

$$|\mu|(\omega) := \sup\{\langle \mu, \varphi\rangle : \|\varphi\|_\infty \leqslant 1, \ \operatorname{supp}(\varphi) \subset \omega\}.$$

Then $|\mu| = |\mu|(\Omega)$ is the total variation of μ. The weak topology of $\mathcal{M}(\Omega, \mathbb{R}^N)$ is as defined as: a sequence (μ_n) converges weakly to *mu* in $\mathcal{M}(\Omega, \mathbb{R}^N)$, and written $\mu_n \rightharpoonup \mu$, if

$$\lim_{n\to+\infty} \int h(x)\, d\mu_n(x) = \int h(x)\, d\mu(x), \quad \forall h \in C_0(\Omega, \mathbb{R}^N).$$

In the sequel, for every $u \in L^1_{loc}(\Omega)$, Du will denote the distributional derivative of u.

$$BV(\Omega) := \{u \in L^1(\Omega, \mathbb{R}) : Du \in \mathcal{M}(\Omega, \mathbb{R}^N)\}.$$

Recall that the strong topology of $BV(\Omega)$ is given by the norm

$$\|u\|_{BV} := \|u\|_{L^1(\Omega)} + |Du|(\Omega)$$

and its weak topology is given by:

$$u_n \rightharpoonup u \text{ in } BV(\Omega) \iff u_u \to u \text{ in } L^1(\Omega) \text{ and } Du_n \rightharpoonup Du \text{ in } \mathcal{M}(\Omega, \mathbb{R}^N).$$

In well-posed minimization problems, the standard definition of the term $\int_\Omega \Phi(|Du|)$ is very restrictive and concerns only convex functions Φ with linear growth at infinity:

$$\Phi_\infty(1) := \lim_{s\to+\infty} \frac{\Phi(s)}{s} \in]0, +\infty[. \qquad (3)$$

Indeed, let $u \in BV(\Omega)$ and let the Lebesgue decomposition of the measure Du with respect to the N-dimensional Lebesgue measure dx :

$$Du = \nabla u\, dx + D^s u,$$

where $\nabla u\ dx$ is the absolutely continuous (regular) part of the measure Du and $D^s u$ its singular part, which is mutually singular with dx. If the function Φ satisfies the growth condition (3) at infinity, then the classical definition of $\int_\Omega \Phi(|Du|)$ is given by:

$$\int_\Omega \Phi(|Du|) := \int_\Omega \Phi(|\nabla u|)dx + \Phi_\infty(1) \int |D^s u|. \qquad (4)$$

The reason behind this definition is that under this restrictive growth condition (3), the lower semi-continuity of the functional: $u \mapsto \int_\Omega \Phi(|Du|)$ for the weak topology of $BV(\Omega)$ holds true. This semi-continuity result is a key ingredient to show that minimizing sequences are relatively compact in $BV(\Omega)$.

In our context, the hypothesis (H2) implies that the recession term $\Phi_\infty(1) = 0$, so the standard definition of $\int_\Omega \Phi(|Du|)$ ignores the singular part of the measure Du.

In [3], Aubert et al. studied two situations in image restoration and decomposition:

- Φ Sublinear at infinity and the energy has no singular part of TV.
- Φ Bounded and the the energy contains generalized singular part of TV.

Thus, when the singular part is ignored in the definition of $\int_\Omega \Phi(|Du|)$, the study is complete. However, when the singular part in the definition of $\int_\Omega \Phi(|Du|)$ is considered, the study is incomplete since only bounded functionals Φ are valid.

In this note we show that minimization problems involving general sublinear regularizing terms are ill-posed, then it is more convenient that our results cover a large class of natural definitions of $\int_\Omega \Phi(|Du|)$.

2. The main result

To provide a more general definition of the regularizing term $\int_\Omega \Phi(|Du|)$, we will recall some fine properties of functions of bounded variation [5,1]. Let $u \in BV(\Omega)$, we define the approximate upper limit u^+ and the approximate lower limit u^- of u on Ω as the following:

$$u^+(x) := \inf\left\{t \in [-\infty, +\infty] : \lim_{r\to 0} \frac{\operatorname{meas}[\{u > t\} \cap B(x, r)]}{r^N} = 0\right\},$$

$$u^-(x) := \sup\left\{t \in [-\infty, +\infty] : \lim_{r\to 0} \frac{\operatorname{meas}[\{u < t\} \cap B(x, r)]}{r^N} = 0\right\},$$

where $B(x, r)$ is the ball of center x and radius r. In particular, Lebesgue points in Ω are those which verify $u^+(x) = u^-(x)$. We denote by S_u the jump set, that is, the complement, up to a set of $\mathcal{H}^{N-1}$ measure zero, of the set of Lebesgue points

$$S_u := \{x \in \Omega : u^+(x) > u^-(x)\},$$

where $\mathcal{H}^{N-1}$ is the $(N-1)$ – dimensional Hausdorff measure. The set S_u is countably rectifiable, and for $\mathcal{H}^{N-1}$ almost everywhere $x \in \Omega$, we can define a normal vector $\mathbf{n}_u(x)$. In [1], L. Ambrosio showed that for every $u \in BV(\Omega)$, the singular part of the finite measure Du can also be decomposed into a *jump* J_u part and a *Cantor* part C_u

$$Du = (\nabla u)\,dx + (u^+ - u^-)\,\mathbf{n}_u\,\mathcal{H}^{N-1}_{|S_u} + C_u. \quad (5)$$

The jump part $J_u = (u^+ - u^-)\,\mathbf{n}_u\,\mathcal{H}^{N-1}_{|S_u}$ and the Cantor part C_u are mutually singular. Moreover, the measure C_u is diffuse, i.e. $C_u(\{x\}) = 0$ for every $x \in \Omega$ and $C_u(B) = 0$ for every $B \subset \Omega$ such that $\mathcal{H}^{N-1}(B) < +\infty$, that is, when the support of C_u is not empty, its Hausdorff dimension is strictly greater than $N-1$.

Now, we can give the more general definition:

Definition 1. Let $u \in BV(\Omega)$ and let $Du \in \mathcal{M}(\Omega, \mathbb{R}^N)$ be its distributional derivative. We define the measure $\Phi(|Du|)$as follows:

$$\Phi(|Du|) := \Phi(|\nabla u|)dx + \Phi_1(u^+ - u^-)\,d\mathcal{H}^{N-1}_{|S_u} + \Phi_2(|\,C_u\,|),$$

where Du is decomposed as in (5) and Φ_i are any nonnegative functions satisfying $\Phi_i(t) = 0$ if and only if $t = 0$, i = 1, 2.

The previous definition extends the standard total variation, i.e. $\Phi(s) = s$

$$\Phi(|Du|) := |\nabla u|\,dx + (u^+ - u^-)\,d\mathcal{H}^{N-1}_{|S_u} + |C_u|,$$

and also general total variations associated to convex functions $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ with linear growth at infinity. Indeed, let Φ^∞ be the recession function of Φ defined by $\Phi^\infty(z) := \lim_{s\to\infty} \Phi(s\,z)/s$ and the standard definition of the measure

$$\Phi(|\,Du\,|) := \Phi(|\nabla u|)dx + \Phi^\infty(1)(u^+ - u^-)\,d\mathcal{H}^{N-1}_{|S_u} + \Phi^\infty(1)\,|\,C_u\,|\,.$$

In this case, the functions Φ_1, Φ_2 in Definition 1 are given by $\Phi_1(s) = \Phi_2(s) = \Phi^\infty(1) \times s$, for every $s \geqslant 0$.

Now let Ω be a bounded domain in $\mathbb{R}^2$ and $f \in L^2(\Omega)$ be an observed image which corresponds to the ideal image u corrupted by a Gaussian noise $\eta \in L^2(\Omega)$; that is $f = R\,u + \eta$. It is well-known since the seminal work of Tikhonov and Arsenin [9], that the restored (ideal) image u is not other than the minimizer of a certain strictly convex energy

$$E(v) := \frac{\lambda}{2}\int_\Omega (f-v)^2\,dx + \int_\Omega \Psi(|\nabla v|)\,dx,$$

on an adequate functional space, where the function Ψ has to be chosen to realize some desired regularization effects. The parameter $\lambda > 0$ can be interpreted as the Lagrange multiplier with respect to the constraint on the variance of the noise η or as a regularizing coefficient.

In what follows, consider the functional

$$J(u) := \frac{\lambda}{2}\int_\Omega (f-u)^2\,dx + \int_\Omega \Phi(|\nabla u|)\,dx + \int_{S_u} \Phi_1(u^+ - u^-)\,d\mathcal{H}^{N-1} + \int_{\Omega\setminus S_u} \Phi_2(|\,C_u\,|), \quad (6)$$

Now we state our result

Theorem 1. *Consider the functional J defined by (6), where Φ satisfies (H1) and (H2). Let f be an arbitrary function in $L^\infty(\Omega)$. Then*

$$\inf_{u\in BV(\Omega)} J(u) = 0.$$

Moreover, the infimum of J on $BV(\Omega)$ is achieved if and only if f is constant.

Idea of the proof. The idea is to discretize the domain Ω into $A^n_{ij} := [x^n_i, x^n_{i+1}] \times [y^n_i, y^n_{i+1}]$ and then consider a sequence (v_n) of affine functions, bounded in infinity norms by vert$f$vert$_\infty$, such that $v_n \to f$ in $L^2(\Omega)$. The key point is a tricky calculation that shows that

$$\int_\Omega \Phi(|\nabla v_n|)\,dx \leqslant \frac{2\|f\|_\infty}{n}\,\mathrm{Per}(\Omega) + \frac{1}{2}[\mathrm{Per}(\Omega)]^2\,n^{2-\beta} \to 0,$$

for some $\beta > 2$. Using the fact $(v_n)_n \in BV(\Omega)$, we conclude that $\inf_{u\in BV(\Omega)} J(u) = 0$. Therefore,

- if f is constant then $f \in BV(\Omega)$ and $J(f) = 0$ and consequently J has a minimizer on $BV(\Omega)$,
- conversely, if J has a minimizer $\widehat{u}$ on $BV(\Omega)$ then necessarily $\widehat{u} = f$, $\nabla\widehat{u} = 0$, $\widehat{u}^+ = \widehat{u}^-$ and $C_{\widehat{u}} = 0$; hence $\widehat{u}$ is constant.

References

[1] L. Ambrosio, Existence theory for a new class of variational problems, Arch. Rational Mech. Anal. 111 (1990) 291–322.

[2] L. Ambrosio, N. Fusco, D. Pallara, Functions of bounded variation and free discontinuity problems, Oxford Mathematical Monographs, Oxford University Press, 2000.

[3] G. Aubert, A. El Hamidi, C. Ghannam, M. Menard, On a class of ill-posed minimization problems in image processing, J. Math. Anal. Appl. 352 (2009) 380–399.

[4] L. Ambrosio, G. Buttazzo, Weak lower semicontinuous envelope of functionals defined on a space of measures, Ann. Mat. Pura Appl. 150 (1988) 311–339.

[5] G. Aubert, P. Kornprobst, Mathematical Problems in Image Processing, Springer, 2006.

[6] G. Bouchitté, G. Buttazzo, New lower semicontinuity results for nonconvex functionals defined on measures, Nonlinear Anal. 15 (1990) 679–692.

[7] G. Bouchitté, G. Buttazzo, Integral representation of nonconvex functionals defined on measures, Ann. Inst. H. Poincar Anal. Non Linaire 9 (1992) 101–117.

[8] G. Bouchitté, M. Valadier, Multifonctions s.c.i. et rgularise s.c.i. essentielle, Ann. Inst. H. Poincar Anal. Non Linaire 6 (1989) 123–149.

[9] A.N. Tikhonov, V.A. Arsenin, Solution of Ill-posed Problems, Winston & Sons, 1977.