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SHORT COMMUNICATION

Existence of uniformly stable solutions of nonautonomous discontinuous dynamical systems

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Abstract We are concerned here with the existence of uniformly Lyapunov stable integrable solution of linear and nonlinear nonautonomous discontinuous dynamical systems.

1. Introduction

The discrete dynamical system [1,4]

$$x_n = ax_{n-1}, \quad n = 1, 2 \ldots \tag{1}$$

$$x_0 = c. \tag{2}$$

has the discrete solution

$$x_n = a^n x_0, \quad n = 1, 2 \ldots. \tag{3}$$

The more general dynamical system

$$x(t) = ax(t-r), \ t \in (0, T] \text{ and } r > 0 \tag{4}$$

$$x(t) = x_o, \ t \leqslant 0. \tag{5}$$

has the discontinuous (integrable) solution

$$x(t) = a^{1+[\frac{t}{r}]} x_o \in L_1(0, T], \quad \text{where } [.] \text{ is the bract function.} \tag{6}$$

The nonlinear discrete dynamical system

$$x_n = f(x_{n-1}), \quad n = 1, 2 \ldots \tag{7}$$

with the initial data (2)

$$x_0 = c$$

has the discrete solution

$$x_n = f^n(x_o), \quad n = 1, 2 \ldots \tag{8}$$

but the nonlinear problem

$$x(t) = f(x(t-r)), \quad r > 0 \tag{9}$$

with the initial data (5) is more general than the problem (7)–(14) and has the discontinuous (integrable) solution

$$x_n = f^{1+[\frac{t}{r}]}(x_o) \in L_1(0, T) \tag{10}$$

So, we can call the systems (4)–(5) and (9)–(5)) are discontinuous dynamical systems.

Definition 1. The discontinuous dynamical system is the problem of the retarded functional equation

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$$x(t) = f(t, x(t-r)), \quad r,\ t > 0 \tag{11}$$

$$x(t) = g(t), \quad t \in (-\infty, 0]. \tag{12}$$

Let $L_1[a,b]$, $-\infty < a < b < \infty$ be the class of Lebesgue integrable function defined on $[a,b]$ with the equivalent norm

$$||x||_{L_1[a,b]} = \int_a^b e^{-Nt}|x(t)|dt,\ x \in L_1[a,b],\ N > 0 \text{ is arbitrary}$$

and let ${}_nL_1[a,b]$, $-\infty < a < b < \infty$ be the class of Lebesgue integrable column vectors $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))'$ defined on $[a,b]$ with the equivalent norm

$$||X||_{{}_nL_1[a,b]} = \sum_{k=0}^{n} \int_a^b e^{-Nt}|x_k(t)|dt,\ x_k \in L_1[a,b],\ N > 0 \text{ is arbitrary}$$

Here we prove the existence of a unique uniformly Lyapunov stable solution $x \in L_1[0,T]$ for the nonautonomous nonlinear discontinuous dynamical system.

$$x_i(t) = f_i(t, x_1(t-r_1),\ x_2(t-r_2), \ldots, x_n(t-r_n)), \quad t,\ r_i > 0 \tag{13}$$

$$x_i(t) = g_i(t) \in L_1(-r, 0],\ \lim_{t\to 0} g_i(t) = x_i(0), \quad i = 1, 2\ldots, n \tag{14}$$

where $r = \min\{r_1, r_2, \ldots, r_n\}$.

The nonautonomous linear discontinuous dynamical system

$$x_i(t) = \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t-r_j) \tag{15}$$

$$x_i(t) = g_i(t) \in L_1(-r, 0],\ \lim_{t\to 0} g_i(t) = x_i(0)$$

where a_{ij} and $r_j > 0$, $r = \min\{r_1, r_2, \ldots, r_n\}$ are constants and b_{ij} are bounded functions on $(0,T], T < \infty$ for $i,j = 1,2\ldots,n$ will be studied.

2. Nonlinear systems

Consider the nonlinear nonautonomous nonlinear discontinuous dynamical system (13) and (14).

Theorem 1. *If the functions f_i, $i = 1,2\ldots,n$ are continuous on $[0,T]$ and satisfy the Lipschitz condition*

$$|f_i(t, x_1, x_2, \ldots, x_n) - f_i(t, y_1, y_2, \ldots, y_n)| \leqslant k_i \sum_{j=1}^{n} |x_j - y_j|,$$

then the discontinuous dynamical system (13) and (14) has a unique integrable solution $x \in L_1(0,T]$, $T < \infty$.

This solution is uniformly Lyapunov stable in the sense that $\forall \epsilon > 0$, there exists $\delta > 0$ such that

$$||G - G^*|| = \sum_{i=1}^{n} ||g_i - g_i^*||_{L_1(-r,0]} < \delta,\ \text{implies} ||X - X^*||_{{}_nL_1(0,T]} < \epsilon.$$

where $G = (g_i)_{n\times 1}$, $G^* = (g_i^*)_{n\times 1}$, $X = (x_i)_{n\times 1}$ and $X^* = (x_i^*)_{n\times 1}$.

Proof. Define the map $F = (F_i)_{n\times 1}$ where F_i: $L_1[0,T] \to L_1[0,T]$, $i = 1,2,\ldots,n$ are defined by

$$F_i x_i(t) = f_i(t, x_1(t-r_1), x_2(t-r_2), \ldots, x_n(t-r_n)),$$

we find that

$$|F_i x_i(t) - F_i y_i(t)| \leqslant k_i \sum_{j=1}^{n} |x_j(t-r_j) - y_j(t-r_j)|$$

and

$$\begin{aligned}
||F_i x_i - F_i y_i||_{L_1(0,T]} &\leqslant k_i \sum_{j=1}^{n} e^{-Nr_j} \int_0^T e^{-N(t-r_j)}|x_j(t-r_j) - y_j(t-r_j)|dt \\
&= k_i \sum_{j=1}^{n} e^{-Nr_j} \int_{r_j}^T e^{-N(t-r_j)}|x_j(t-r_j) - y_j(t-r_j)|dt \\
&= k_i \sum_{j=1}^{n} e^{-Nr_j} \int_0^{T-r_j} e^{-N(s_j)}|x_j(s_j) - y_j(s_j)|ds \\
&\leqslant k_i \sum_{j=1}^{n} e^{-Nr_j} \int_0^T e^{-N(\tau)}|x_j(\tau) - y_j(\tau)|d\tau \\
&\leqslant (k_i \sum_{j=1}^{n} e^{-Nr_j})||X - Y||_{{}_nL_1(0,T]}.
\end{aligned}$$

This implies that

$$\begin{aligned}
||FX - FY||_{{}_nL_1(0,T]} &= \sum_{i=1}^{n} ||F_i x_i - F_i y_i||_{L_1(0,T]} \\
&\leqslant \left(\sum_{i=1}^{n} k_i \sum_{j=1}^{n} e^{-Nr_j}\right)||X - Y||_{{}_nL_1(0,T]}.
\end{aligned}$$

Choose N so large enough such that $(\sum_{i=1}^{n} k_i \sum_{j=1}^{n} e^{-Nr_j}) < 1$ we deduce by the contraction fixed point theorem that there exists a unique solution $x \in L_1(0,T]$of the problem (13)–(14) [3].

Consider now the problem consisting of Eq. (13) and the initial data

$$x_i(t) = g_i^*(t) \in L_1(-r, 0],\ \lim_{t\to 0} g_i^*(t) = x_i^*(0), \quad i = 1, 2\ldots, n. \tag{16}$$

Then we have

$$\begin{aligned}
||x_i - x_i^*|| &\leqslant k_i \sum_{j=1}^{n} e^{-Nr_j} \int_0^T e^{-N(t-r_j)}|x_j(t-r_j) - y_j(t-r_j)|dt \\
&= k_i \sum_{j=1}^{n} e^{-Nr_j} \left[\int_0^{r_j} e^{-N(t-r_j)}|g_j(t-r_j) - g_j^*(t-r_j)|dt \right. \\
&\quad \left. + \int_0^T e^{-N(t-r_j)}|x_j(t-r_j) - y_j(t-r_j)|dt\right] \\
&= k_i \sum_{j=1}^{n} e^{-Nr_j} \left[\int_{-r_j}^0 e^{-N\tau_j}|g_j(\tau_j) - g_j^*(\tau_j)|d\tau \right. \\
&\quad \left. + \int_0^{T-r_j} e^{-N(s_j)}|x_j(s_j) - x_j^*(s_j)|ds\right] \\
&\leqslant k_i \sum_{j=1}^{n} e^{-Nr_j} \left[\int_{-r}^0 e^{-N\theta}|g_j(\theta) - g_j^*(\theta_j)|d\theta \right. \\
&\quad \left. + \int_0^T e^{-N(u)}|x_j(u) - x_j^*(u)|du\right] \\
&= k_i \sum_{j=1}^{n} e^{-Nr_j} \left[||g_j - g_j^*||_{L_1(-r,0]} + ||x_j - x_j^*||_{L_1(0,T]}\right] \\
&\leqslant \left(k_i \sum_{j=1}^{n} e^{-Nr_j}\right)[||G - G^*||_{L_1(-r,0]} + ||X - X^*||_{{}_nL_1(0,T]}].
\end{aligned}$$

This implies that

$$||X - X^*||_{_nL_1(0,T]} \leqslant \left(\sum_{i=1}^n k_i \sum_{j=1}^n e^{-Nr_j}\right)\left[||G - G^*||_{L_1(-r,0]} + ||X - X^*||_{_nL_1(0,T]}\right]. \quad (17)$$

Hence

$$||X - X^*||_{_nL_1(0,T]} \leqslant \left(1 - \sum_{i=1}^n k_i \sum_{j=1}^n e^{-Nr_j}\right)^{-1}\left(\sum_{i=1}^n k_i \sum_{j=1}^n e^{-Nr_j}\right) ||G - G^*||_{L_1(-r,0]} \quad (18)$$

and the result follows. Consider the problem consisting of Eq. (13)

$$x_i(t) = f_i(t, x_1(t - r_1), x_2(t - r_2), \ldots, x_n(t - r_n)), \quad t,\ r_i > 0$$

and the initial data

$$x_i(t) = x_{io}, \quad t \leqslant 0, \quad i = 1, 2 \ldots, n. \quad (19)$$

Then the following corollary can be proved. □

Corollary 1. *If the assumptions of Theorem 1 are satisfied, then the discontinuous dynamical system (13) and (19) has a unique integrable solution $x \in L_1(0,T]$, $T < \infty$.*

This solution is uniformly Lyapunov stable [2] in the sense that $\forall \epsilon > 0$, there exists $\delta > 0$ such that

$$||X_o - X_o^*||^* = \sum_{i=1}^n |x_{io} - x_{io}^*| < \delta, \text{ implies } ||X - X^*||_{_nL_1(0,T]} < \epsilon.$$

where $X_o = (x_{i\ o})_{n\times 1}$ and $X_o^* = (x_{io}^*)_{n\times 1}$.

Corollary 2. *If the function f satisfies the Lipschitz condition*

$$|f(t, x_1, x_2, \ldots, x_n) - f_i(t, y_1, y_2, \ldots, y_n)| \leqslant k \sum_{j=1}^n |x_j - y_j|,$$

then the discontinuous dynamical system

$$x(t) = f(t, x(t - r_1), x(t - r_2), \ldots, x(t - r_n)), \quad t,\ r_i > 0 \quad (20)$$

$$x(t) = g(t) \in L_1(-r, 0],\ \lim_{t\to 0} g(t) = x(0), r = \min\{r_1, r_2, \ldots, r_n\} \quad (21)$$

has a unique integrable solution $x \in L_1(0,T]$. This solution is uniformly Lyapunov stable.

Example 1. Let ρ be a bounded function on $(0,T]$. The nonautonomous nonlinear discontinuous dynamical system of the Logistic equation

$$x(t) = \rho(t)x(t - r_1)(1 - x(t - r_2), t > 0 \text{ and } r_2 > r_1 > 0 \quad (22)$$

with the initial data

$$x(t) = g(t) \in L_1(-r_1, 0], \lim_{t\to 0} g(t) = x(0)$$

has a unique integrable solution $x \in L_1(0,T]$. This solution is uniformly Lyapunov stable.

3. Linear systems

Consider the linear nonautonomous nonlinear discontinuous dynamical system of the equation

$$x_i(t) = \sum_{j=1}^n b_{ij}(t)x_j(t - r_j) \quad (23)$$

with the data (14)

$$x_i(t) = g_i(t) \in L_1(-r, 0],\ \lim_{t\to 0} g_i(t) = x_i(0)$$

where $r_j > 0$, $r = \min\{r_1, r_2, \ldots, r_n\}$ are constants and b_{ij} are bounded functions on $(0,T]$, $T < \infty$ for $i,j = 1,2\ldots,n$.

Let f_i in (13) be given by

$$f_i(t, x_1(t - r_1), x_2(t - r_2),\ dots,\ x_n(t - r_n)) = \sum_{j=1}^n b_{ij}(t)x_j(t - r_j).$$

Then, by the same way as in Theorem 1, we can prove the following theorem

Theorem 2. *Let $b_{ij}(t)$ are bounded on $(0,T]$. Then the discontinuous dynamical system (23) and (14) has a unique integrable solution $x \in L_1(0,T]$, $T < \infty$. This solution is uniformly Lyapunov stable.*

Consider now the discontinuous dynamical system (15)–(14)

$$x_i(t) = \sum_{j=1}^n a_{ij}x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t - r_j)$$

$$x_i(t) = g_i(t) \in L_1(-r, 0],\ \lim_{t\to 0} g_i(t) = x_i(0)$$

where a_{ij} and $r_j > 0$, $r = min\{r_1, r_2, \ldots, r_n\}$ are constants and b_{ij} are bounded functions on $(0,T]$, $T < \infty$ for $i,j = 1,2\ldots,n$ will be studied.

As a corollary of Theorem 2, we can prove the following theorem for the discontinuous dynamical system (15) and (14).

Theorem 3. *Let $b_{ij}(t)$are bounded on $(0,T]$. If the matrix $I - A, A = (a_{ij}), i,j = 1,2\ldots,n$ is nonsingular matrix, then the discontinuous dynamical system (15)–(14) has a unique integrable solution $x \in L_1(0,T]$, $T < \infty$. This solution is uniformly Lyapunov stable.*

Proof. Let $B(t) = (b_{ij}(t))_{n\times n}$ and $C(t) = (I - A)^{-1}B(t) = \left(\sum_{i,j=1}^n a_{ij}b_{ij}(t)\right) = (c_{ij}(t))$, then $c_{ij}(t)$ are bounded on $(0,T]$.

Let $'$ means the transpose of the matrix and write (15) in the form

$$X(t) = AX(t) + B(t)(x_1(t - r_1), x_2(t - r_2), \ldots, x(t - r_n))',$$

then

$$(I - A)X(t) = B(t)(x_1(t - r_1), x_2(t - r_2), \ldots, x(t - r_n))',$$

which implies that

$$X(t) = (I - A)^{-1}B(t)(x_1(t - r_1), x_2(t - r_2), \ldots, x(t - r_n))'$$

and (15) can be written as

$$x_i(t) = \sum_{j=1}^n c_{ij}(t)x_j(t - r_j)$$

and the proof follows from Theorem 2. □

Corollary 3. *The discontinuous dynamical system (15)–(19)*

$$x_i(t) = \sum_{j=1}^n a_{ij}x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t - r_j)$$

$$x_i(t) = x_{io}, \quad t \leqslant 0, \quad i = 1, 2 \ldots, n$$

has a unique integrable solution $x \in L_1(0, T]$, $T < \infty$. This solution is uniformly Lyapunov stable.

References

[1] S. Elaydi, An Introduction to Difference Equations, third ed., Springer, 2004.
[2] G.A. Coddington, N. Levinson, Theory of Differential Equations, McGaraw-Hill Book Company, New York, 1955.
[3] R.F. Curtain, A.J. Pritchard, Functional Analysis in Modern Applied Mathematics, Academic press, 1977.
[4] Oded. Galor, Discrete Dynamical System, Springer, 2007.