Journal of the Egyptian Mathematical Society (2016) **24**, 244–249

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

A new 2-inner product on the space of *p*-summable sequences

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Received 14 February 2015; revised 29 May 2015; accepted 1 July 2015

Keywords

2-inner product spaces;
2-normed spaces;
p-summable sequences;
Weights

Abstract In this paper, we wish to define a 2-inner product, non-standard, possibly with weights, on ℓ^p. For this purpose, we aim to obtain a different 2-norm $\|.,.\|_{2,v,w}$, which is not equivalent to the usual 2-norm $\|.,.\|_p$ on ℓ^p (except with the condition $p=2$), satisfying the parallelogram law. We discuss the properties of the induced 2-norm $\|.,.\|_{2,v,w}$ and its relationships with the usual 2-norm on ℓ^p. We also find that the 2-inner product $\langle .,.|.\rangle_{v,w}$ is actually defined on a larger space.

2010 MATHEMATICS SUBJECT CLASSIFICATION: Primary 46C50; Secondary 46B20; 46C05; 46C15; 46B99

1. Introduction

By $\ell^p = \ell^p(\mathbb{R})$ we denote the space of all p-summable sequences of real numbers. For $p \neq 2$, $(\ell^p, \|x\|_p)$ is not an inner product space, since the usual norm $\|x\|_p = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$ on ℓ^p does not satisfy the parallelogram law. One alternative is to define a semi-inner product on ℓ^p as in [1], but having a semi-inner product is not as nice as having an inner product.

In [2], Gunawan defined a usual 2-norm $\|.,.\|_p$ on the space of p-summable sequences of real numbers. The usual 2-norm $\|.,.\|_p$ also is not a 2-inner product with $p \neq 2$ because it does not satisfy the parallelogram law.

In this paper, we eventually wish to define a 2-inner product $\langle .,.|.\rangle_{v,w}$, non-standard, possibly with weights, on ℓ^p , so that orthogonality and many other notions on this space can be defined. For this purpose, we aim to obtain a different 2-norm $\|.,.\|_{2,v,w}$, which is not equivalent to the usual 2-norm $\|.,.\|_p$ on ℓ^p (except with the condition $p=2$), not only satisfies the parallelogram law, but also it is non-standard. For $p > 2$, we also obtain a result which describes how the weighted 2-inner product space is associated to the weights. We discuss the properties of the induced 2 -norm $\|.,.\|_{2,v,w}$ and its relationships with

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2015.07.001

the usual 2-norm on ℓ^p. We also find that the 2-inner product $\langle ., . | . \rangle_{v,w}$ is actually defined on a larger space.

2. Definitions and preliminaries

Let X be a real vector space of dimension $d \geq 2$. The real-valued function $\langle \cdot, \cdot | \cdot \rangle$ which satisfies the following properties on X^3 is called 2-inner product on X, and the pair $(X, \langle \cdot, \cdot | \cdot \rangle)$ is called a 2-inner product space:

1. $\langle x, x | z\rangle \geq 0$; $\langle x, x|z\rangle = 0$ if and only if x and z are linearly dependent,
2. $\langle x, y|z\rangle = \langle y, x|z\rangle$,
3. $\langle x, x|z\rangle = \langle z, z|x\rangle$,
4. $\langle \alpha x, y|z\rangle = \alpha\langle x, y|z\rangle$, for $\alpha \in R$,
5. $\langle x_1 + x_2, y|z\rangle = \langle x_1, y|z\rangle + \langle x_2, y|z\rangle$.

The function $\|., .\|$: $X \times X \to [0, \infty)$, which follows four properties, is called a 2-norm and the pair $(X, \| \cdot, \cdot\|)$ is called a 2-normed space:

1. $\|x, z\| = 0$ if and only if x and z are linearly dependent,
2. $\|x, z\| = \|z, x\|$, for $x, z \in X$,
3. $\|\alpha x, z\| = |\alpha|\|x, z\|$, for $x, z \in X$ and $\alpha \in \mathbb{R}$,
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$, for $x, y, z \in X$.

Definition 2.1. A sequence $(x^{(n)})$ in a linear 2-normed space $(X, \|., .\|)$ is called a Cauchy sequence, if for every y in X, $\lim_{n,m\to\infty}\left\|x^{(n)} - x^{(m)}, y\right\| = 0$, [3].

Definition 2.2. Let $\{a_1, a_2\}$ be a linearly independent set on a 2-normed space $(X, \|., .\|)$. A sequence $(x^{(n)})$ in X is called a Cauchy sequence with respect to the set $\{a_1, a_2\}$ if $\|x^{(n)} - x^{(m)}, a_1\| \to 0$ and $\|x^{(n)} - x^{(m)}, a_2\| \to 0$ as $m, n \to \infty$, [4].

Definition 2.3. A sequence $(x^{(n)})$ in a linear 2-normed space $(X, \|., .\|)$ is called a convergent sequence, if there is an x in X such that for every y in X, $\lim_{n\to\infty}\left\|x^{(n)} - x, y\right\| = 0$, [4].

Definition 2.4. Let $\{a_1, a_2\}$ be a linearly independent set on a 2-normed space $(X, \|., .\|)$. A sequence $(x^{(n)})$ in X is called a convergent sequence with respect to the set $\{a_1, a_2\}$ if there exists an $x \in X$ such that $\left\|x^{(n)} - x, a_1\right\| \to 0$ and $\left\|x^{(n)} - x, a_2\right\| \to 0$ as $n \to \infty$.

Definitions 2.1 and 2.3 are clearly stronger than Definitions 2.2 and 2.4. But in some cases, like in finite dimensional case and the standard case the respective two definitions are equivalent, which is not clear in the infinite dimensional case. But from the results in [5], we understand that the respective two definitions are still equivalent in the spaces ℓ^p and L^p (the space of p-integrable functions).

As we work with sequence spaces of real numbers, we will use the sum notation $\sum_k$ instead of $\sum_{k=1}^{\infty}$, for brevity. Throughout the paper we will use the following inequalities, (see, [6]):

$$\sum_k |a_k + b_k|^p \leq \sum_k |a_k|^p + \sum_k |b_k|^p \quad (0 < p \leq 1). \tag{2.1}$$

$$\left(\sum_{k=1}^{n} |a_k|\right)^p \leq n^{p-1} \sum_{k=1}^{n} |a_k|^p \quad (p \geq 1). \tag{2.2}$$

Note: The inequality (2.2) is a case of Hölder's inequality in the finite dimensional.

3. Main results

In this section, we begin with observation on ℓ^p, $1 \leq p < \infty$. It is well known that there exists $x \in \ell^q$ but $x \in \ell^p$ while $1 \leq p < q \leq \infty$. As sets, we have $\ell^p \subseteq \ell^q$ and the inclusion is strict. So, for $1 \leq p < q$, ℓ^p can actually be considered as a subspace of ℓ^q. For $q = 2$, we know that the norm $\| \cdot \|_2$ satisfies the parallelogram law. Then we can equip ℓ^p, $1 \leq p < 2$, with $\| \cdot \|_2$, so that ℓ^p became an inner product space with the inner product

$$\langle x, y\rangle := \sum_k x_k y_k.$$

Similarly, we realize that ℓ^p is a subspace of $(\ell^2, \|., .\|_2)$. Here, ℓ^p can be equipped with the 2-inner product

$$\langle x, y|z \rangle := \frac{1}{2} \sum_{k_1} \sum_{k_2} \begin{vmatrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix} \begin{vmatrix} y_{k_1} & y_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix}$$

and the 2-norm

$$\|x, z\|_2 : = \left(\frac{1}{2} \sum_{k_1} \sum_{k_2} \begin{vmatrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix}^2 \right)^{\frac{1}{2}}.$$

We can check that the 2-norm $\|., .\|_2$ satisfies the parallelogram law:

$$\|x + y, z\|_2^2 + \|x - y, z\|_2^2 = 2\|x, z\|_2^2 + 2\|y, z\|_2^2$$

for every $x, y, z \in \ell^p$.

Next, we work on ℓ^p, $2 < p < \infty$. We note that the space ℓ^p is now larger than ℓ^2. Consequently, the usual 2-inner product and 2-norm on ℓ^2 are not used for all sequences in ℓ^p. Here, we present a new definition of 2-inner product and 2-norm on ℓ^p, which satisfies the parallelogram law, using weights.

For arbitrary $v = (v_k) \in \ell^p$, $v_k > 0$ ($\forall k \in \mathbb{N}$), set of ℓ_v^2 is defined by

$$\ell_v^2 := \left\{ x = (x_k) : \sum_k v_k^{p-2} x_k^2 < \infty, \ v = (v_k) \in \ell^p, \ v_k > 0 \ (\forall k \in \mathbb{N}) \right\}.$$

As set, we observe $\ell^p \subset \ell_v^2$ and the inclusion is strict. It is also known that v is not unique. Thus we have V_p, the collection of all sequences $v = (v_k) \in \ell^p$ where $v_k > 0$ for every $k \in \mathbb{N}$. Let v, w be in V_p, then v and w are *equivalent*, write $v \sim w$, if and only if there exists a constant $C > 0$ such that

$$\frac{1}{C} v_k \leq w_k \leq C v_k$$

for every $k \in \mathbb{N}$.

Let $v \sim w$ and $\begin{vmatrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{vmatrix} \neq 0$ if $k_1 \neq k_2$. Next we define the mapping which maps every triple of sequences $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ from ℓ^p to

$$\langle x, y|z\rangle_{v,w} := \frac{1}{2} \sum_{k_1} \sum_{k_2} \left\| \begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{matrix} \right\|^{p-2} \begin{vmatrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2} \end{vmatrix} \begin{vmatrix} y_{k_1} & z_{k_1} \\ y_{k_2} & z_{k_2} \end{vmatrix} \tag{3.1}$$

and the mapping $\|.,.\|_{2,v,w}$ which maps every pair of sequences $x = (x_k)$ and $z = (z_k)$ from ℓ^p to

$$\|x, z\|_{2,v,w} := \left[\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2}\end{matrix}\right\|^{p-2}\left|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right|^2\right]^{\frac{1}{2}}. \tag{3.2}$$

We observe that the mappings are well-defined on ℓ^p. Indeed, for $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ in ℓ^p, it follows from Hölder's Inequality that

$$\begin{aligned}
&\left|\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2}\end{matrix}\right\|^{p-2}\left|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right|\left|\begin{matrix} y_{k_1} & z_{k_1} \\ y_{k_2} & z_{k_2}\end{matrix}\right|\right| \\
&\leq \left[\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2}\end{matrix}\right\|^{p}\right]^{\frac{p-2}{p}} \\
&\quad\times\left[\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right\|^{\frac{p}{2}}\left\|\begin{matrix} y_{k_1} & z_{k_1} \\ y_{k_2} & z_{k_2}\end{matrix}\right\|^{\frac{p}{2}}\right]^{\frac{2}{p}} \\
&\leq \|v, w\|_p^{p-2}\left[\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right\|^{p}\right]^{\frac{1}{p}} \\
&\quad\times\left[\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} y_{k_1} & z_{k_1} \\ y_{k_2} & z_{k_2}\end{matrix}\right\|^{p}\right]^{\frac{1}{p}}.
\end{aligned} \tag{3.3}$$

Thus the two mappings are defined on ℓ^p. Moreover, we have the following proposition, whose proof is left to the reader.

Proposition 3.1. *The mappings in (3.1) and (3.2) define a weighted 2-inner product and a weighted 2-norm, respectively, on ℓ^p.*

From (3.3), we see that the following inequality

$$\|x, z\|_{2,v,w} \leq \|v, w\|_p^{\frac{p-2}{2}}\,\|x, z\|_p \tag{3.4}$$

holds for every $x, z \in \ell^p$. It is then tempting to ask whether the two 2-norms are equivalent on ℓ^p. The answer, however, is negative, due to the following result.

Proposition 3.2. *There is no constant $C > 0$ such that*

$$\|x, z\|_p \leq C\|x, z\|_{2,v,w}$$

for every $x, z \in \ell^p$.

Proof. Let $\{z_1, z_2\}$ be a linearly independent set where $z_1 = (1, 0, \ldots)$ and $z_2 = (0, 1, 0, \ldots)$. Suppose that such a constant exists. Then, for $x := e_n = (0, \ldots, 0, 1, 0, \ldots)$, where the 1 is the nth term, we have

$$1 \leq C|v_i w_n - w_i v_n|^{\frac{p-2}{2}}$$

for each $n \in \mathbb{N}$ and for each $i = 1, 2$, since $\|x, z_i\|_p = 1$ and $\|x, z_i\|_{2,v,w} = |v_i w_n - w_i v_n|^{\frac{p-2}{2}}$. But this cannot be true, since v_n, $w_n \to 0$ as $n \to \infty$. $\square$

According to Proposition 3.2, it is possible to find a sequence in ℓ^p which is divergent with respect to the 2-norm $\|.,.\|_p$, but convergent with respect to the 2-norm $\|.,.\|_{2,v,w}$.

Example 3.1. Let $x^{(n)} := e_n \in \ell^p$, where $e_n = (0, \ldots, 0, 1, 0, \ldots)$ (the 1 is the nth term) and let $\{z_1, z_2\}$ be a linearly independent set where $z_1 = (1, 0, \ldots)$ and $z_2 = (0, 1, 0, \ldots)$, then $\left\|x^{(n)} - x^{(m)}, z_i\right\|_p = 2^{\frac{1}{p}} \nrightarrow 0$ as $m, n \to \infty$. Since $(x^{(n)})$ is not a Cauchy sequence with respect to $\|.,.\|_p$, it is not convergent with respect to $\|.,.\|_p$. However, $\left\|x^{(n)}, z_i\right\|_{2,v,w} = |v_i w_n - w_i v_n|^{\frac{p-2}{2}} \to 0$ as $n \to \infty$, since $v_n, w_n \to 0$ as $n \to \infty$. Hence, $(x^{(n)})$ is convergent with respect to the 2-norm $\|.,.\|_{2,v,w}$.

If we wish, we can also define another weighted 2-norm $\|.,.\|_{\beta,v,w}$ on ℓ^p, where $1 \leq \beta \leq p < \infty$, by

$$\|x, z\|_{\beta,v,w} := \left[\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2}\end{matrix}\right\|^{p-\beta}\left\|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right\|^{\beta}\right]^{\frac{1}{\beta}}.$$

Here p may be less than 2. Note that if $\beta = p$, then $\|.,.\|_{\beta,v,w} = \|.,.\|_p$.

The following proposition gives a relationship between two such weighted 2-norms on ℓ^p.

Proposition 3.3. *Let $1 \leq \beta < \gamma \leq p$. Then we have*

$$\|x, z\|_{\beta,v,w} \leq \|v, w\|_p^{\frac{p(\gamma-\beta)}{\gamma\beta}}\,\|x, z\|_{\gamma,v,w}$$

for every $x, z \in \ell^p$.

Proof. Suppose $x, z \in \ell^p$ we compute

$$\begin{aligned}
\|x, z\|_{\beta,v,w}^{\beta} &= \frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2}\end{matrix}\right\|^{p-\beta}\left\|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right\|^{\beta} \\
&= \frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2}\end{matrix}\right\|^{\frac{p(\gamma-\beta)}{\gamma}}\left\|\begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2}\end{matrix}\right\|^{\frac{(p-\gamma)\beta}{\gamma}}\left\|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right\|^{\beta} \\
&\leq \frac{1}{2}\left[\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2}\end{matrix}\right\|^{p}\right]^{\frac{\gamma-\beta}{\gamma}} \\
&\quad\times\left[\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2}\end{matrix}\right\|^{p-\gamma}\left\|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right\|^{\gamma}\right]^{\frac{\beta}{\gamma}} \\
&= \|v, w\|_p^{\frac{p(\gamma-\beta)}{\gamma}}\,\|x, z\|_{\gamma,v,w}^{\beta}.
\end{aligned}$$

Taking βth roots of both sides, we obtain $\|x, z\|_{\beta,v,w} \leq \|v, w\|_p^{\frac{p(\gamma-\beta)}{\gamma\beta}}\,\|x, z\|_{\gamma,v,w}$. $\square$

Corollary 3.4. *If $1 \leq \beta < 2 < \gamma \leq p$. Then there are constants $C_1, C_2 > 0$ such that*

$$C_1\|x, z\|_{\beta,v,w} \leq \|x, z\|_{2,v,w} \leq C_2\|x, z\|_{\gamma,v,w}$$

for every $x, z \in \ell^p$.

4. Further results

In the main result, we have $\ell^p \subset \ell^2_v$ (as sets), for $2 < p < \infty$, and the inclusion is strict. With respect to the 2-norms on these spaces as we have seen in the beginning of this section, for every

$x, z \in \ell^p$ we have $\|x, z\|_{2,v,w} < \infty$. This suggests that ℓ^p lives inside a larger space, consisting all x, z with $\|x, z\|_{2,v,w} < \infty$.

Proposition 4.1.

1. *If $x, z \in \ell^p$ with $\|x, z\|_p < \infty$, then $x, z \in \ell^2_v$ with $\|x, z\|_{2,v,w} < \infty$.*
2. *The converse is not true.*

Proof. Let $x, z \in \ell^p$ with $\|x, z\|_p < \infty$. It follows from (3.4) that $\|x, z\|_{2,v,w} \le \|v, w\|_p^{\frac{p-2}{2}} \|x, z\|_p$, which means that $x, z \in \ell^2_v$ with $\|x, z\|_{2,v,w} < \infty$. To show that the converse is not true, we need to find $\|x, z\|_{2,v,w} < \infty$ but $\|x, z\|_p = \infty$. We know that $v_k > 0$, $w_k > 0$ for all $k \in \mathbb{N}$, and $v_k \to 0$, $w_k \to 0$ as $k \to \infty$. So, choose $m_1 \in \mathbb{N}$ such that $v_{m_1}^{p-2} < \frac{1}{2}$ and $w_{m_1}^{p-2} < \frac{1}{3}$, $m_2 > m_1$ such that $v_{m_2}^{p-2} < \frac{1}{2^2}$ and $w_{m_2}^{p-2} < \frac{1}{3^2}$, $m_3 > m_2$ such that $v_{m_3}^{p-2} < \frac{1}{2^3}$ and $w_{m_3}^{p-2} < \frac{1}{3^3}$, and so on. Since the process never stops, we obtain an increasing sequence of nonnegative integers (m_j) such that $v_{m_j}^{p-2} < 2^{-j}$ and $w_{m_j}^{p-2} < 3^{-j}$ for every $j \in \mathbb{N}$. Now put $x_k = 1$ for $k = m_1, m_2, m_3, \ldots$ and $x_k = 0$ otherwise. Let $\{e_1, e_2\}$ be a linearly independent set where $e_1 = (1, 0, \ldots)$ and $e_2 = (0, 1, 0, \ldots)$. We will give the proof for $1 < p - 2 < \infty$, by using the inequality (2.2) and the triangle inequality. The proof for $0 < p - 2 \le 1$ can be done similarly by using the inequality (2.1) and the triangle inequality. Hence for $i = 1, 2$, we have

$$
\begin{aligned}
\|x, e_i\|^2_{2,v,w} &= \frac{1}{2}\sum_{k_1}\sum_{k_2} \left\| \begin{vmatrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{vmatrix} \right\|^{p-2} \left\| \begin{vmatrix} x_{k_1} & e_{ik_1} \\ x_{k_2} & e_{ik_2} \end{vmatrix} \right\|^2 \\
&\le \frac{1}{2}\sum_{k} (|w_i||v_k| + |v_i||w_k|)^{p-2}|x_k|^2 \\
&\quad + \frac{1}{2}\sum_{k} (|v_i||w_k| + |v_k||w_i|)^{p-2}|x_k|^2 \\
&\le 2^{p-3}{v_i}^{p-2}\sum_j {w_{m_j}}^{p-2} + 2^{p-3}{w_i}^{p-2}\sum_j {v_{m_j}}^{p-2} \\
&< 2^{p-3}{v_i}^{p-2}\sum_j \frac{1}{3^j} + 2^{p-3}{w_i}^{p-2}\sum_j \frac{1}{2^j} \\
&= 2^{p-4}{v_i}^{p-2} + 2^{p-3}{w_i}^{p-2} < \infty,
\end{aligned}
$$

where $i = 1, 2$

$$
\|x, e_i\|^p_p = \frac{1}{2}\sum_{k_1}\sum_{k_2} \left\| \begin{vmatrix} x_{k_1} & e_{ik_1} \\ x_{k_2} & e_{ik_2} \end{vmatrix} \right\|^p = \infty. \qquad \square
$$

Theorem 4.2. *The space $\left(\ell^2_v, \|., .\|_{2,v,w}\right)$ is a 2-Banach space. Accordingly, $\left(\ell^2_v, \langle ., .|.\rangle_{v,w}\right)$ is a 2-Hilbert space.*

Proof. It is easy to see that the space $\left(\ell^2_v, \|., .\|_{2,v,w}\right)$ is a linear 2-normed space, so we omit the details. To prove the completeness, let $(x^{(n)})$ be any Cauchy sequence in the space $(l^2_v, \|., .\|_{2,v,w})$ where $x^{(n)} = \left(x^{(n)}_k\right) = \left(x^{(n)}_1, x^{(n)}_2, \ldots\right)$. Then for every nonzero $z \in \ell^2_v$, for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$

$$
\begin{aligned}
&\left\|x^{(n)} - x^{(m)}, z\right\|_{2,v,w} \\
&= \left(\frac{1}{2}\sum_{k_1}\sum_{k_2} \left\| \begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{matrix} \right\|^{p-2} \left| \begin{matrix} x^{(n)}_{k_1} - x^{(m)}_{k_1} & z_{k_1} \\ x^{(n)}_{k_2} - x^{(m)}_{k_2} & z_{k_2} \end{matrix} \right|^2 \right)^{\frac{1}{2}} < \frac{\varepsilon}{2}.
\end{aligned} \tag{4.1}
$$

Since this is true for every nonzero $z \in l^2_v$, then we can choose privately, $u = u_k := (-1)^k z_k$. The vectors u and z are clearly linearly independent. Then from the definition of Cauchy sequence in a 2-normed space, we have

$$
\begin{aligned}
&\left\|x^{(n)} - x^{(m)}, z\right\|^2_{2,v,w} \\
&= \frac{1}{2}\sum_{k_1}\sum_{k_2} \left\| \begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{matrix} \right\|^{p-2} \left| \begin{matrix} x^{(n)}_{k_1} - x^{(m)}_{k_1} & z_{k_1} \\ x^{(n)}_{k_2} - x^{(m)}_{k_2} & z_{k_2} \end{matrix} \right|^2 < \frac{\varepsilon^2}{4}
\end{aligned} \tag{4.2}
$$

and

$$
\begin{aligned}
&\left\|x^{(n)} - x^{(m)}, u\right\|^2_{2,v,w} \\
&= \frac{1}{2}\sum_{k_1}\sum_{k_2} \left\| \begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{matrix} \right\|^{p-2} \left| \begin{matrix} x^{(n)}_{k_1} - x^{(m)}_{k_1} & u_{k_1} \\ x^{(n)}_{k_2} - x^{(m)}_{k_2} & u_{k_2} \end{matrix} \right|^2 < \frac{\varepsilon^2}{4}.
\end{aligned} \tag{4.3}
$$

From (4.2) and (4.3) we can write

$$
\begin{aligned}
I_1 + I_2 := &\frac{1}{2} \sum_{\substack{k_1 \\ k_1+k_2=\text{odd}}} \sum_{k_2} \left\| \begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{matrix} \right\|^{p-2} \\
&\times \left[\left(x^{(n)}_{k_1} - x^{(m)}_{k_1}\right)z_{k_1} - \left(x^{(n)}_{k_2} - x^{(m)}_{k_2}\right)z_{k_2}\right]^2 \\
&+ \frac{1}{2} \sum_{\substack{k_1 \\ k_1+k_2=\text{even}}} \sum_{k_2} \left\| \begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{matrix} \right\|^{p-2} \\
&\times \left[\left(x^{(n)}_{k_1} - x^{(m)}_{k_1}\right)z_{k_1} - \left(x^{(n)}_{k_2} - x^{(m)}_{k_2}\right)z_{k_2}\right]^2 \\
&< \frac{\varepsilon^2}{4}
\end{aligned}
$$

and

$$
\begin{aligned}
H_1 + H_2 := &\frac{1}{2} \sum_{\substack{k_1 \\ k_1+k_2=\text{odd}}} \sum_{k_2} \left\| \begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{matrix} \right\|^{p-2} \\
&\times \left[\left(x^{(n)}_{k_1} - x^{(m)}_{k_1}\right)u_{k_1} - \left(x^{(n)}_{k_2} - x^{(m)}_{k_2}\right)u_{k_2}\right]^2 \\
&+ \frac{1}{2} \sum_{\substack{k_1 \\ k_1+k_2=\text{even}}} \sum_{k_2} \left\| \begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{matrix} \right\|^{p-2} \\
&\times \left[\left(x^{(n)}_{k_1} - x^{(m)}_{k_1}\right)u_{k_1} - \left(x^{(n)}_{k_2} - x^{(m)}_{k_2}\right)u_{k_2}\right]^2 \\
&< \frac{\varepsilon^2}{4}.
\end{aligned}
$$

Clearly, $I_1 + H_1 < \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4} = \frac{\varepsilon^2}{2}$, that is;

$$
\begin{aligned}
&\frac{1}{2} \sum_{\substack{k_1 \\ k_1+k_2=\text{odd}}} \sum_{k_2} \left\| \begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2} \end{matrix} \right\|^{p-2} \left(\left[\left(x^{(n)}_{k_1} - x^{(m)}_{k_1}\right)z_{k_2} - \left(x^{(n)}_{k_1} - x^{(m)}_{k_1}\right)z_{k_1}\right]^2\right. \\
&\left. + \left[\left(x^{(n)}_{k_1} - x^{(m)}_{k_1}\right)u_{k_1} - \left(x^{(n)}_{k_2} - x^{(m)}_{k_2}\right)u_{k_2}\right]^2\right) \\
&< \frac{\varepsilon^2}{2}.
\end{aligned} \tag{4.4}
$$

Now, check H_1. If $k_1 + k_2$ is odd, then we have two cases:

1. If k_1 is odd and k_2 is even;
2. If k_1 is even and k_2 is odd.

For both cases, we obtain the following.

$$\begin{aligned}
&\frac{1}{2}\sum_{\substack{k_1\\ k_1+k_2=\text{odd}}}\sum_{k_2}\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\\
&\quad\times\left[\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)u_{k_2}-\left(x_{k_2}^{(n)}-x_{k_2}^{(m)}\right)u_{k_1}\right]^2\\
&=\frac{1}{2}\sum_{\substack{k_1\\ k_1+k_2=\text{odd}}}\sum_{k_2}\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\\
&\quad\times\left[\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)(-1)^{k_2}z_{k_2}-\left(x_{k_2}^{(n)}-x_{k_2}^{(m)}\right)(-1)^{k_1}z_{k_1}\right]^2\\
&=\frac{1}{2}\sum_{\substack{k_1\\ k_1+k_2=\text{odd}}}\sum_{k_2}\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\\
&\quad\times\left[\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)z_{k_2}+\left(x_{k_2}^{(n)}-x_{k_2}^{(m)}\right)z_{k_1}\right]^2.
\end{aligned}$$

Then for every k_1 and k_2, we conclude that

$$\begin{aligned}
&\frac{1}{2}\sum_{\substack{k_1\\ k_1+k_2=\text{odd}}}\sum_{k_2}\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\\
&\quad\times\left[\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)u_{k_2}-\left(x_{k_2}^{(n)}-x_{k_2}^{(m)}\right)u_{k_1}\right]^2\\
&=\frac{1}{2}\sum_{\substack{k_1\\ k_1+k_2=\text{odd}}}\sum_{k_2}\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\\
&\quad\times\left[\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)z_{k_2}+\left(x_{k_2}^{(n)}-x_{k_2}^{(m)}\right)z_{k_1}\right]^2.
\end{aligned}\tag{4.5}$$

By (4.4) and (4.5), we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{\substack{k_1\\ k_1+k_2=\text{odd}}}\sum_{k_2}\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\\
&\quad\times\left(\left[\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)z_{k_2}-\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)z_{k_1}\right]^2\right.\\
&\quad\left.+\left[\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)z_{k_2}+\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)z_{k_1}\right]^2\right)\\
&<\frac{\varepsilon^2}{2}.
\end{aligned}\tag{4.6}$$

We know that, for every $a,\ b\in\mathbb{R}$ we have $(a-b)^2+(a+b)^2=2a^2+2b^2$. Then from (4.6), for all $n, m>n_0$ we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{\substack{k_1\\ k_1+k_2=\text{odd}}}\sum_{k_2}\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\left(\left[\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)z_{k_2}\right]^2\right.\\
&\quad\left.+\left[\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)z_{k_1}\right]^2\right)<\frac{\varepsilon^2}{4}.
\end{aligned}$$

Since for all $n, m>n_0$

$$\frac{1}{2}\sum_{\substack{k_1\\ k_1+k_2=\text{odd}}}\sum_{k_2}\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)^2 z_{k_2}^2<\frac{\varepsilon^2}{4}$$

and

$$\frac{1}{2}\sum_{\substack{k_1\\ k_1+k_2=\text{odd}}}\sum_{k_2}\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\left(x_{k_2}^{(n)}-x_{k_2}^{(m)}\right)^2 z_{k_1}^2<\frac{\varepsilon^2}{4}.$$

Then for every $k_1,\ k_2\in\mathbb{N}$ and for every $n, m>n_0$ we have

$$\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\left(x_{k_1}^{(n)}-x_{k_1}^{(m)}\right)^2 z_{k_2}^2<\frac{\varepsilon^2}{4}$$

and

$$\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\left(x_{k_2}^{(n)}-x_{k_2}^{(m)}\right)^2 z_{k_1}^2<\frac{\varepsilon^2}{4}.$$

Hence for every $n,\ m>n_0$ and for every $k_1,\ k_2\in\mathbb{N}$ $|x_{k_1}^{(n)}-x_{k_1}^{(m)}|^2<\varepsilon^2C^2_{1(k_1,k_2)}=\varepsilon_1$ and $|x_{k_2}^{(n)}-x_{k_2}^{(m)}|^2<\varepsilon^2C^2_{2(k_1,k_2)}=\varepsilon_2$ where $C^2_{1(k_1,k_2)}=\frac{1}{4\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}z_{k_2}^2}$ and $C^2_{2(k_1,k_2)}=\frac{1}{4\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}z_{k_1}^2}$. Thus for each fixed $k_1,\ k_2\in\mathbb{N}$, $(x_k^{(n)})$ is a Cauchy sequence of real numbers. Hence it is convergent, say $x_k^{(n)}\to x_k$ as $n\to\infty$ for each $k=k_1,\ k_2\in\mathbb{N}$. Using these infinitely many limits $x_1, x_2, \ldots$, we define the sequence $x:=(x_1, x_2, \ldots)$. Then we have constructed a candidate limit for the sequence $(x^{(n)})$. However, so far, we only have that each individual component of $x^{(n)}$ converges to the corresponding component of x, i.e., $(x^{(n)})$ converges componentwise to x. To prove that $(x^{(n)})$ converges to x in 2-norm, we go back (4.1) and pass it to the limit $m\to\infty$. We obtain for every nonzero $z\in l_v^2$ and for all $n>n_0$

$$\begin{aligned}
&\left\|x^{(n)}-x, z\right\|_{2,v,w}\\
&=\left(\frac{1}{2}\sum_{k_1}\sum_{k_2}\begin{Vmatrix} v_{k_1} & w_{k_1}\\ v_{k_2} & w_{k_2}\end{Vmatrix}^{p-2}\begin{vmatrix} x_{k_1}^{(n)}-x_{k_1} & z_{k_1}\\ x_{k_2}^{(n)}-x_{k_2} & z_{k_2}\end{vmatrix}^2\right)^{\frac{1}{2}}<\frac{\varepsilon}{2}.
\end{aligned}$$

Since the space l_v^2 is linear, we also get $x=(x-x^{(n)})+x^{(n)}\in\ell_v^2$. This completes the proof. □

The following proposition tells us that ℓ^p is not "too far" from ℓ_v^2.

Proposition 4.3. *As a subspace of* ℓ_v^2, $(\ell^p, \|.,.\|_{2,v,w})$ *is not complete, but dense in* ℓ_v^2.

Proof. Since $(\ell_v^2, \|.,.\|_{2,v,w})$ is complete, it suffices to show that ℓ^p is not closed in ℓ_v^2. As in the proof of Proposition 4.1, we construct an increasing sequence of non-negative integers (k_j) such that $v_{k_j}^{p-2}<2^{-j}$ and $w_{k_j}^{p-2}<3^{-j}$ for every $j\in\mathbb{N}$. Next, for each $j\in\mathbb{N}$, we define $x^{(n)}=(x_k^{(n)})$ by $x_k^{(n)}=1$ for $k=k_1,\ k_2,\ldots,k_n$ and $x_k^{(n)}=0$ otherwise. Let $\{e_1, e_2\}$ be a linearly independent set where $e_1=(1,0,\ldots)$ and $e_2=(0,1,0,\ldots)$. We will give the proof for the case $1<p-2<\infty$, by using the inequality (2.2) and the triangle inequality. The proof for the case $0<p-2\le 1$ can be done similarly by using the inequality (2.1) and the triangle inequality. Then we see that $(x^{(n)})$ forms a Cauchy sequence

in ℓ^2_v since for $m > n$ we have

$$\|x^{(n)} - x^{(m)}, e_i\|^2_{2,v,w}$$
$$= \frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2}\end{matrix}\right\|^{p-2}\left\|\begin{matrix} x^{(n)}_{k_1} - x^{(m)}_{k_1} & e_{ik_1} \\ x^{(n)}_{k_2} - x^{(m)}_{k_2} & e_{ik_2}\end{matrix}\right\|^2$$
$$\leq \frac{1}{2}\sum_{k}(|w_i||v_k| + |v_i||w_k|)^{p-2}\left|x^{(n)}_k - x^{(m)}_k\right|^2$$
$$+ \frac{1}{2}\sum_{k}(|v_i||w_k| + |v_k||w_i|)^{p-2}\left|x^{(n)}_k - x^{(m)}_k\right|^2$$
$$\leq 2^{p-3}\sum_{j=n+1}^{m}({w_i}^{p-2}v^{p-2}_{k_j} + w^{p-2}_{k_j}{v_i}^{p-2})$$
$$+ 2^{p-3}\sum_{j=n+1}^{m}({v_i}^{p-2}w^{p-2}_{k_j} + v^{p-2}_{k_j}{w_i}^{p-2})$$
$$= 2^{p-3}{v_i}^{p-2}\sum_{j=n+1}^{m}\frac{1}{3^j} + 2^{p-3}{w_i}^{p-2}\sum_{j=n+1}^{m}\frac{1}{2^j} \to 0$$

as $m, n \to \infty$ for each $i = 1, 2$. Since ℓ^2_v is complete, $(x^{(n)})$ is convergent and we know that the limit is the sequence $x = (x_k)$ where $x_{k} = 1$ for $k = k_1, k_2, k_3, \ldots$ and $x_k = 0$ otherwise. While $x^{(n)} \in \ell^p$ for every $n \in \mathbb{N}$, the limit $x \notin \ell^p$. This shows that ℓ^p is not closed in $(\ell^2_v, \|.,.\|_{2,v,w})$. The fact that ℓ^p is dense in $\left(\ell^2_v, \|.,.\|_{2,v,w}\right)$ is easy to see, since every $x = (x_k) \in \ell^2_v$ can be approximated by $x^{(n)} := (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ for sufficiently large values of $n \in \mathbb{N}$ with $\left\|x^{(n)} - x, e_1\right\|_{2,v,w} \to 0$ and $\left\|x^{(n)} - x, e_2\right\|_{2,v,w} \to 0$ as $n \to \infty$. □

Proposition 4.1 motivates us to study ℓ^2_v further as the ambient space, replacing ℓ^p. So far, we have fixed the weights $v = (v_k)$ and $w = (w_k)$. We now would like to know how the space ℓ^2_v depends on the choice of pairs (v, w).

Let V_p be the collection of all sequences. Let $v = (v_k)$ and $w = (w_k) \in \ell^p$ with $v_k > 0$, $w_k > 0$ for every $k \in \mathbb{N}$ and $\left\|\begin{smallmatrix} v_{k_1} & w_{k_1} \\ v_{k_2} & w_{k_2}\end{smallmatrix}\right\| \neq 0$ if $k_1 \neq k_2$. Let $v_1, w_1, v_2, w_2 \in V_p$. We say that the pairs (v_1, w_1) and (v_2, w_2) are equivalent, write $(v_1, w_1) \sim (v_2, w_2)$, if and only if there exists a constant $C > 0$ such that

$$\frac{1}{C}\left\|\begin{matrix} v_{1k_1} & w_{1k_1} \\ v_{1k_2} & w_{1k_2}\end{matrix}\right\| \leq \left\|\begin{matrix} v_{2k_1} & w_{2k_1} \\ v_{2k_2} & w_{2k_2}\end{matrix}\right\| \leq C\left\|\begin{matrix} v_{1k_1} & w_{1k_1} \\ v_{1k_2} & w_{1k_2}\end{matrix}\right\|$$

for every $k_1, k_2 \in \mathbb{N}$.

Theorem 4.4. *Let $v_1, w_1, v_2, w_2 \in V_p$. Then, the following statements are equivalent:*

1. $(v_1, w_1) \sim (v_2, w_2)$.
2. *There exists a constant $C > 0$ such that*

$$\frac{1}{C}\|x, z\|_{2,v_1,w_1} \leq \|x, z\|_{2,v_2,w_2} \leq C\|x, z\|_{2,v_1,w_1} \qquad x, z \in \ell^p.$$

Proof. (1) ⇒ (2): Let $x, z \in \ell^p$ and $(v_1, w_1) \sim (v_2, w_2)$, then

$$\|x, z\|_{2,v_2,w_2} = \left[\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} v_{2k_1} & w_{2k_1} \\ v_{2k_2} & w_{2k_2}\end{matrix}\right\|^{p-2}\left\|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right\|^2\right]^{\frac{1}{2}}$$
$$\leq \left[\frac{1}{2}\sum_{k_1}\sum_{k_2}C^{p-2}\left\|\begin{matrix} v_{1k_1} & w_{1k_1} \\ v_{1k_2} & w_{1k_2}\end{matrix}\right\|^{p-2}\left\|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right\|^2\right]^{\frac{1}{2}}$$
$$= C^{\frac{p-2}{2}}\|x, z\|_{2,v_1,w_1}.$$

On the other hand, we have

$$\|x, z\|_{2,v_2,w_2} = \left[\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} v_{2k_1} & w_{2k_1} \\ v_{2k_2} & w_{2k_2}\end{matrix}\right\|^{p-2}\left\|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right\|^2\right]^{\frac{1}{2}}$$
$$\geq \left[\frac{1}{2}\sum_{k_1}\sum_{k_2}\frac{1}{C^{p-2}}\left\|\begin{matrix} v_{1k_1} & w_{1k_1} \\ v_{1k_2} & w_{1k_2}\end{matrix}\right\|^{p-2}\left\|\begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2}\end{matrix}\right\|^2\right]^{\frac{1}{2}}$$
$$= \frac{1}{C^{\frac{p-2}{2}}}\|x, z\|_{2,v_1,w_1}.$$

Hence, the implication (1) ⇒ (2) holds.

(2) ⇒ (1): Let $x, z \in \ell^p$ and $\frac{1}{C}\|x, z\|_{2,v_1,w_1} \leq \|x, z\|_{2,v_2,w_2} \leq C\|x, z\|_{2,v_1,w_1}$ where $C > 0$. Take $x = e_n = (0, \ldots, 0, 1, 0, \ldots)$ and $z = e_m = (0, \ldots, 0, 1, 0, \ldots)$, $n, m \in \mathbb{N}$, where the 1 is the nth and mth term, respectively. Then $x, z \in \ell^p$, so that $x, z \in \ell^2_v$.

$$\|x, z\|_{2,v_1,w_1} = \left\|\begin{matrix} v_{1n} & v_{1m} \\ w_{1n} & w_{1m}\end{matrix}\right\|^{\frac{p-2}{2}} \quad \text{and}$$
$$\|x, z\|_{2,v_2,w_2} = \left\|\begin{matrix} v_{2n} & v_{2m} \\ w_{2n} & w_{2m}\end{matrix}\right\|^{\frac{p-2}{2}}.$$

Hence, from our assumption, we obtain

$$\frac{1}{C}\left\|\begin{matrix} v_{1n} & v_{1m} \\ w_{1n} & w_{1m}\end{matrix}\right\|^{\frac{p-2}{2}} \leq \left\|\begin{matrix} v_{2n} & v_{2m} \\ w_{2n} & w_{2m}\end{matrix}\right\|^{\frac{p-2}{2}} \leq C\left\|\begin{matrix} v_{1n} & v_{1m} \\ w_{1n} & w_{1m}\end{matrix}\right\|^{\frac{p-2}{2}},$$

and this holds for every $n, m \in \mathbb{N}$. Taking the $(\frac{p-2}{2})$th roots, we conclude that $(v_1, w_1) \sim (v_2, w_2)$. □

Acknowledgements

The authors are grateful to anonymous referees for their careful reading of the paper which improved it greatly. The first author's research was supported by Scientific and Technological Research Council of Turkey (TUBITAK), 2214-A International Doctoral Research Fellowship Programme (BIDEB).

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