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Reliability of multi-component stress–strength models

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Abstract In this paper, we estimate the reliability of some parallel and series multi-component stress–strength models. We determine the reliability of a system composed of k dependent components subjected to n dependent stresses. We study the cases, when the components are either arranged in series or in parallel. The components strengths are assumed to have $(k + 1)$-parameter multivariate Marshall–Olkin exponential distribution, while the stresses are $(n + 1)$-parameter multivariate Marshall–Olkin exponentially distributed.

1. Introduction

Estimation of the reliability of stress–strength models has been discussed in the literature extensively. For example Hanagal [1] obtained the estimation of the reliability of a series system under the assumption of a multivariate Pareto distribution for the strengths of the components and subjected to exponential common stress. Hanagal [2] obtained the estimation of system reliability of a stress–strength model with k components either parallel or series. He assumed that the distributions of the strengths of the k components and the distribution of the common stress are all independent and are two parameter exponential. Hanagal [3] obtained the estimation of system reliability in multi-component series stress–strength models. He considered the estimation of $R = P(X_{k+1} < \min(X_1, X_2, \cdots, X_k))$ when X_i, $i = 1, 2, \ldots, k + 1$, all follow independent Gamma, Weibull and Pareto distributions. For the case of non-independent components, Hanagal [4] estimated the reliability of a parallel system with two components having a bivariate exponential distribution subjected to a common stress, which can be either exponential or gamma. Also Ba'akkel [5] discussed the reliability of a system with two components with strengths having a bivariate exponential distribution and subjected to different strategies of stresses. Ebrahimi [6] discussed series stress–strength models having bivariate Marshall–Olkin exponential strengths subjected to q stresses. The stresses are independent and exponentially distributed.

Modern engineering systems may have more than two components. The components may fail separately or simultaneously. The $(k + 1)$-parameter multivariate exponential distribution and the absolutely continuous multivariate exponential (ACMVE) distribution may represent the lifetimes or strengths

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of such components. Hanagal [7] discussed the reliability of an s-out of k system. The strengths of the k-components follow the ACMVE distribution and are subjected to a common stress which is exponentially distributed.

In some situations the system may be imposed to different stresses that could not be independent. Not too much work considered this case. In the present article, we consider the problem of estimating the reliability of a system with k components subjected to n stresses. The components could be arranged either parallel or series. The strengths of the components and the stresses imposed on the system are all assumed to have different multivariate Marshall–Olkin exponential distributions (MVE) [8]. The strengths and the stresses are independent and are assumed to have $(k+1)$ and $(n+1)$-parameter multivariate Marshall–Olkin exponential distributions, respectively.

As in Proschan and Sullo [9] a set of random variables T_1, $T_2, \ldots, T_r$ is said to have an $(r+1)$-parameter MVE distribution if the survival function of $T_1, T_2, \ldots, T_r$ is given by

$$\begin{aligned}\overline{F}(t) &= P(T_1 > t_1, \cdots, T_r > t_r)\\ &= \exp\left[-\sum_{i=1}^{r}\beta_i t_i - \beta_0 \max(t_1, \cdots, t_r)\right],\\ &\quad t_i \geqslant 0, i = 1,2,\cdots,r, \boldsymbol{\beta} \in A\end{aligned}$$

$$\text{where } \boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_r, \beta_0) \tag{1.1}$$

$$\text{and } A = \{\boldsymbol{\beta} : 0 \leqslant \beta_i < \infty; i = 0,1,2,\cdots,r; \; \beta_0 + \beta_i > 0, i = 1,2,\cdots,r\}.$$

This distribution will be denoted from now on as MVE$(r+1)$. The MVE$(r+1)$arises in the following context: suppose that $T_1, T_2, \ldots, T_r$ represent failure times or strengths of components labeled 1,2,..., r, respectively, and $\{Z_i(t),\ t \succeq 0; \beta\}$, $i = 0,1,\ldots,r$, be $r+1$ mutually independent Poisson processes with corresponding intensities $\beta_i, \beta \in A$. A shock in $Z_i(t)$ process is selectively fatal to component i, $i = 0,1,\ldots,r$, while a shock in $Z_0(t)$ process is simultaneously fatal to all r components. Hence, if $U_0, U_1, U_2, \ldots, U_r$ represent the times to the first events in $Z_0(t), Z_1(t), \ldots, Z_r(t)$, respectively, $T_i = \min(U_0, U_i)$, where U_0 and U_i are independent exponential random variables. Thus, it is evident that MVE$(r+1)$can be represented in terms of independent exponential random variables. This property is used in generating samples from MVE$(r+1)$. As mentioned by Marshall and Olkin [8], the marginal distribution of T_i is exponential with parameter $\gamma_i = \beta_0 + \beta_i$, $i = 1,2,\ldots,r$, while the joint marginal distribution of T_i, T_j is bivariate exponential Marshall–Olkin distribution with parameters β_i, β_j, β_0, where $i, j = 1,2,\ldots,r$ and $i \neq j$. In general the joint marginal distribution of $T_{i_1}, T_{i_2}, \cdots, T_{i_s}$ is MVE$(s+1)$, with parameters, $\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_s}$, and β_0.

2. Reliability of the system

In this section we derive the reliability of a system consisting of k components subjected to n stresses. The strengths of the components $X_1, X_2, \ldots X_k$, have MVE$(k+1)$ with parameters λ_i, $i = 0,1,2,\ldots, k$. Each component is subjected to any one of the n stresses $(Y_1, Y_2, \ldots, Y_n)$. The stresses $(Y_1, Y_2, \ldots, Y_n)$, are assumed to have MVE$(n+1)$ with parameters μ_i, $i = 0,1,2,\ldots, n$. The stresses $(Y_1, Y_2, \ldots, Y_n)$and the strengths $(X_1, X_2, \ldots, X_k)$, are assumed to be independent. We determine the reliability of the system for both parallel and series arrangements of the components.

2.1. Reliability of the parallel system

For the parallel case, the reliability of the system is given by

$$\begin{aligned}R_1 &= P[\max(X_1, X_2, \cdots, X_k) > \max(Y_1, Y_2, \cdots, Y_n)]\\ &= P[Z > H] = \int_0^\infty \overline{F}_Z(h)\, dF_H(h),\end{aligned} \tag{2.1.1}$$

where $Z = \max(X_1, X_2, \ldots, X_k)$ and $H = \max(Y_1, Y_2, \ldots, Y_n)$. The survival function of Z is given by

$$\begin{aligned}\overline{F}_z(z) &= P[Z > z]\\ &= P(X_1 > z \text{ or } X_2 > z \text{ or} \cdots \text{or } X_k > z)\\ &= \sum_{l=1}^{k}(-1)^{l+1}\sum_{1 \leqslant i_1 < \cdots < i_l \leqslant k} P(X_{i_1} > z,\\ &\quad X_{i_2} > z, \cdots, X_{i_l} > z).\end{aligned} \tag{2.1.2}$$

Thus, using (1.1), we get

$$\overline{F}_z(z) = \sum_{l=1}^{k}(-1)^{l+1} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant k} \exp(-(\lambda_0 + \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_l})z). \tag{2.1.3}$$

Similarly, the cumulative distribution of H is given by

$$F_H(h) = 1 - \sum_{s=1}^{n}(-1)^{s+1} \sum_{1 \leqslant j_1 < \cdots < j_s \leqslant n} \exp(-(\mu_0 + \mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_s})h) \tag{2.1.4}$$

Substituting with (2.1.3) and (2.1.4) into (2.1.1), we get

$$\begin{aligned}R_1 = \sum_{s=1}^{n}(-1)^{s+1} \sum_{1 \leqslant j_1 < \cdots < j_s \leqslant n} (\mu_0 + \mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_s}) \Bigg\{ \sum_{l=1}^{k}(-1)^{l+1}\\ \times \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant k} (\lambda_0 + \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_l} + \mu_0 + \mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_s})^{-1} \Bigg\}.\end{aligned} \tag{2.1.5}$$

2.2. Reliability of the series system

The reliability of the system for the series case is

$$\begin{aligned}R_2 &= P(\min(X_1, \cdots, X_k) > H) = P(M > H)\\ &= \int_0^\infty \overline{F}_M(h)\, dF_H(h),\end{aligned} \tag{2.2.1}$$

where $M = \min(X_1, \ldots, X_k)$.

Noticing that M is exponentially distributed with parameter $\lambda = \sum_{i=0}^{k}\lambda_i$,the survival function of M is given by

$$\overline{F}_M(h) = e^{-\lambda h}. \tag{2.2.2}$$

Using (2.2.2) and (2.1.4) in (2.2.1) we get

$$R_2 = \sum_{s=1}^{n}(-1)^{s+1} \sum_{1 \leqslant j_1 < \cdots < j_s \leqslant n} \frac{(\mu_0 + \mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_s})}{\left(\lambda + \mu_0 + \mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_s}\right)}. \tag{2.2.3}$$

3. Special cases

In this section we consider some special cases of the results of Section 2.

(i) When $n = 1$, the k components will be subjected to a common stress Y. This stress is distributed exponentially with mean μ^{-1}, and independent of the strengths of the components. According to (2.1.5) and putting ($\mu_1 = \mu$ and $\mu_0 = 0$) the reliability of the parallel system will be

$$R_1 = \mu \sum_{l=1}^{k} (-1)^{l+1} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant k} (\lambda_0 + \lambda_{i_1 +} \lambda_{i_2 + \cdots +} \lambda_{i_l} + \mu)^{-1}. \tag{3.1}$$

For the series case, according to (2.2.3) the reliability of the system will be

$$R_2 = \frac{\mu}{\lambda + \mu}. \tag{3.2}$$

(ii) When $n = 2$, each of the k components is subjected to any one of two dependent stresses, say, Y_1 and Y_2. That is, Y_1 and Y_2, are two dependent stresses having a bivariate Marshall–Olkin exponential distribution (BVE), and are independent of the strength of the system. According to (2.1.5) the reliability of the parallel system is given by

$$\begin{aligned} R_1 = \sum_{s=1}^{2} \Bigg[(\mu_0 + \mu_s) \sum_{l=1}^{k} (-1)^{l+1} \\ \times \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant k} (\lambda_0 + \lambda_{i_1 +} \lambda_{i_2 + \cdots +} \lambda_{i_l} + \mu_0 + \mu_s)^{-1} \Bigg] \\ - \mu \sum_{l=1}^{k} (-1)^{l+1} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant k} (\lambda_0 + \lambda_{i_1 +} \lambda_{i_2 + \cdots +} \lambda_{i_l} + \mu)^{-1}, \end{aligned} \tag{3.3}$$

here $\mu = \sum_{i=0}^{2} \mu_i$.
According to (2.2.3) the reliability of the series system is given by

$$R_2 = \frac{\mu_0 + \mu_1}{\lambda + \mu_0 + \mu_1} + \frac{\mu_0 + \mu_2}{\lambda + \mu_0 + \mu_2} - \frac{\mu_0 + \mu_1 + \mu_2}{\lambda + \mu_0 + \mu_1 + \mu_2}. \tag{3.4}$$

(iii) When $k = 2$, $n = 1$ then (X_1, X_2) follows a bivariate Marshall–Olkin [8] exponential distribution (BVE) and subjected to a common stress Y, which is distributed exponentially with mean μ^{-1}, and independent of the strengths of the components. According to (2.1.5) the reliability of the parallel system is given by

$$R_1 = \frac{\mu}{\lambda_0 + \lambda_1 + \mu} + \frac{\mu}{\lambda_0 + \lambda_2 + \mu} - \frac{\mu}{\lambda_0 + \lambda_1 + \lambda_2 + \mu}, \tag{3.5}$$

which is the same as the result obtained by Hanagal [4] and Ba'akkel [5]. Similarly using (2.2.3) we get the reliability of the series system

$$R_2 = \frac{\mu}{\lambda_0 + \lambda_1 + \lambda_2 + \mu}, \tag{3.6}$$

which is the same as the result obtained by Ba'akkel [5].

(iv) When $k = 2$, and $n = 2$, the system consists of two components with strengths (X_1, X_2). Each component is subjected to any one of two dependent stresses (Y_1, Y_2). According to (2.1.5) the reliability of the parallel system is given by

$$\begin{aligned} R_1 = \sum_{s=1}^{2} (\mu_0 + \mu_s) \left[\sum_{i=1}^{2} (\lambda_0 + \lambda_i + \mu_0 + \mu_s)^{-1} - (\lambda + \mu_0 + \mu_s)^{-1} \right] \\ - \mu \left[\sum_{i=1}^{2} (\lambda_0 + \lambda_i + \mu)^{-1} - (\lambda + \mu)^{-1} \right], \end{aligned} \tag{3.7}$$

here $\lambda = \sum_{i=0}^{2} \lambda_i$, $\mu = \sum_{i=0}^{2} \mu_i$.According to (2.2.3) the reliability of the series system is given by

$$R_2 = \frac{\mu_0 + \mu_1}{\lambda + \mu_0 + \mu_1} + \frac{\mu_0 + \mu_2}{\lambda + \mu_0 + \mu_2} - \frac{\mu_0 + \mu_1 + \mu_2}{\lambda + \mu_0 + \mu_1 + \mu_2}. \tag{3.8}$$

4. Estimation of the reliability

Appropriate non-parametric estimators of R_i, $i = 1,2$, could be obtained by estimating the probability $P(\max(X_1, X_2, \ldots, X_k) > \max(Y_1, Y_2, \ldots, Y_n))$, for the parallel case and estimating the probability $P(\min(X_1, X_2, \ldots, X_k) > \max(Y_1, Y_2, \ldots, Y_n))$, for the series case. Using the data of a sample of N observations from the MVE($k + 1$)and MVE($n + 1$), appropriate estimators of R_i $i = 1,2$,are obtained by counting the proportion of the sample observations such that $\max(x_{1j}, x_{2j}, \ldots, x_{kj}) > \max(y_{1j}, y_{2j}, \ldots, y_{nj})$ for the parallel case and $\min(x_{1j}, x_{2j}, \ldots, x_{kj}) > \max(y_{1j}, y_{2j}, \ldots, y_{nj})$ for the series case, to the total number of observations in the sample, where $(x_{1j}, x_{2j}, \ldots, x_{kj})$ and $(y_{1j}, y_{2j}, \ldots, y_{nj})$ are the j-th observation corresponding to the strengths and the stresses, respectively, and $j = 1, \ldots, N$.

For parametric estimation, the estimation of reliability for each model could be obtained by replacing the parameters in the equation of R_i, $i = 1, 2$, by their corresponding estimators. Several estimators of the parameters of the bivariate and multivariate Marshall–Olkin exponential distributions have been suggested in the literature, see for example Prochan and Sullo[9], Bhattacharyya and Johnson [10], Arnold [11], Bemis, Bain and Higgins [12], Kundu and Kumar [13], and Karlis [14] and others.

Here, we shall mention the estimators that we will use for estimating R_i, $i = 1,2$.

Arnold [11] has suggested consistent estimators of the parameters using the distribution properties of the model with the form

$$\beta_i^{(A)} = \frac{(N-1)N_i}{N \sum_{j=1}^{N} \min(t_{1j}, t_{2j}, \cdots, t_{rj})}, i = 0, 1, \cdots, r, \tag{4.1}$$

where N_i denotes the number of observations for which $t_i = \min(t_1, t_2, \ldots, t_r)$, N_0 denotes the number of observations for which $t_1 = t_2 = \cdots = t_r$, in a sample of N observations from the distribution given by (1.1).

For obtaining the maximum likelihood estimates (MLE) of the parameters of MVE ($r + 1$), Proschan and Sullo [9] showed that, if$(T_1, \ldots, T_r)$ is random vector having MVE ($r + 1$)distribution, and n_0 denotes the number of observations for which at least $t_i = t_j$, for $i \neq j$, n_i denotes the number of observations for which $t_i < \max(t_1, t_2, \ldots, t_r)$, and $n_i^{(c)}$ denotes the number of observations for which only $t_i = \max(t_1, t_2, \ldots, t_r)$, the log-likelihood function for a given sample of size N is given by

$$l(\beta) = \sum_{i=0}^{r} n_i \log(\beta_i) + \sum_{i=1}^{r} n_i^{(c)} \log(\beta_0 + \beta_i) - \sum_{j=1}^{N}\sum_{i=1}^{r} \beta_i t_{ij} - \beta_0 \sum_{j=1}^{N} \max(t_{1j}, t_{2j}, \cdots, t_{rj}). \quad (4.1)$$

If the usual conventions are adopted, e.g. $0^0 = 1, 0\log(0) = 0$, etc., (4.1) is well-defined for all possible values of $\beta_i > 0$, n_i and $n_i^{(c)}$. The resulting system of equations cannot be solved in a closed form. If all $n_i > 0$ the MLE of $\beta_i > 0$, exists uniquely. If $n_s = 0$, for some $s = 0, 1, \ldots, r$, an explicit form of the MLE of β_i, exists (see Theorem 4.1 in [9]).

Proschan and Sullo [9] proposed simple estimators, which they called INT estimators, defined by

$$\beta_i^{(T)} = \frac{n_i}{N - n_i^{(c)}} N \Big/ \sum_{j=1}^{N} t_{ij}, \quad i = 1, \cdots, r, \quad (4.2)$$

$$\beta_0^{(T)} = \left[N - \sum_{i=1}^{r} \frac{n_i}{N - n_i^{(c)}} n_i^{(c)}\right] \Bigg/ \sum_{j=1}^{N} \max(t_{1j}, t_{2j}, \cdots, t_{rj}).$$

These estimators are developed from intuitive considerations of the distribution. They used these estimates as the first iterate in solving the likelihood equations iteratively using the method of successive approximations, applied by putting the likelihood equations in the form $\beta = g(\beta)$ and then using the functional iteration $\beta^{(m+1)} = g(\beta^{(m)})$, $m = 0, 1, 2, \ldots$. Specifically, let

$$\beta_i^{(m+1)} = \left[n_i + \xi_i^{(m)} n_i^{(c)}\right] \Big/ \sum_{j=1}^{N} t_{ij}, \; i = 1, \cdots, r, \quad (4.3)$$

$$\beta_0^{(m+1)} = \left[N - \sum_{i=1}^{r} \xi_i^{(m)} n_i^{(c)}\right] \Big/ \sum_{j=1}^{N} \max(t_{1j}, t_{2j}, \cdots, t_{rj}),$$

where

$$\xi_i^{(0)} = \frac{n_i}{N - n_i^{(c)}}, i = 1, \cdots, r$$

$$\xi_i^{(m)} = \beta_i^{(m)} / (\beta_i^{(m)} + \beta_0^{(m)}) i = 1, \cdots, r; \quad m = 1, 2 \cdots.$$

The iteration is terminated when some convergence criterion is met. Karlis [14] developed an EM type algorithm for the computation of the MLE's based on the multivariate reduction technique. He used the consideration that $T_i = \min(U_0, U_i)$, $i = 1, \ldots, r$. According to the multivariate reduction derivation, the missing data consist of the non-observable random variables U_i, $i = 0, \ldots, r$ while the observed data are the values T_i, $i = 1, \ldots, r$. The EM algorithm proceeds by calculating the conditional expectation of U_i given T_i and the current values of the parameters $\beta^{(m)} = (\beta_0^{(m)}, \beta_1^{(m)}, \cdots, \beta_r^{(m)})$ which called the E-step, while the M-step just calculates the MLE's for a sample from exponential distributions, using the expectations of the E-step.

The conditional expectations of U_i's given the T_i's (see Karlis [14]) are as follow:

First case when $t_1 = t_2 = \ldots = t_r$.

$$E(U_0|T_1, \cdots T_r, \boldsymbol{\beta}) = t_1,$$
$$E(U_i|T_1, \cdots T_r, \boldsymbol{\beta}) = t_1 + \beta_i^{-1}, i = 1, \cdots, r.$$

Second case if some of the t_i's are equal but there are some other with smaller values, i.e. $t_{i_1}, t_{i_2}, \cdots, t_{i_k} < t_{j_1} = \cdots = t_{j_p} = t^{(0)}$ for some k and p,

$$E(U_0|T_1, \cdots T_r, \boldsymbol{\beta}) = t^{(0)},$$
$$E(U_{ib}|T_1, \cdots T_r, \boldsymbol{\beta}) = t_{ib}, \; b = 1, \cdots, k,$$
$$E(U_{jb}|T_1, \cdots T_r, \boldsymbol{\beta}) = t^{(0)} + \beta_{jb}^{-1}, \; b = 1, \cdots, p.$$

The last case if some t_j is larger than the rest,

$$E(U_0|T_1, \cdots T_r, \boldsymbol{\beta}) = t_j + \frac{\beta_j}{\beta_j + \beta_0}\frac{1}{\beta_0},$$
$$E(U_j|T_1, \cdots T_r, \boldsymbol{\beta}) = t_j + \frac{\beta_0}{\beta_j + \beta_0}\frac{1}{\beta_j},$$
$$E(U_i|T_1, \cdots T_r, \boldsymbol{\beta}) = t_i, i = 1, \cdots, r, \quad i \neq j.$$

5. Numerical illustration

For a numerical illustration of the results obtained, a simulation study is performed. Two thousand samples each of size 10, 30 and 100 are generated from the strengths and stresses distributions.

Taking $k = 3$, $\lambda_1 = 0.06$, $\lambda_2 = 0.03$, $\lambda_3 = 0.07$, and $\lambda_0 = 0.04$, two cases are considered:

Case 1: the common stress ($n = 1$), we take $\mu = .25$.

Table 1 System reliability under common exponential stress.

$k = 3$ and $n = 1$	$N = 10$		$N = 30$		$N = 100$	
	Parallel case	Series case	Parallel case	Series case	Parallel case	Series case
R	0.8513772	0.5555556	0.8513772	0.5555556	0.8513772	0.5555556
$R^{(T)}$	0.8433652	0.5472533	0.8482669	0.5524275	0.850093	0.5555835
$R^{(I)}$	0.8425224	0.547117	0.8479657	0.5523857	0.8500074	0.5555626
$R^{(EM)}$	0.8425224	0.547117	0.8479657	0.5523857	0.8500075	0.5555626
$R^{(A)}$	0.8634436	0.5796057	0.8546703	0.5627921	0.8522117	0.5580528
$R^{(N)}$	0.85155	0.55715	0.8503833	0.5555	0.850275	0.556575
$MSE^{(T)}$	0.00448015	0.00856184	0.00140409	0.003140519	0.00043428	0.00096121
$MSE^{(I)}$	0.00447337	0.00857663	0.00139800	0.003142227	0.00043441	0.00096050
$MSE^{(EM)}$	0.00447337	0.00857664	0.00139800	0.003142228	0.00043441	0.00096050
$MSE^{(A)}$	0.00763443	0.01208204	0.00290438	0.004177479	0.00084677	0.00123744
$MSE^{(N)}$	0.01231763	0.02561142	0.00426529	0.008694198	0.00128019	0.00252636
$b^{(T)}$	−0.00801209	−0.00830224	−0.00311035	−0.003128105	−0.00128418	2.79440e-05
$b^{(I)}$	−0.00885482	−0.00843853	−0.00341151	−0.003169838	−0.00136979	7.04821e-06
$b^{(EM)}$	−0.00885479	−0.00843852	−0.00341150	−0.003169835	−0.00136979	7.05479e-06
$b^{(A)}$	0.01206636	0.02405013	0.00329307	0.007236515	0.00083442	0.00249727
$b^{(N)}$	0.00017276	0.00159444	−0.00099391	−5.55555e-05	−0.00110224	0.00101944

Case 2: three stresses ($n = 3$), we take

(i) The expected value of each one of the three stresses equals to the expected value of the common stress in Case 1 ($\mu^{-1} = \alpha_1^{-1} = \alpha_2^{-1} = \alpha_3^{-1}$, where $\alpha_i = \mu_0 + \mu_i$), namely $\mu_0 = 0.2, \mu_1 = 0.05$, $\mu_2 = 0.05$, $\mu_3 = 0.05$.

(ii) The sum of the expected value of the three stresses equals to the expected value of the common stress in Case 1 ($\mu^{-1} = \alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1}$, where $\alpha_i = \mu_0 + \mu_i$), namely $\mu_0 = 0.3$, $\mu_1 = 0.4$, $\mu_2 = 0.2$, $\mu_3 = 1.45$.

It is to be noted that these values are chosen arbitrary just for illustrating the results obtained. Tables 1–3 show the true values of R_i, $i = 1, 2$, and their corresponding estimates by the INT method (R^T), the iterative method (R^I), the EM algorithm (R^{EM}), Arnold's method (R^A)and non parametric method (R^N). The simulated R^T, R^I, R^{EM}, $R^A$and R^N are the mean of the 2000 replicates of the corresponding estimates. For computing R^IandR^{EM} we used the same convergence criterion, which is the change on the parameter values at successive iterations is less than10^{-6}, and the same initials which are the INT estimators. For comparison the bias (b) and MSE, of the different estimates in each case are calculated. Where bias (b)is the difference of the mean of the 2000 replicates estimates from the true values of R and MSE is the mean of the squares of the differences of the 2000 replicates estimates from the true values of R.

Clearly as is known, the reliability of the parallel system is greater than that of the series system. We find that the reliability of the system under the common stress is greater than that under three stresses when the expected value of each one of the three stresses equals to the expected value of the common stress and less than that when the sum of the expected values of the three stresses equals to the expected value of the common stress. In general as N increases all estimates $R^{(T)}$, $R^{(I)}$, $R^{(\mathrm{EM})}$, $R^{(A)}$, and $R^{(N)}$,converge to R and MSE decreases. All estimates give good results even for small $N(N = 10)$.

Table 2 System reliability under three stresses model when $\mu^{-1} = \alpha_1^{-1} = \alpha_2^{-1} = \alpha_3^{-1}$.

$k = 3$ and $n = 1$	$N = 10$		$N = 30$		$N = 100$	
	Parallel case	Series case	Parallel case	Series case	Parallel case	Series case
R	0.8217638	0.5030303	0.8217638	0.5030303	0.8217638	0.5030303
$R^{(T)}$	0.814075	0.4973373	0.8205231	0.5036056	0.8219642	0.5030831
$R^{(I)}$	0.8130545	0.4972407	0.8202186	0.5035475	0.8218886	0.503055
$R^{(\mathrm{EM})}$	0.8130546	0.4972407	0.8202186	0.5035475	0.8218886	0.503055
$R^{(A)}$	0.8386531	0.5288856	0.8178332	0.5018213	0.8240517	0.5059867
$R^{(N)}$	0.82355	0.50435	0.8226833	0.5065167	0.82224	0.50225
$\mathrm{MSE}^{(T)}$	0.00574701	0.00940572	0.00175104	0.0031197	0.00057453	0.00095941
$\mathrm{MSE}^{(I)}$	0.00575849	0.0094048	0.00174967	0.00312101	0.00057326	0.00095933
$\mathrm{MSE}^{(\mathrm{EM})}$	0.00575849	0.00940480	0.00174967	0.00312106	0.00057326	0.00095933
$\mathrm{MSE}^{(A)}$	0.01062211	0.01282729	0.00438878	0.00580826	0.00113751	0.00130028
$\mathrm{MSE}^{(N)}$	0.01482359	0.02652782	0.00471909	0.00834135	0.00147781	0.00260054
$b^{(T)}$	−0.00768869	−0.00569304	−0.00124065	0.00057531	0.00020040	5.28411e-05
$b^{(I)}$	−0.00870924	−0.00578965	−0.00154515	0.00051717	0.00012483	2.47098e-05
$b^{(\mathrm{EM})}$	−0.00870919	−0.00578965	−0.00154514	0.00051719	0.00012484	2.47159e-05
$b^{(A)}$	0.01688935	0.02585532	−0.00393052	−0.00120895	0.00228790	0.00295640
$b^{(N)}$	0.001786233	0.00131970	0.00091956	0.00348636	0.00047623	−0.0007803

Table 3 System reliability under three stresses model when $\mu^{-1} = \alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1}$.

$k = 3$ and $n = 1$	$N = 10$		$N = 30$		$N = 100$	
	Parallel case	Series case	Parallel case	Series case	Parallel case	Series case
R	0.910531	0.6710158	0.910531	0.6710158	0.910531	0.6710158
$R^{(T)}$	0.9038087	0.6561326	0.9079135	0.6691879	0.9096677	0.6686644
$R^{(I)}$	0.9032454	0.6566265	0.907744	0.6692647	0.9096362	0.6687045
$R^{(\mathrm{EM})}$	0.9032455	0.6566265	0.907744	0.6692647	0.9096362	0.6687045
$R^{(A)}$	0.7999717	0.5684885	0.8914304	0.6471798	0.9064146	0.6615557
$R^{(N)}$	0.9125	0.67215	0.9116167	0.6750167	0.909295	0.66964
$\mathrm{MSE}^{(T)}$	0.00200379	0.00736045	0.00063014	0.00223967	0.00016482	0.00067897
$\mathrm{MSE}^{(I)}$	0.00200256	0.00730018	0.00062636	0.00223893	0.00016451	0.00067822
$\mathrm{MSE}^{(\mathrm{EM})}$	0.00200256	0.00730018	0.00062636	0.00223892	0.00016451	0.00067822
$\mathrm{MSE}^{(A)}$	0.09290336	0.06977117	0.00569541	0.01171752	0.00068396	0.00286692
$\mathrm{MSE}^{(N)}$	0.00773762	0.02265066	0.00281345	0.00735684	0.00082498	0.00223186
$b^{(T)}$	−0.0067223	−0.0148832	−0.0026175	−0.00182795	−0.00086337	−0.00235146
$b^{(I)}$	−0.0072856	−0.0143893	−0.0027870	−0.00175116	−0.00089479	−0.00231131
$b^{(\mathrm{EM})}$	−0.0072856	−0.0143893	−0.0027867	−0.00175115	−0.00089479	−0.00231130
$b^{(A)}$	−0.1105593	−0.1025273	−0.0191006	−0.02383599	−0.00089479	−0.00946009
$b^{(N)}$	0.00196896	0.00113416	0.00108563	0.00400083	−0.00123603	−0.00137584

Table 4 Average number of iterations until convergence criterion is met.

	$N = 10$		$N = 30$		$N = 100$	
	INT	(1,1,1,1)	INT	(1,1,1,1)	INT	(1,1,1,1)
Iterative method	9.907	13.032	8.8915	11.392	8.044	10.5865
EM-algorithm	27.9155	77.948	21.596	32.9665	18.3155	30.065

We see from Tables 1–3that, the iterative method and the EM-algorithm method give the same results for estimating the values of R_i, $i = 1,2$, concerning biasness and MSE's. However, for the same convergence criterion the average number of iterations using the iterative method is less than that using the EM-algorithm whether using INT estimates as initials or any other initials (for example putting all initial values equal to 1). Table (4) shows the average number of iterations until convergence criterion is met when the initial values of the parameters are equal to the INT estimates or all equal to 1.

Concerning biasness, we find that for a small sample size ($N = 10$) the non parametric method gives the smallest bias, and the differences in $b^{(T)}$, $b^{(I)}$ and $b^{(\mathrm{EM})}$ appear after 3-rd decimal place. While for large samples the differences in bias decrease. Concerning mean squared errors, we find that the differences in MSE oF $R^{(T)}$, $R^{(I)}$ and $R^{(\mathrm{EM})}$ appear after 4-th decimal place. For large samples the MSE are almost the same for $R^{(T)}$, $R^{(I)}$ and $R^{(\mathrm{EM})}$. In general Arnold estimates give the largest bias. We can say that the non-parametric method gives acceptable results. We also see that the differences between the $R^{(T)}$, $R^{(I)}$ and $R^{(\mathrm{EM})}$ are very small.

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References

[1] D.D. Hanagal, A multivariate Pareto distribution, Communi cations in statistics, Theory and Methods 25 (7) (1996) 1471–1488.

[2] D.D. Hanagal, On the estimation of system reliability in stress–strength models, Economic Quality Control 12 (1998) 17–22.

[3] D.D. Hanagal, Estimation of system reliability in multicomponent series stress–strength model, Journal of the Indian Statistical Association 41 (2003) 1–7.

[4] D.D. Hanagal, Estimation of system reliability from stress–strength relationship, Communications in statistics, Theory and Methods 25 (8) (1996) 1783–1797.

[5] H.M. Ba'akkel, Parametric inference about reliability for stress–strength Models, M.Sc. Thesis, Faculty of Science, King Abdulaziz University, 2003.

[6] N. Ebrahimi, Estimation of reliability for a series stress–strength system, IEEE Transactions on Reliability R-31 (1982) 202–205.

[7] D.D. Hanagal, Estimation of system reliability in s-out-of-k system, Statistical Papers 40 (1999) 99–106.

[8] A.W. Marshall, I. Olkin, A multivariate exponential distribution, Journal of American Statistical Association 62 (1967) 30–44.

[9] F. Proschan, P. Sullo, Estimating the parameters of multivariate exponential distribution, Journal of American Statistical Association 71 (1976) 465–472.

[10] G.K. Bhattacharyya, R.A. Johnson, Maximum Likelihood Estimation and Hypothesis Testing in the Bivariate Exponential Model of Marshal and Olkin, Department of Statistics, University of Wisconsin, 1971.

[11] B.C. Arnold, Parameter estimation for a multivariate exponential distribution, Journal of American Statistical Association 63 (1968) 848–852.

[12] B.M. Bemis, L.J. Bain, J.J. Higgins, Estimation and hypothesis testing for the parameters of a bivariate exponential distribution, Journal of American Statistical Association 67 (1972) 927–929.

[13] D. Kundu, D.A. Kumar, Estimating the parameters of the Marshall–Olkin bivariate Weibull distribution by EM Algorithm, Computational Statistics & Data Analysis 53 (2009) 956–965.

[14] D. Karlis, ML estimation for multivariate shock models via an EM algorithm, Annals of the Institute of Statistical Mathematics 55 (2003) 817–830.