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On minimal sets and strictly weaker topologies

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Abstract In [1,2] Farrag characterized the stirictly weaker principal topologies than any given principal topology on a nonempty set by using the minimal open sets which are defined by Steiner [3]. This paper mainly generalizes this result by using the minimal sets, which are defined in the paper with respect to the given topology τ on a nonempty set.

1. Introduction

Let τ_1 and τ_2 be two topologies on a nonempty set X then (1) τ_1 is weaker than τ_2 or τ_2 is stronger than τ_1 if $\tau_1 \subset \tau_2$ (2) τ_1 is strictly weaker than τ_2 if $\tau_1$is weaker than τ_2 and $\tau_1 \subset \tau \subset \tau_2$ such that $\tau \notin \{\tau_1, \tau_2\}$ implies that τ is not a topology on X. In [4] Frohlich defined an ultratopology on a set X to be a strictly weaker topology than the discrete topology D on X. The ultratopologies on X are divided into two classes the principal and the nonprincipal ultratopologies on X, $E_z \cup\ P_y$ and $E_z \cup F$, where E_z is the excluding, P_y is the particular point topologies on X, F is an ultrafilter on X and y, z are any two distinct points of X. In [5] Mashhour and Farrag showed that the principal ultratopology $E_z \cup P_y$ on a set X is the topology on X having the minimal basis $\beta_{yz} = \{\{x\}, \{y, z\}: x \in (X - \{z\})\}$where y and z are two distinct points of X and denoted by D_{yz}. In [3] Steiner defined a minimal open set at a point $x \in X$ in a topological space (X, τ) to be the open set containing x and is contained in each open set containing x. The author defined also a principal topology on a set X to be the topology on X having the minimal basis that consists only of open sets minimal at each point $x \in X$. It is Proved that a topology τ on a set X is principal iff arbitrary intersections of members of τ are members of τ. In [1] the authors gave a necessary and sufficient conditions for the principal topology τ^* on X to be strictly weaker than a given principal topology τ on X and proved that τ^* must be of the form $\tau^* = \tau \cap D_{yz}$ which is denoted by τ_{yz} where y and z are two distinct points of X satisfying three conditions depending on the minimal open sets in τ. In M[c]Clusky and M[c]Cartan [6,7] and Kennedy and M[c]Cartan [8] defined the $\tau-$ kernel $\widehat{\{x\}}$ of $\{x\}$ to be the intersection of all open sets which contain the point x.

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2. The minimal sets

Definition 2.1. Let (X, τ) be a topological space and $x \in X$. Then, $\widehat{\{x\}} = \cap\{G \in \tau : x \in G\}$ is called the minimal set at the point x with respect to τ on X.

The set $\widehat{\{x\}}$ is defined in [6] to be the $\tau-$ kernel of $\{x\}$.

As a direct consequence of the definition of the minimal sets at the points of a nonempty set X with respect to a topology τ on X, if $x, y \in X$ are any two distinct points, then we have the following remarks and theorems:

Remark 2.2. If τ is a principal topology, then $\widehat{\{x\}} \in \tau$ is the minimal open set at the point x as it is defined in [3].

Remark 2.3. $x \in \widehat{\{y\}}$ implies that $\widehat{\{x\}} \subset \widehat{\{y\}}$. Therefore, $\widehat{\{x\}} = \widehat{\{y\}}$ iff $x \in \widehat{\{y\}}$ and $y \in \widehat{\{x\}}$.

Remark 2.4. $y \in \widehat{\{x\}}$ iff $x \in \overline{\{y\}}$ iff $\tau \subset D_{yx}$. Therefore, $\widehat{\{x\}} = \{y \in X : x \in \overline{\{y\}}\} = \{y \in X : \tau \subset D_{yx}\}$ and $\overline{\{x\}} = \{y \in X : x \in \widehat{\{y\}}\} = \{y \in X : \tau \subset D_{xy}\}$.

Remark 2.5. $\widehat{\{x\}} = \widehat{\{y\}}$ iff $\overline{\{x\}} = \overline{\{y\}}$. For, $\widehat{\{x\}} = \widehat{\{y\}}$ iff $x \in \widehat{\{y\}}$ and $y \in \widehat{\{x\}}$ iff $y \in \overline{\{x\}}$ and $x \in \overline{\{y\}}$ iff $\overline{\{x\}} = \overline{\{y\}}$.

Remark 2.6. $\widehat{\{x\}} = x$ for each point $x \in X$ iff $\overline{\{x\}} = x$ for each point $x \in X$.

Remark 2.7. $\widehat{\{x\}} \cap \widehat{\{y\}} = \phi$ for any two distinct points x, $y \in X$ iff $\overline{\{x\}} \cap \overline{\{y\}} = \phi$ for any two distinct points x, $y \in X$.

Remark 2.8. If X is an infinite set, then we may have the same family of minimal sets at the points of X with respect to each of a class of topologies on X. For example, each of the class of the T_1- topologies has the same family of the minimal sets, that is $\widehat{\beta} = \{\widehat{\{x\}} : x \in X\} = \{\{x\} : x \in X\}$.

Remark 2.9. If (X, τ) is a topological space, then $\widehat{\{x\}} \notin \tau$ implies that each open set containing x is infinite. For let G be a finite subset of X, $G \in \tau$ and $x \in G$, then either $G - \widehat{\{x\}} = \phi$ which implies that $G = \widehat{\{x\}}$ or $G - \widehat{\{x\}} = \{x_1, x_2, \ldots, x_n\}$ and so for each $i \in \{1, 2, \ldots, n\}$ there exists an open set $G^i \in \tau$ such that $x \in G^i$ and $x^i \notin G^i$. Then, $\widehat{\{x\}} = G \cap G_1 \cap G_2 \cap \ldots \cap G_n$ which implies in both cases that $\widehat{\{x\}} \in \tau$.

Remark 2.10. If τ^* and τ are two topologies on a nonempty set X and $x \in X$ such that $\widehat{\{x\}}^* \neq \widehat{\{x\}}$ where $\widehat{\{x\}}^*_*$ and $\widehat{\{x\}}$ are the minimal sets at the point x with respect to τ^* and to τ respectively, then $\tau^* \neq \tau$ but not conversely as it shown by Remark 2.8. For, $t \in \widehat{\{x\}}^* - \widehat{\{x\}}$ implies that there is an open set $G \in \tau$ such that $x \in G$ and $t \notin G$ then $G \notin \tau^*$ because any open set in τ^* containing x, contains t because $t \in \widehat{\{x\}}^*$. Hence, $\tau^* \neq \tau$. Similarly: $t \in \widehat{\{x\}} - \widehat{\{x\}}^*$ implies that $\tau^* \neq \tau$.

Remark 2.11. Let (X, τ) be a topological space, then the following statements are equivalent:

(1) (X, τ) is T_0,
(2) $\widehat{\{x\}} \neq \widehat{\{y\}}$ for any two distinct points x, $y \in X$ and
(3) $\widehat{\{x\}} \cap \overline{\{x\}} = \{x\}$, for each point $x \in X$.

Remark 2.12. By using Remarks 2.6 and 2.7 a topological space (X, τ) is T^1 iff $\widehat{\{x\}} = \{x\}$ for each point $x \in X$ iff $\widehat{\{x\}} \cap \widehat{\{y\}} = \phi$ for any two distinct points x, $y \in X$.

Theorem 2.13. *Let (X, τ) be a regular topological space, then $\widehat{\{x\}} = \overline{\{x\}}$ for each $x \in X$. Moreover, the family $\widehat{\beta} = \{\widehat{\{x\}} : x \in X\}$ of the minimal sets at the points of X with respect to the topology τ on X is a partition of X.*

Proof. Suppose that (X,τ) is a regular topological space. Then, every open set containing x also contains $\overline{\{x\}}$, hence $\overline{\{x\}} \subset \widehat{\{x\}}$. If $y \in \widehat{\{x\}} - \overline{\{x\}}$, then there is an open set $G \in \tau$ such that $y \in G$ and $x \notin G$. Since τ is regular then there is an open set $V \in \tau$ such that $y \in V \subset \overline{V} \subset G$. Then, $X - \overline{V}$ is an open set containing x but not y, a contradiction. Thus, $\overline{\{x\}} = \widehat{\{x\}}$. If $y \in \widehat{\{x\}}$ then $x \in \widehat{\{y\}}$, and so $\widehat{\{x\}} \subset \widehat{\{y\}}$. On the other hand, $y \in \widehat{\{x\}} = \overline{\{x\}}$ implies that $\overline{\{y\}} \subset \overline{\{x\}}$, and thus $\overline{\{x\}} = \overline{\{y\}}$ whenever $y \in \widehat{\{x\}}$. This clearly shows that $\widehat{\beta} = \{\widehat{\{x\}} : x \in X\}$ must be a partition of X.

As a direct consequence of Theorem 2.13 we have the following corollary.

Corollary 2.14. *(X,τ) is a regular principal topological space iff $\widehat{\beta} = \{\widehat{\{x\}} : x \in X\}$ is a partition of X where each minimal set at x is open iff each open set is closed.*

Remark 2.15. As a direct consequence of Remarks 2.11, 2.12 and Theorem 2.13, if (X, τ) is regular and not T_1, then it is not T_0 i.e, a regular T_0 is T_3 which is an old and well known result.

Theorem 2.16. *Let (X, τ) be a topological space, then $\widehat{\beta} = \{\widehat{\{x\}} : x \in X\}$ is the minimal basis for a principal topology $\widehat{\tau}$ on X stronger than τ. If, τ is a principal topology on X, then $\widehat{\tau} = \tau$.*

Proof. Clearly; $\cup\{\widehat{\{x\}} : x \in X\} = X$, if x, y, $z \in X$ are distinct such that $x \in \widehat{\{y\}} \cap \widehat{\{z\}}$, then $\widehat{\{x\}} \subset \widehat{\{y\}} \cap \widehat{\{z\}}$. Therefore, $\widehat{\beta} = \{\widehat{\{x\}} : x \in X\}$ is a basis for some topology $\widehat{\tau}$ on X. If $x \in X$ and $G \in \widehat{\tau}$ such that $x \in G$, then there exists $y \in X$ such that $x \in \widehat{\{y\}} \subset G$ which implies that $\widehat{\{x\}} \subset G$. Hence, $\widehat{\{x\}} \in \widehat{\tau}$ is the minimal open set at the point x. Hence, $\widehat{\tau}$ is a principal topology on X and $\widehat{\beta}$ is its minimal basis. If τ is principal, then $\widehat{\beta} = \{\widehat{\{x\}} : x \in X\}$ is its minimal basis and $\widehat{\tau} = \tau$.

Example $\widehat{\beta} = \{\{x\} : x \in X\}$ is the family of the minimal sets at the points of X with respect to the minimal T_1- topology C on X, i.e the topology on X in which each proper nonempty subset is member iff its complement if finite. Hence, $\widehat{C} = D$ where D is the discrete topology on X.

3. Strictly weaker topologies

Theorem 3.1. *Let (X, τ) and (X, τ^*) be two principal topological spaces. Then, τ^* is a strictly weaker principal topology than τ iff there are two distinct points y and $z \in X$ satisfying the conditions:*

(1) $y \notin U_z$,
(2) $z \in U_x$ and $x \notin U_Z$ imply that $y \in U_x$ and
(3) $x \in U_y$ and $y \notin U_x$ imply that $x \in U_z$

and $\tau^* = \tau \cap D_{yz}$ having the minimal basis $\beta_{yz} = \{U_x, U_y \cup U_z : U_x \in (\beta - \{U_z\})\}$ where β is the minimal basis for τ, U_x is the minimal open set at the point x for each $x \in X$.

If (X,τ) is a topological space and $A \subset X$ in [9] defined the τ – minimal set at A to be $\widehat{A} = \bigcup\{\widehat{\{x\}} : x \in A\}$ and proved that $\widehat{A} = \bigcap\{G \in \tau : A \subset G\}$ and clearly $A \subset B \subset X$ implies that $\widehat{A} \subset \widehat{B}$. If (X,τ) and (X, τ^*) are two topological spaces and $A \subset X$ then $\tau^* \subset \tau$ implies that $\widehat{A} \subset \widehat{A}^*$ and $\overline{A} \subset \overline{A}^*$ where $\widehat{A}^*$ and $\overline{A}^*$ are $\widehat{A}$ and $\overline{A}$ with respect to τ^*, respectively. In this article a generalization of Theorem 3.1 will be given for any topology τ principal or nonprincipal on a nonempty set X. Theorem 3.1 will be a special case.

Lemma 3.2. *Let (X,τ) be a topological space, $y,z \in X$ be two distinct points, $\tau_{yz} = \tau \cap D_{yz}$ and $A \subset X$. Then*

(a) $\widehat{A}_{yz} \subset \widehat{A} \cup \widehat{\{y\}}$ and either $y \in \widehat{A}$ which implies that $\widehat{A}_{yz} = \widehat{A}$ or $y \notin \widehat{A}$ which implies that.

$$\widehat{A}_{yz} = \begin{cases} \widehat{A} & \text{if} \quad z \notin \widehat{A} \\ \widehat{A} \cup \widehat{y} & \text{if} \quad z \in \widehat{A} \end{cases}$$

(b) $\overline{A}_{yz} \subset \overline{A} \cup \overline{\{z\}}$ and either $z \in \overline{A}$ which implies that $\overline{A}_{yz} = \overline{A}$ or $z \notin \overline{A}$ which implies in [6] that

$$\overline{A}_{yz} = \begin{cases} \overline{A} & \text{if} \quad y \notin \overline{A} \\ \overline{A} \cup \overline{\{z\}} & \text{if} \quad y \in \overline{A} \end{cases}$$

Proof. (a) $t \notin \widehat{A} \cup \widehat{\{y\}}$ implies that there are two open sets $U, V \in \tau$ such that $A \subset U$, $y \in V$ and $t \notin U \cup V$which imply that $U \cup V \in \tau_{yz}$ and $A \subset U \cup V$ which imply that $t \notin \widehat{A}_{yz}$. Then $\widehat{A}_{yz} \subset \widehat{A} \cup \widehat{\{y\}}$. Also

(1) $z \notin \widehat{A}$ implies that there exists an open set $U \in \tau$ such that $A \subset U$ and $z \notin U$. Then $t \notin \widehat{A}$ implies that there is an open set $V \in \tau$ such that $A \subset V$ and $t \notin V$. Then $U \cap V \in \tau_{yz}$ since $z \notin U \cap V$, $A \subset U \cap V$ and $t \notin U \cap V$ which implies that $t \notin \widehat{A}_{yz}$ and so $\widehat{A}_{yz} = \widehat{A}$.

(2) If $z \in \widehat{A}$ then $G \in \tau_{yz}$ such that $A \subset G$ implies that $z \in G$ implies that $y \in G$ implies that $y \in \widehat{A}_{yz}$ implies that $\widehat{\{y\}} \subset \widehat{\{y\}}_{yz} \subset \widehat{A}_{yz}$. Hence $\widehat{A}_{yz} = \widehat{A} \cup \widehat{\{y\}}$.(b) $z \in \overline{A} \cup \overline{\{z\}}$ implies that $\overline{A} \cup \overline{\{z\}} \in \tau_{yzc}$ where $\tau^{yzc} = \{X - G : G \in \tau^{yz}\}$ implies that $\overline{A}_{yz} \subset \overline{A} \cup \overline{\{y\}}$. Also

(1) $y \notin \overline{A}$ implies that $\overline{A} \in \tau_{yzc}$ implies that $\overline{A}_{yz} = \overline{A}$.

(2) $y \in \overline{A}$ implies that $z \in \overline{A}_{yz}$ implies that $\overline{\{z\}} \subset \overline{\{z\}}_{yz} \subset \overline{A}_{yz}$ implies that $\overline{A}_{yz} = \overline{A} \cup \overline{\{z\}}$.

Theorem 3.3. *Let (X, τ) be a topological space, $\widehat{\beta} = \{\widehat{\{x\}} : x \in X\}$ be the family of the minimal sets at the points of X with respect to the topology τ and $y, z \in X$ be two distinct points satisfying the following conditions:*

(1) $y \notin \widehat{\{z\}}$,
(2) $z \in \widehat{\{x\}}$ and $x \notin \widehat{\{z\}}$ imply that $y \in \widehat{\{x\}}$ and
(3) $x \in \widehat{\{y\}}$ and $y \notin \widehat{\{x\}}$ imply that $x \in \widehat{\{z\}}$Then, $\widehat{\beta}_{yz} = \{\widehat{\{x\}}, \widehat{\{y\}} \cup \widehat{\{z\}} : \widehat{\{x\}} \in (\widehat{\beta} - \widehat{\{z\}})\}$ is the family of the minimal sets at the points of X with respect to the topology $\tau_{yz} = \tau \cap D_{yz}$ on X. If τ^* is a topology on X such that $\tau^* \neq \tau_{yz}$ and $\tau_{yz} \subset \tau^* \subset \tau$, then $\tau^*_{yz} = \tau^* \cap D_{yz} = \tau_{yz}$ and the families of the minimal sets at the points of X with respect to τ and τ^* are the same.

Proof By the condition (1) $y \notin \widehat{\{z\}}$ implies that $\widehat{\{z\}} \neq \widehat{\{y\}} \cup \widehat{\{z\}}$. Since $\tau_{yz} \subset \tau$ then $\widehat{\{x\}} \subset \widehat{\{x\}}_{yz}$ for each point $x \in X$. If, $G \in \tau$ such that $y \in G$, then $G \in \tau_{yz}$ which implies that $\widehat{\{y\}}_{yz} = \widehat{\{y\}}$. If $x \in X$ such that $\widehat{\{x\}} \notin \{\widehat{\{y\}}, \widehat{\{z\}}\}$, then we have two cases $z \in \widehat{\{x\}}$ or $z \notin \widehat{\{x\}}$. If $z \in \widehat{\{x\}}$,then by the condition (2) $y \in \widehat{\{x\}}$ and so $G \in \tau$ such that $x \in G$ implies that $G \in \tau_{yz}$ implies that $\widehat{\{x\}}_{yz} = \widehat{\{x\}}$.If, $z \notin \widehat{\{x\}}$ then by Lemma 3.2(a) $\widehat{\{x\}}_{yz} = \widehat{\{x\}}$. Also by Lemma 3.2(a) $\widehat{\{z\}}_{yz} = \widehat{\{y\}} \cup \widehat{\{z\}}$. Therefore, $\widehat{\beta}_{yz} = \{\widehat{\{x\}}, \widehat{\{y\}} \cup \widehat{\{z\}} : \widehat{\{x\}} \in (\widehat{\beta} - \{\widehat{\{z\}}\})\}$. By the condition (1) $y \notin \widehat{\{z\}}$ implies that $\tau \cap D_{yz} \neq \tau$ and clearly $\tau_{yz} \subset \tau$. If τ^* is a topology on X such that $\tau^* \neq \tau_{yz}$ then, $\tau_{yz} \subset \tau^* \subset \tau$ implies that $\tau_{yz} \subset \tau^* \cap D_{yz} \subset \tau \cap D_{yz} = \tau_{yz}$implies that $\tau^*_{yz} = \tau_{yz}$. Then, $\widehat{\{x\}}^*_{yz} = \widehat{\{x\}}_{yz}$ and so $\widehat{\beta}_{yz} = \{\widehat{\{x\}}, \widehat{\{y\}} \cup \widehat{\{z\}} : \widehat{\{x\}} \in (\widehat{\beta} -, \widehat{\{z\}})\} = \{\widehat{\{x\}}^*, \widehat{\{y\}}^* \cup \widehat{\{z\}}^* : \widehat{\{x\}}^* \in (\widehat{\beta}^* - \widehat{\{z\}}^*)\} = \widehat{\beta}^*_{yz}$ and $\widehat{\beta}^* = \{\widehat{\{x\}}^* : x \in X\}$. Since $\tau_{yz} \subset \tau^* \subset \tau$, $\widehat{\{x\}} \subset \widehat{\{x\}}^* \subset \widehat{\{x\}}_{yz} = \widehat{\{x\}}$ which implies that $\widehat{\{x\}}^* = \widehat{\{x\}}$ for each point $x \in X$ such that $\widehat{\{x\}} \neq \widehat{\{z\}}$ and $\widehat{\{z\}}_{yz} = \widehat{\{z\}}^*_{yz}$ implies that $\widehat{\{y\}} \cup \widehat{\{z\}} = \widehat{\{y\}}^* \cup \widehat{\{z\}}^*$. If, $t \in \widehat{\{z\}}^* - \widehat{\{z\}}$, then $t \in \widehat{\{z\}}^*$ implies that $t \in \widehat{\{y\}} \cup \widehat{\{z\}}$ implies that $t \in \widehat{\{y\}}$ because $t \notin \widehat{\{z\}}$ and we have two cases: (i) $y \notin \widehat{\{t\}}$ which implies by condition (3) that $t \in \widehat{\{z\}}$ and this contradicts that $t \notin \widehat{\{z\}}$. (ii) $y \in \widehat{\{t\}}$ which implies that $\widehat{\{y\}} \subset \widehat{\{t\}} \subset \widehat{\{t\}}^* \subset \widehat{\{z\}}^*$ because $\tau^* \subset \tau$ and $t \in \widehat{\{z\}}^*$which implies that $y \in \widehat{\{z\}}^*$ which implies that $\tau^* \subset D_{yz}$ which implies that $\tau^* = \tau^* \cap D_{yz} = \tau^*_{yz} = \tau_{yz}$ which contradicts that $\tau^* \neq \tau_{yz}$. Hence, such point t does not exist and so, $\widehat{\{z\}}^* = \widehat{\{z\}}$ because $\widehat{\{z\}} \subset \widehat{\{z\}}^*$. This completes the Proof.

Remark 3.4. Let X be an infinite set, y and z be two distinct points of X satisfy the conditions (1), (2) and (3) of Theorem 3.3 and τ be a nonprincipal topology on X. Then, $\widehat{\beta}_{yz} = \{\widehat{\{x\}}, \widehat{\{y\}} \cup \widehat{\{z\}} : \widehat{\{x\}} \in (\widehat{\beta} - \{\widehat{\{z\}}\})\}$ may be a family with respect to more than one topology on X. For, $\tau = \{G \subset X : y \notin G$ or $\{y,z\} \subset G$ such that $X - G$ is finite$\}$ $= E_y \cup (P_{\{y,z\}} \cap C)$ is a nonprincipal topology on X in which $\widehat{\{y\}} = \{y,z\}$ and $\widehat{\{z\}} = \{z\}$. So, $\widehat{\beta}_{yz} = \{\{x\}, \{y,z\} : x \in (X - \{y,z\})\}$ is the family of the minimal sets with respect to the topologies:

$\tau_1 = \{G \subset X : \{y,z\} \cap G = \phi$ or $\{y, z\} \subset G$ such that $X - G$ is finite$\}$ $= E_{\{y,\ z\}} \cup (P_{\{y,z\}} \cap C) = \tau_{yz}$ and
$\tau_2 = \{G \subset X : X - G$ is finite and either $\{y,z\} \cap G = \phi$ or $\{y,z\} \subset G\} \cup \{\phi\} = (E_{\{y,z\}} \cap C) \cup (P_{\{y,z\}} \cap C) = \tau^*_{yz}$
Clearly $\tau_1 = \tau \cap D_{yz} = \tau_{yz} \neq \tau_2 = \tau^* \cap D_{yz} = \tau^*_{yz}$ where
$\tau^* = \{G \subset X : X - G$ is finite and either $y \notin G$ or $\{y,z\} \subset G\} \cup \{\phi\} = (E_y \cap C) \cup (P_{\{y,\ z\}} \cap C)$

and clearly the minimal sets with respect to τ and τ^* are coincided, $\tau^* \subset \tau$ and $\tau^*_{yz} \subset \tau_{yz}$.In fact $\tau_1 = \tau \cap D_{yz}$ is not strictly weaker than τ for, if $\tau^{+} = \{G \subset X: \{y,z\} \cap G = \phi$ or $X - G$ is finite and either $\{y,z\} \cap G = \{z\}$ or $\{y,z\} \cap G = \{y,z\}\} = E_{\{y,z\}} \cup (E_y \cap P_z \cap C) \cup (P_{\{y,z\}} \cap C)$. Then, $\tau_1 = \tau_{yz} \subset \tau^{+} \subset \tau$ where $E_A = \{G \subset X: G \cap A = \phi\} \cup \{X\}$, $P_A = \{G \subset X: G \cap A = A\} \cup \{\phi\}$ and C is the cofinite topology on X. While $\tau_2 = \tau^* \cap D_{yz} = \tau^*_{yz}$ is a strictly weaker topology on X than τ^* for, if τ^{**} is a topology on X such that $\tau^*_{yz} \subset \tau^{**} \subset \tau^*$, then $G \in \tau^{**} - \tau^*_{yz}$ implies that $z \in G$ and $y \notin G$. Hence, $G \cup (X - \{y,z\}) = X - \{y\} \in \tau^{**}$ since $X - \{y,z\} \in \tau^*_{yz}$.So, if $G \subset X$ such that $X - G$ is finite, $z \in G$ and $y \notin G$, then $X - (G \cup \{y\}) = \{x_1, x_2, \ldots, x_n\}$ and$X - \{x_i\} \in \tau^*_{yz}$ for each $i \in \{1,2,\ldots,n\}$. So, $G = (X - \{y\}) \cap (X - \{x_1\}) \cap (X - \{x_2\}) \cap \ldots \cap (X - \{x_n\}) \in \tau^{**}$. Hence, $\tau^{**} = \tau^*$.

Remark 3.5. Let τ and τ^* be two topologies on a nonempty set X, $\tau^* \subset \tau$, $\tau \neq \tau^*$ and $y, z \in X$ be two distinct points satisfying the conditions (1), (2) and (3) of Theorem 3.3 then:

(1) $G \in \tau - \tau^*$ such that $y \in G$ or $y, z \notin G$ imply that $\tau_{yz} \neq \tau^*_{yz}$,
(2) $\tau_{yz} \subset \tau^* \subset \tau$ imply that $\tau - \tau^* \subset \{G \in \tau: z \in G$ and $y \notin G\} = \tau - \tau_{yz}$ and $\tau_{yz} = \tau^*_{yz}$.
(3) $\widehat{\tau}_{yz}$ is a strictly weaker principal topology than $\widehat{\tau}$ on X where $\widehat{\tau}$ is the topology on X defined by Theorem 2.16.

Corollary 3.6. *Let τ be a principal topology on X, β be the minimal basis for τ and $y, z \in X$ be two distinct points satisfying the conditions (1), (2) and (3) of Theorem* 3.1. Then, $\beta_{yz} = \{U_x, U_y \cup U_z : U_x \in (\beta - \{U_z\})\}$ *is the minimal basis for the principal topology $\tau_{yz} = \tau \cap D_{yz}$ which is strictly weaker than τ where U_x is the minimal open set at x for each point $x \in X$.*

Proof. It is a direct consequence of Theorem 3.3.

Lemma 3.7. *Let (X, τ) be a topological space and $x, y, z, t \in X$ such that $x \notin \widehat{\{z\}}$ and $y \notin \widehat{\{t\}}$. Then, $\widehat{\beta}_{xz} \neq \widehat{\beta}_{yt}$ implies that $\tau_{xz} \neq \tau_{yt}$.*

Proof. By Theorem 3.3; if $\widehat{\{t\}} \in \widehat{\beta}_{xz} - \widehat{\beta}_{yt}$ and $\widehat{\{z\}} \in \widehat{\beta}_{yt} - \widehat{\beta}_{xz}$ then there are two cases: (i) $\widehat{\{z\}} = \widehat{\{t\}}$ in this case if $G \in \tau$ then $z \in G$ iff $t \in G$. Since $\widehat{\{z\}}_{xz} = \cap\{G \in \tau_{xz} : z \in G\} = \cap\{G \in \tau : z, t, x \in G\}$ and $\widehat{\{t\}}_{yt} = \cap\{G \in \tau_{yt} : t \in G\} = \cap\{G \in \tau : z, t, y \in G\}$ hence $\widehat{\beta}_{xz} \neq \widehat{\beta}_{yt}$ implies that $\widehat{\{z\}}_{xz} \neq \widehat{\{t\}}_{yt}$ implies that $\{G \in \tau: z, t, x \in G\} \neq \{G \in \tau: z, t, y \in G\}$ implies that there exists $G \in \tau$ such that $z \in G$ and either (1) $x \in G$ and $y \notin G$ which implies that $G \in \tau_{xz} - \tau_{yt}$ or (2) $y \in G$ and $x \notin G$ which implies that $G \in \tau_{yt} - \tau_{xz}$. Therefore $\tau_{xz} \neq \tau_{yt}$. (ii) $\widehat{\{z\}} \neq \widehat{\{t\}}$ which implies that either $z \notin \widehat{\{t\}}$ or $t \notin \widehat{\{z\}}$. If $z \notin \widehat{\{t\}}$ then by Lemma 3.2(a) $\widehat{\{t\}}_{xz} = \widehat{\{t\}}$ and $\widehat{\{t\}}_{yt} = \widehat{\{y\}} \cup \widehat{\{t\}} \neq \widehat{\{t\}}$ which implies that $\widehat{\{t\}}_{xz} \neq \widehat{\{t\}}_{yt}$. Hence by Remark 2.10 $\tau_{xz} \neq \tau_{yt}$. Similarly, $t \notin \widehat{\{z\}}$implies that $\tau_{xz} \neq \tau_{yt}$.

Lemma 3.8. *Let (X,τ) be a topological space and $x, y, z, t \in X$ be such that $x \notin \widehat{\{z\}}$ and $y \notin \widehat{\{t\}}$. Then, $\tau_{xz} = \tau \cap D_{xz} = \tau \cap D_{yt} = \tau_{yt}$ iff $\widehat{\{x\}} = \widehat{\{y\}}$ and $\widehat{\{z\}} = \widehat{\{t\}}$.*

Proof. Clearly, $x \notin \widehat{\{z\}}$ and $y \notin \widehat{\{t\}}$ iff $\tau \notin \{\tau_{xz}, \tau_{yt}\}$. Suppose that $\widehat{\{x\}} = \widehat{\{y\}}$ and $\widehat{\{z\}} = \widehat{\{t\}}$. Then, by Remark 2.4 $y \in \widehat{\{x\}}$ and $z \in \widehat{\{t\}}$ implies that $\tau \subset D_{yx} \cap D_{zt}$ which implies that $\tau = \tau \cap D_{yx} \cap D_{zt}$ which implies that $\tau \cap D_{xz} = \tau \cap D_{yx} \cap D_{xz} \cap D_{zt} \subset \tau \cap D_{yz} \cap D_{zt} \subset \tau \cap D_{yt}$. Similarly, one can show that $\tau \cap D_{yt} \subset \tau \cap D_{xz}$.

Conversely; by Lemma 3.2(a) $\widehat{\{z\}}_{xz} = \widehat{\{x\}} \cup \widehat{\{z\}}$ and $\widehat{\{t\}}_{yt} = \widehat{\{y\}} \cup \widehat{\{t\}}$. Then, $\widehat{\{z\}} \neq \widehat{\{t\}}$ implis that $z \notin \widehat{\{t\}}$ which implies by Lemma 3.2(a) that $\widehat{\{t\}} \in \widehat{\beta}_{xz}$ which implies by Lemma 3.7 that $\tau_{xz} \neq \tau_{yt}$ because $\widehat{\{t\}} \notin \widehat{\beta}_{yt}$ or $t \notin \widehat{\{z\}}$ which implies by Lemma 3.7 that $\tau_{xz} \neq \tau_{yt}$ because $\widehat{\{z\}} \notin \widehat{\beta}_{xz}$. Hence, $\tau_{xz} = \tau_{yt}$ implies that $\widehat{\{z\}} = \widehat{\{t\}}$. If, $\widehat{\{z\}} = \widehat{\{t\}}$ and $\widehat{\{x\}} \neq \widehat{\{y\}}$, then either $x \notin \widehat{\{y\}}$ or $y \notin \widehat{\{x\}}$. Now $\widehat{\{z\}}_{xz} = \widehat{\{t\}}_{yt}$ implies that $\widehat{\{x\}} \cup \widehat{\{z\}} = \widehat{\{y\}} \cup \widehat{\{t\}}$ and there are two cases:

(1) $x \notin \widehat{\{y\}}$ implies that $x \in \widehat{\{t\}}$ which implies that $x \in \widehat{\{z\}}$ this contradicts the assumption that $x \notin \widehat{\{z\}}$.

(2) $y \notin \widehat{\{x\}}$ implies that $y \in \widehat{\{z\}}$ which implies that $y \in \widehat{\{t\}}$ this contradicts the assumption that $y \notin \widehat{\{t\}}$.

Hence, $\widehat{\{z\}} = \widehat{\{t\}}$ and $\widehat{\{x\}} \neq \widehat{\{y\}}$ imply that $\widehat{\{z\}}_{xz} \neq \widehat{\{t\}}_{yt}$ implies that $\widehat{\beta}_{xz} \neq \widehat{\beta}_{yt}$ implies by Lemma 3.7 that $\tau_{xz} \neq \tau_{yt}$. Its contra positive is if $x \notin \widehat{\{z\}}$ and $y \notin \widehat{\{t\}}$ then, $\tau_{xz} = \tau_{yt}$ implies that $\widehat{\{x\}} = \widehat{\{y\}}$. This completes the proof.

Remark 3.9. In general Lemma 3.8 is not true for, let X be an infinite set, x, y, z and t be distinct points of X and $X^* = X - \{t\}$. Then, $\tau = \{G \subset X^*: z \notin G$ or $\{x,z\} \subset G$ and $X - G$ is finite$\} \cup \{X\}$is a topology on X in which $\widehat{\{x\}} = \{x\}$, $\widehat{\{z\}} = \{x,z\}$, $\widehat{\{y\}} = \{y\}$ and $\widehat{\{t\}} = X$ which implies that $\tau_{xz} = \tau \cap D_{xz} = \tau = \tau \cap D_{yt} = \tau_{yt}$ while $\widehat{\{x\}} \neq \widehat{\{y\}}$ and $\widehat{\{z\}} \neq \widehat{\{t\}}$. Because of which the conditions $x \notin \widehat{\{z\}}$ and $y \notin \widehat{\{t\}}$ equivalently $\tau \notin \{\tau_{xz}, \tau_{yt},\}$ are given in Lemma 3.8.

Theorem 3.10. *Let (X,τ) be a topological space and y, z be two distinct points of X such that $\tau_{yz} = \tau \cap D_{yz}$ is strictly weaker than τ. Then the points y and z satisfy the conditions (1), (2) and (3) of Theorem* 3.3.

Proof. If $y \in \widehat{\{z\}}$, then $\tau = \tau \cap D_{yz}$ and accordingly $y \notin \widehat{\{z\}}$.If $z \in \widehat{\{x\}}$ then, $\tau = \tau \cap D_{zx}$ and so $\tau \cap D_{yz} = \tau \cap D_{yz} \cap D_{zx} \subset \tau \cap D_{yx} \subset \tau$ If $x \notin \widehat{\{z\}}$, then by Lemma 3.2(a) $\widehat{\{z\}}_{yx} = \widehat{\{z\}}$ implies that $\widehat{\{z\}}_{yz} \neq \widehat{\{z\}}_{yx}$ since, $\widehat{\{z\}}_{yz} \neq \widehat{\{z\}}$ because $y \notin \widehat{\{z\}}$ which implies by Remark 2.10 that $\tau \cap D_{yz} \neq \tau \cap D_{yx}$ which implies that $\tau \cap D_{yx} = \tau$ because τ_{yz} is strictly weaker than τ and so $y \in \widehat{\{x\}}$. Clearly by Lemma 3.2(a), $\widehat{\{z\}}_{yz} = \widehat{\{y\}} \cup \widehat{\{z\}}$ and if $x \in \widehat{\{y\}}$, then $x \in \widehat{\{z\}}_{yz}$ which implies that $\tau \cap D_{yz} \subset D_{xz}$ which implies that $\tau \cap D_{yz} \subset \tau \cap D_{xz} \subset \tau$. Now $\widehat{\{z\}}_{yz} = \widehat{\{z\}}_{xz}$ implies that $\widehat{\{y\}} \cup \widehat{\{z\}} = \widehat{\{x\}} \cup \widehat{\{z\}}$ and so $y \notin \widehat{\{x\}}$ implies that $y \in \widehat{\{z\}}$ implies that $\tau \subset D_{yz}$ implies that $\tau \cap D_{yz} = \tau$ which contradicts that $\tau \cap D_{yz}$ is strictly weaker than τ. Then, $\widehat{\{z\}}_{yz} \neq \widehat{\{z\}}_{xz}$ which implies by remark (2.10) that $\tau \cap D_{yz} \neq \tau \cap D_{xz}$ which implies that $\tau \cap D_{xz} = \tau$ which implies that $x \in \widehat{\{z\}}$. This completes the proof.

Corollary 3.11. *If, (X,τ) is a principal topological space, then $\tau_{yz} = \tau \cap D_{yz}$ is a strictly weaker principal topology on X than τ iff y and z satisfy the conditions (1), (2) and (3) of Theorem* 3.3.

Theorem 3.12. *Let (X,τ) and (X,τ^*) be two topological spaces and τ^* be strictly weaker than τ such that τ and τ^* have different families $\widehat{\beta}$ and $\widehat{\beta}^*$ of minimal sets. Then, there are two distinct points $y,z \in X$ satisfy the conditions (1), (2) and (3) of Theorem* 3.3 *such that $\tau^* = \tau \cap D_{yz} = \tau_{yz}$.*

Proof. For each point $x \in X$, let $\widehat{\{x\}}$ and $\widehat{\{x\}}^*$ be the minimal sets at x with respect to τ and τ^*, respectively. Since $\widehat{\beta}^* \neq \widehat{\beta}$ then there is a point $z \in X$ such that $\widehat{\{z\}}^* \neq \widehat{\{z\}}$, then there is a point $y \in \widehat{\{z\}}^* - \widehat{\{z\}}$ since $\widehat{\{z\}} \subset \widehat{\{z\}}^*$ because $\tau^* \subset \tau$. Then, τ is not contained in D_{yz} because $y \notin \widehat{\{z\}}$ which implies that $\tau \neq \tau \cap D_{yz}$ and $\tau^* \subset D_{yz}$ because $y \in \widehat{\{z\}}^*$ and so, $\tau^* \subset \tau$ implies that $\tau^* \subset \tau \cap D_{yz} \subset \tau$. If, τ^* is strictly weaker than τ then, $\tau^* = \tau \cap D_{yz} = \tau_{yz}$. If, there is a point $t \in X - \{z\}$ such that $\widehat{\{t\}}^* \neq \widehat{\{t\}}$, then using the same argument there is a point $x \in \widehat{\{t\}}^* - \widehat{\{t\}}$ such that$\tau^* = \tau \cap D_{xt}$. Hence, $\tau \cap D_{yz} = \tau \cap D_{xt}$ and so by Lemma 3.8 $\widehat{\{x\}} = \widehat{\{y\}}$ and $\widehat{\{z\}} = \widehat{\{t\}}$. Clearly by Theorem 3.10 y and z satisfy the conditions (1), (2) and (3) of Theorem 3.3.

Corollary 3.13. *Theorem* 3.1 *is a direct consequence of Corollary* 3.11 *and Theorem* 3.12.

Remark 3.14. By using Remark 2.4 one can write the conditions (1), (2) and (3) of Theorem 3.3 as follows:

(1) $z \notin \overline{\{y\}}$,
(2) $x \in \overline{\{z\}}$ and $z \notin \overline{\{x\}}$ imply that $x \in \overline{\{y\}}$ and
(3) $y \in \overline{\{x\}}$ and $x \notin \overline{\{y\}}$ imply that $z \in \overline{\{x\}}$.

Proposition 3.15. *Let τ be a topology on a nonempty set X and $\tau_{yz} = \tau \cap D_{yz}$ be a topology on X satisfies the condition (3) of Theorem* 3.3. Then, $\overline{\{x\}}_{yz} = \overline{\{x\}}$ *for each $x \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$.*

Proof. If $\overline{\{x\}} \neq \overline{\{y\}}$ then either $y \notin \overline{\{x\}}$ which implies by Lemma 3.2(b) that $\overline{\{x\}}_{yz} = \overline{\{x\}}$ or $x \notin \overline{\{y\}}$ and $y \in \overline{\{x\}}$ which implies by condition (3) of Theorem 3.3 that $z \in \overline{\{x\}}$ and so again by Lemma 3.2(b) $\overline{\{x\}}_{yz} = \overline{\{x\}}$.

Theorem 3.16. *Let (X,τ) be a T_0 topological space and $y,z \in X$ be two distinct points satisfying the conditions (1), (2) and (3) of Theorem* 3.3. Then, (X,τ_{yz}) *is T_o iff $z \notin \widehat{\{y\}}$*

Proof. If, (X,τ_{yz}) is T_o, then by Remark 2.11 $\widehat{\{y\}}_{yz} \neq \widehat{\{z\}}_{yz}$ and by Lemma 3.2(a) $\widehat{\{y\}}_{yz} = \widehat{\{y\}}$ and $\widehat{\{z\}}_{yz} = \widehat{\{y\}} \cup \widehat{\{z\}}$ which implies that $z \notin \widehat{\{y\}}$.

Conversely; if $z \notin \widehat{\{y\}}$ then, $\widehat{\{y\}} \neq \widehat{\{y\}} \cup \widehat{\{z\}}$ and so $\widehat{\{y\}}_{yz} \neq \widehat{\{z\}}_{yz}$. If (X,τ) is T_o and $x \in X$ then, $\widehat{\{x\}} \neq \widehat{\{z\}}$ and by Theorem 3.3 $\widehat{\beta}_{yz} = \{\widehat{\{x\}}, \widehat{\{y\}} \cup \widehat{\{z\}} : \widehat{\{x\}} \notin (\widehat{\beta} - \{\widehat{\{z\}}\})\}$. Then, $\widehat{\{x\}}_{yz} \neq \widehat{\{t\}}_{yz}$ for any two distinct points $t,x \in X - \{z\}$. If there is a point $x \in X - \{y,z\}$ such that $\widehat{\{x\}}_{yz} = \widehat{\{z\}}_{yz}$, then $\widehat{\{x\}} = \widehat{\{y\}} \cup \widehat{\{z\}}$. Then, $y,z \in \widehat{\{x\}}$ and either $x \in \widehat{\{y\}}$ which implies that $\widehat{\{x\}} = \widehat{\{y\}}$ or $x \in \widehat{\{z\}}$ which implies that $\widehat{\{x\}} = \widehat{\{z\}}$ which contradicts that (X,τ) is T_o. This contradiction means that $\widehat{\{x\}}_{yz} \neq \widehat{\{z\}}_{yz}$ for each point $x \in X$. Hence (X,τ_{yz}) is T_o.

Corollary 3.17. *If, (X,τ) is T_1 and $y,z \in X$ are any two distinct points, then:*

(1) by Remark 2.11 (X,τ_{yz}) is T_o and $(X, (\tau_{yz})_{zy})$ is not T_o.
(2) by Remark 2.12 (X,τ_{yz}) is not T_1.

Theorem 3.18. *Let (X, τ) be a regular topological space and $y,z \in X$ be two distinct points satisfying the conditions (1), (2) and (3) of Theorem* 3.3. Then, (X,τ_{yz}) *is not regular and $(X, (\tau_{yz})_{zy})$ is regular.*

Proof. It is a direct consequence of Theorems 2.13 and 3.3.

Theorem 3.19. *Let X be an infinite set, $p \in X$ and $y,z \in X - \{p\}$ be any two distinct points. Then (1) $\tau = C \cup E_p = \{G \subset X: p \notin G$ or $X - G$ is finite$\}$ is a topology on X where C is the cofinite topology and E_p is the excluding point topology onX with the excluding point p and (2) $\tau_{yz} = \tau \cap D_{yz}$ is a strictly weaker topology on X than τ.*

Proof. $\tau_{yz} = \tau \cap D_{yz} = (C \cap D_{yz}) \cup (E_p \cap D_{yz}) = C_{yz} \cup (E_p)_{yz}$. If $\tau_{yz} \subset \tau^* \subset \tau$ then $G \in \tau^* - \tau_{yz}$ implies that $G \in \tau$ such that $z \in G$ and $y \notin G$. Now $G \in \tau^*$ implies that $X - \{y,z\} \cup G = X - \{y\} \in \tau^*$ because $X - \{y, z\} \in C_{yz} \subset \tau^*$ and $x \in X$ such that $x \neq y$ implies that $X - \{x\} \in C_{yz}$ and so $\{X - \{x\}: x \in X\} \subset \tau^*$ which implies that $C \subset \tau^*$ and $\{y,z\} \cap G = \{z\} \in \tau^*$ since $\{y,z\} \in (E_p)_{yz} \subset \tau^*$ and hence $\{z\} \in \tau^*$ which implies that $E_p \subset \tau^*$. So, $\tau^* = \tau$. Therefore τ_{yz} is a strictly weaker topology on X than τ.

Theorem 3.20. *Let X be an infinite set, (X,C) be the minimal T_1 topological space and $y,z \in X$ be any two distinct points. Then, $C_{yz} = C \cap D_{yz}$ is a strictly weaker topology on X than C*

Remark 3.21. If (X,τ) is T_1, then $C_{yz} \subset \tau_{yz}$ for any two points $y,z \in X$.

Remark 3.22. If (X,τ) is a topological space, then $\tau_{yz} \subset \widehat{\tau}_{yz} \subset \widehat{\tau}$ and $\tau_{yz} \subset \tau \subset \widehat{\tau}$. If τ is a principal topology on X, then $\tau_{yz} = \widehat{\tau}_{yz}$ is a strictly weaker topology on X than τ.

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