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ORIGINAL ARTICLE

A small time solutions for the KdV equation using Bubnov-Galerkin finite element method

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Abstract A Bubnov-Galerkin finite element method with quintic B-spline functions taken as element shape and weight functions is presented for the solution of the KdV equation. To demonstrate the accuracy, efficiency and reliability of the method three experiments are undertaken for both the evolution of a single solitary wave and the interaction of two solitary waves. The numerical results are compared with analytical solutions and the numerical results in the literature. It is shown that the method presented is accurate, efficient and can be used at small times when the analytical solution is not known.

1. Introduction

In this paper we consider the Korteweg-de Vries (KdV) equation in the form,

$$U_t + \varepsilon U U_x + U_{xxx} = 0 \quad a \leqslant x \leqslant b \tag{1}$$

where $U(x,t)$ is an appropriate field variable, ε and μ are positive parameters, and the subscripts t and x denote differentiation with respect to the time and the space, respectively. The KdV Eq. (1) is a one-dimensional non-linear partial differential equation (PDE) of third order, which plays a major role in the study of non-linear dispersive waves. This equation was originally derived by Korteweg-de Vries [1] to describe the behavior of one-dimensional shallow water solitary waves. Solitary waves are wave packets or pulses which propagate in non-linear dispersive media. For stable solitary wave solutions the non-linear and dispersive terms in the KdV Eq. (1) must balance, and in this case the KdV equation has traveling wave solutions called solitons. A soliton is a very special type of solitary waves which keeps its waveform after collision with other solitons.

A small time solutions using a heat balance integral (HBI) method to solve the KdV equation was obtained by Kutluay et al. [2]. In their paper, extensive comparisons with the analytical values over the defined interval are given. Bahadir [3] used the exponential finite-difference (EFD) technique to solve the KdV equation. This method has been shown to provide higher accuracy than the classical explicit finite difference and the HBI method. Ozer and Kutluay [4] used an analytical–numerical

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(AN) method, for solving the KdV equation and the obtained results are compared with that of the HBI method and the corresponding analytical solution. Irk et al. [5] used a second order spline approximation (SA) technique and made comparisons with earlier methods. Ozdes and Aksan [6] used the method of lines (MOL) for solving the KdV equation and also in [7] used a quadratic B-spline Galerkin finite element (QBGFE) method and compared these techniques with the analytical solutions and other numerical solutions that are obtained earlier using various numerical techniques.

In this paper, we present an algorithm for solving Eq. (1) by applying Bubnov Galerkin finite element method. The time integration of the resulting system is carried out using Crank–Nicholson scheme. Evolution and interaction of solitary waves with various amplitudes are undertaken.

The presence of the third spatial derivative in Eq. (1) requires that the interpolation functions and their first and second derivatives must be continuous throughout the region of solution. When using Bubnov Galerkin, the quintic B-splines interpolation functions can be used with partial differential equations containing derivatives up to order four.

The results obtained are compared with their corresponding analytical solutions and also with the various numerical methods mentioned above. To check accuracy, efficiency and reliability of the scheme presented we evaluate the invariants and error norms for the simulations undertaken.

2. Finite element scheme

A numerical solution to the KdV Eq. (1) is sought over the finite region $[a, b]$ with boundary conditions as will be prescribed. Let $a = x_0 < x_1 < \ldots < x_N = b$ be a partition of $[a, b]$ by the equally spaced knots x_i and let $\phi_i(x)$ be those quintic B-splines with knots at the points x_i, $0 < i < N$. The set of splines $\{\phi_{i-2}, \phi_{i-1}, \phi_i, \phi_{i+1}, \phi_{i+2}, \phi_{i+3}\}$ forms a basis for functions defined over the finite region $[a, b]$. We seek the approximation $U_N(x, t)$ to the solution $U(x, t)$ which uses

$$U_N(x,t) = \sum_{i=-2}^{N+2} \phi_i(x) u_i(t) \tag{2}$$

where the u_i are time dependent parameters to be determined from the boundary conditions and from conditions to be determined herein.

$$U(a,t) = U(b,t) = 0, \quad U_x(a,t) = U_x(b,t) = 0 \tag{3}$$

We identify the finite elements with the intervals $[x_i, x_{i+1}]$ with nodes at x_i and x_{i+1}. Each quintic B-splines covers six elements: consequently each element $[x_i, x_{i+1}]$ is covered by six splines $(\phi_{i-2}, \phi_{i-1}, \phi_i, \phi_{i+1}, \phi_{i+2}, \phi_{i+3})$ which are given in terms of a local coordinate system ζ given by $h\zeta = (x - x_i)$ where $h = x_{i+1} - x_i$ and $0 \leqslant \zeta \leqslant 1$. Leads to the following expressions for these splines over the element $[x_i, x_{i+1}]$ are [8,9],

$$\begin{aligned}
\phi_{i-2} &= 1 - 5\zeta + 10\zeta^2 - 10\zeta^3 + 5\zeta^4 - \zeta^5 \\
\phi_{i-1} &= 26 - 50\zeta + 20\zeta^2 + 20\zeta^3 - 20\zeta^4 + 5\zeta^5 \\
\phi_i &= 66 - 60\zeta^2 + 30\zeta^4 - 10\zeta^5 \\
\phi_{i+1} &= 26 + 50\zeta + 20\zeta^2 - 20\zeta^3 - 20\zeta^4 + 10\zeta^5 \\
\phi_{i+2} &= 1 + 5\zeta + 10\zeta^2 + 10\zeta^3 + 5\zeta^4 - 5\zeta^5 \\
\phi_{i+3} &= \zeta^5
\end{aligned} \tag{4}$$

The spline $\phi_i(x)$ and its three derivatives vanish outside the interval $[x_{i-3}, x_{i+3}]$. These spline act like "shape" functions for the element when we set up equations in terms of the element parameters u_i^e using Eq. (4). The variation of $U_N(x, t)$ over the element $[x_{i-3}, x_{i+3}]$ is given by

$$u^e(x,t) = \sum_{j=i-2}^{i+3} \phi_j(x) u_j(t) \tag{5}$$

The nodal value of $U_N(x, t)$ and the derivatives at the knots are given in terms of the element parameters by

$$\begin{aligned}
U_i &= u_{i-2} + 26u_{i-1} + 66u_i + 26u_{i+1} + u_{i+2}, \\
hU_i' &= 5(u_{i+2} + 10u_{i+1} - 10u_{i-1} - u_{i-2}), \\
h^2 U_i'' &= 20(u_{i-2} + 2u_{i-1} - 6u_i + 2u_{i+1} + u_{i+2}), \\
h^3 U_i''' &= 60(u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}), \\
h^4 U_i'''' &= 120(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}),
\end{aligned} \tag{6}$$

where the dashes denote differentiation with respect to x. An application of the Galerkin's method to Eq. (1) with weight functions $W(x)$, leads to

$$\int_a^b W(U_t + \varepsilon U U_x + \mu U_{xxx}) dx = 0 \tag{7}$$

Now, we set up the relevant element matrices. For typical element $[x_i, x_{i+1}]$ we have the contribution,

$$\int_e W(u_t^e + \varepsilon u^e u_x^e + \mu u_{xxx}^e) dx$$

Replacing the weight function $W(x)$ and the unknown values $u(t)$ from (5) by B-spline shape functions (4),

$$\sum_{i=l-2}^{l+3} \left(\int_{x_l}^{x_{l+1}} \phi_k \phi_i dx \right) \dot{u}_i^e + \varepsilon \sum_{j=l-2}^{l+3} \sum_{i=l-2}^{l+3} \left(\int_{x_l}^{x_{l+1}} \phi_k \phi_i \phi_j' dx \right) u_i^e u_j^e + \mu \sum_{i=l-2}^{l+3} \left(\int_{x_l}^{x_{l+1}} \phi_k phi_i''' dx \right) u_i^e \tag{8}$$

which in matrix form is

$$A^e \dot{u}^e + \varepsilon u^{eT} F^e u^e + \mu D^e u^e \tag{9}$$

Where

$$u^e = (u_{l-2}, u_{l-1}, u_l, u_{l+1}, u_{l+2}, u_{l+3})^T. \tag{10}$$

The element matrices are given by the integrals

$$\begin{aligned}
A_{ij}^e &= \textstyle\int_{x_l}^{x_{l+1}} \phi_i \phi_k dx, \\
F_{ijk}^e &= \textstyle\int_{x_l}^{x_{l+1}} \phi_i \phi_j' \phi_k dx, \\
D_{ij}^e &= \textstyle\int_{x_l}^{x_{l+1}} \phi_i''' \phi_j dx,
\end{aligned} \tag{11}$$

where i, j, k take only $l-2$, $l-1$, l, $l+1$, $l+2$, $l+3$ for this element $[x_l, x_{l+1}]$. The matrices A^e, D^e are therefore 6×6 and F^e is $6 \times 6 \times 6$. We use the associated 6×6 matrix L^e instead of F^e in our algorithm

$$L_{ij}^e = \sum_{k=l-2}^{l+3} F_{ijk}^e u_k^e, \tag{12}$$

which depends upon the parameters u_k^e. The element matrices A^e, F^e, D^e can be determined algebraically from Eq. (11), (see Appendix A), where u_k^e is given by Eq. (10). The assembly of the element Eq. (9) leads to the equation

$$Au + (\varepsilon L + \mu D)u = 0 \tag{13}$$

where the matrices A, D, L are assembled from the element matrices A^e, D^e, L^e in the usual way and

$$u = (u_{-2}, u_{-1}, u_0, \cdots, u_{N+1}, u_{N+2})^T. \tag{14}$$

3. Crank–Nicholson scheme

To obtain a recurrence relationship for the solution of Eq. (1), time center on $(n + \frac{1}{2})\Delta t$, where Δt is the time step and use Crank–Nicholson approach [10] with

$$u = \frac{1}{2}\left(u^n + u^{n+1}\right), \quad \dot{u} = \frac{1}{\Delta t}\left(u^{n+1} - u^n\right) \tag{15}$$

Substituting (15) into (13), we obtain

$$\frac{A}{\Delta t}\left(u^{n+1} - u^n\right) + \frac{1}{2}(\varepsilon L + \mu D)\left(u^{n+1} + u^n\right) = 0 \tag{16}$$

And then

$$\left(A + \frac{\Delta t}{2}\varepsilon L + \frac{\Delta t}{2}\mu D\right)u^{n+1} = \left(A - \frac{\Delta t}{2}\varepsilon L + \frac{\Delta t}{2}\mu D\right)u^n, \tag{17}$$

where the superscripts n and $n + 1$ are time labels. The system (17) consists of $N + 1$ linear equations in $N + 5$ unknowns. To obtain a unique solution to this system we need four additional conditions. These are obtained from the boundary conditions and can be used to eliminate u_{-2}, u_{-1}, $u_0, \cdots, u_{N+1}$, u_{N+2} from the recurrence relationships (17) so that the solution set becomes an 11 banded $(N + 5) \times (N + 5)$ matrix equation. An inner iteration at each time step is carried out to ensure convergence of the non-linear term. The iteration algorithm is as follows:

1. At startup u^0 is known from the initial conditions. From Eq. (17) we calculate a first approximation u_1^1 to u using $u = u^0$. A second approximation u_2^1 is then found with $u = \frac{1}{2}(u^0 + u_1^1)$, and a third u_3^1 with $u = \frac{1}{2}(u^0 + u_2^1)$. On this first step we find 10 iterations are usually sufficient to obtain a good approximation for u^1.
2. On the general step to find a first approximation u_1^{n+1} to u^{n+1}, we use $u = u^n + \frac{1}{2}(u^n + u^{n-1})$ A second approximation u_2^{n+1} is then found from, $u = \frac{1}{2}(u^n + u^{n+1})$ and so on. Two or three iterations are usually sufficient to obtain convergence.

The time evolution of u^n and hence $U_N(x, t)$ can be started once the initial vector of the parameters u^0 is obtained.

4. The initial state

From the initial condition we can determine the vector u^0 so that the time evaluation of u^{n+1} using recurrence relationships (17) can be started. Rewrite the global trial functions Eq. (2), we get

$$U_N(x, 0) = \sum_{i=-2}^{N+2} \phi_i(x) u_i^0 \tag{18}$$

where u_i^0 are unknown parameters to be determined. To determine the initial vector u_i^0, we require U_N satisfying the following constraints:

1. It must agree with the analytical initial condition at the knots x_j; using Eq. (6) Lead to $N + 1$ conditions, and
2. The first and second derivatives of the approximate initial condition shall be zero at both ends of the range; Eq. (6) produce four further equations.

The start up vector u^0 is then determined as the solution of matrix equations.

$$Mu^0 = b \tag{19}$$

Where

$$M = \begin{bmatrix} 3 & 30 & 27 & & & & & & \\ 1 & 18 & 33 & 8 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & \\ & & . & . & . & . & . & & \\ & & & 1 & 26 & 66 & 26 & 1 & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 8 & 33 & 18 & 1 \\ & & & & & & 27 & 30 & 3 \end{bmatrix} \tag{20}$$

$$b = (U''(x_0), U'(x_0), U(x_0), U(x_1), \cdots, U(x_N), U'(x_N), U''(x_N))^T \tag{21}$$

$$u^0 = (u_-^0 2, u_-^0 1, u_0^0, \cdots, u_N^0, u_{N+1}^0, u_{N+2}^0)^T$$

The initial vector u^0 is then determined as the solution of the penta-diagonal matrix Eq. (19), the system is solved by a variant of Thomas algorithm.

5. KdV simulations

A numerical algorithm for the KdV equation should possess the following properties:

1. The migration of solutions should be adequately described.
2. The numerical scheme should exhibit the same conservation laws as the differential equation.

The numerical algorithm developed in Section 3 will be validated by studying test problems concerned with the migration and interaction of solitons. We use the L_2 and L_∞ error norms to measure the difference between the numerical and analytical solutions and hence to show how well the scheme predicts the position and amplitude of the solution as the simulation proceeds. The L_2 and L_∞ error norms of the solution are defined by

$$\begin{aligned} L_2 &= \|U^{\text{exact}} - U^n\|_2 = [h\sum_{i=1}^{N}|U_i^{\text{exact}} - U_i^n|^2]^{\frac{1}{2}}, \\ L_\infty &= \|U^{\text{exact}} - U^n\|_\infty = max_i|U_i^{\text{exact}} - U_i^n| \end{aligned} \tag{22}$$

Also, the conservation properties of the numerical scheme will be examined by calculating the lowest three invariants corresponding to conservation of mass I_1, momentum I_2 and energy I_3,

$$\begin{aligned} I_1 &= \textstyle\int_a^b U dx \approx h\sum_{i=1}^{N} U_i^n, \\ I_2 &= \textstyle\int_a^b U^2 dx \approx h\sum_{i=1}^{N}(U_i^n)^2, \\ I_3 &= \textstyle\int_a^b (U^3 - \frac{3}{\varepsilon}\mu U_x^2) dx \approx h\sum_{i=1}^{N}(U_i^n)^3 - \frac{3}{\varepsilon}\mu((U_x)_i^n)^2 \end{aligned} \tag{23}$$

The integrals are approximated by sums and the variables U_i^n and their derivatives are found from Eq. (6). Accuracy of the method will also be measured with percentage error (PE) defined by:

$$PE = \frac{|\text{Exact value} - \text{Approximate value}|}{\text{Exact value}} \times 100 \quad (24)$$

Experiment 1. *Single solitary wave simulations*

To allow comparison with earlier works, we solve the KdV Eq. (1) with the boundary conditions

$$U(0,t) = U(2,t) = 0, \quad t > 0 \quad (25)$$

And the initial condition

$$U(x,0) = 3c\,sech^2(ax - x_0), \quad 0 \leqslant x \leqslant 2 \quad (26)$$

The analytical solution is given by

$$U(x,t) = 3c\,sech^2(ax - bt - x_0), \quad 0 \leqslant x \leqslant 2 \quad (27)$$

where c and x_0 are constants, a and b defined by $a = \frac{1}{2}\sqrt{(\varepsilon c/\mu)}$ and $b = a\varepsilon c$. We choose the parameters $\mu = 4.84 \times 10^{-4}$ and $\varepsilon = 1.0$. For assessing the accuracy of the present scheme, we use $c = 0.3$, $x_0 = 6.0$, $\Delta t = 0.005$, $\Delta x = 0.01$ and carry out the simulation up to $t = 3.0$. The error norms L_2 and L_∞ as well as the first three invariants I_1, I_2 and I_3 are recorded in Table 1, for times up to $t = 3.0$. As it can be seen in Table 1, all three invariants I_1, I_2 and I_3 are satisfactorily constant changing by 5.1×10^{-4}%, 3.4×10^{-3}% and 1.3×10^{-1}% of their original values respectively during the simulation. The L_2 and L_∞ error norms are also recorded and the L_2 norm is less than 7×10^{-8}, while the L_∞ norm is less than 4.4×10^{-7}.

To compare the results of the simulations obtained by the present algorithm with their corresponding analytical values and also with the numerical solutions obtained by other numerical methods in the literature [2–7], Eq. (17) is solved at times $t = 0.005$ and $t = 0.01$ with $x_0 = 6.0$, $c = 0.3$, $\Delta t = 0.001$ and $\Delta x = 0.0125$. To measure the difference between the numerical and exact solutions, the percentage error, defined by Eq. (24), is used and displayed in Table 2, together with a comparison with earlier results in the literature [2–7]. It is seen that the present method produces less percentage error than those in the literature.

Experiment 2. We consider the KdV Eq. (1) subject to the boundary conditions

$$U(0,t) = U(4,t) = 0, \quad t > 0 \quad (28)$$

and the initial condition which will be derived from the analytical solution [6] given as

$$U(x,t) = 12\mu(\log F)_{xx}, \quad 0 \leqslant x \leqslant 4 \quad (29)$$

Where

Table 1 Invariants and error norms for the single solitary wave of the KdV equation at $t = 0, 0.5, \cdots, 3$.

t	I_1	I_2	I_3	$I_2 \times 10^8$	$I_\infty \times 10^7$
0	0.144598097	0.0867593065	0.0467906334	1.62397562	0.59604644
0.5	0.144598857	0.0867593363	0.0468500480	2.42225333	1.19209290
1.0	0.144598886	0.0867593437	0.0468500815	2.45920830	1.19209290
1.5	0.144598842	0.0867593139	0.0468500555	1.76728587	0.59604644
2.0	0.144598871	0.0867593288	0.0468500741	1.88711393	0.59604644
2.5	0.144598886	0.0867593437	0.0468500778	1.61578591	0.59604644
3.0	0.144598842	0.0867593363	0.0468499027	6.80048728	4.30688829

Table 2 Percentage errors of experiment (1) for some selected values of x.

Time	Method	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
0.005	HBI [2]	3.7987	2.9327	3.2960	3.6626	3.6652
	EFD [3]	3.7752	2.9319	3.2940	3.6382	3.6286
	AN [4]	0.0271	0.1105	0.0216	0.0470	0.1067
	SA [5]	0.0016	0.0884	0.0392	0.0007	0.0000
	MOL [6]	0.0129	0.0137	0.0024	0.0154	0.0000
	QBGFE [7]	0.0107	0.0104	0.0104	0.0096	0.0000
	Present	0.0000	0.0002	0.0006	0.0007	0.0000
0.01	HBI [2]	7.7419	5.9806	6.4701	7.1909	7.1961
	EFD [3]	30.753	0.6980	0.6379	32.357	30.864
	AN [4]	0.0701	0.2344	0.0393	0.0984	0.1029
	SA [5]	0.0020	0.3647	0.1503	0.0050	0.0000
	MOL [6]	0.0251	0.0232	0.0026	0.0311	0.0000
	EQBGF [7]	0.0242	0.0042	0.0195	0.0240	0.0000
	Present	0.0000	0.0002	0.0007	0.0007	0.0000

$$F = 1 + \exp(\eta_1) + \exp(\eta_2) + \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\right)^2 \exp(\eta_1 + \eta_2)$$

$$\eta_i = \alpha_i x - \alpha_i^3 \mu t + b_i, \quad (i = 1, 2),$$

$$\alpha_1 = \sqrt{\frac{0.3}{\mu}}, \quad \alpha_2 = \sqrt{\frac{0.1}{\mu}}, \quad b_1 = -0.48\alpha_1, \quad b_2 = -1.07\alpha_2$$

Taking the parameters $\mu = 4.84 \times 10^{-4}$, $\varepsilon = 1.0$, $x_0 = 6.0$, $c = 0.3$, $\Delta t = 0.001$ and $\Delta x = 0.0125$.

Table 3 records the invariants and error norms for the same experiment.

As it can be seen in Table 3, all three invariants I_1, I_2 and I_3 are satisfactorily constant changing by $3.6 \times 10^{-4}\%$, 0.0% and $4.3 \times 10^{-2}\%$ of their original values respectively during the

Table 3 Invariants and error norms for the experiment 2 of the KdV equation at $t = 0, 1, \cdots, 6$.

t	I_1	I_2	I_3	$L_2 \times 10^8$	$L_\infty \times 10^7$
0	0.0990547816	0.0195130249	0.0023114084	9.51884576	2.65299403
1	0.0990551323	0.0195130249	0.0023240522	9.51884576	2.64979965
2	0.0990551338	0.0195130249	0.0025207695	9.51884576	2.59474802
3	0.0990551341	0.0195130249	0.0026153580	9.51884578	1.95004410
4	0.0990551341	0.0195130249	0.0027354542	9.51884577	2.5307659
5	0.0990551341	0.0195130249	0.0023408480	9.51884576	2.65437841
6	0.0990551341	0.0195130249	0.0023123914	9.51884576	2.66182834

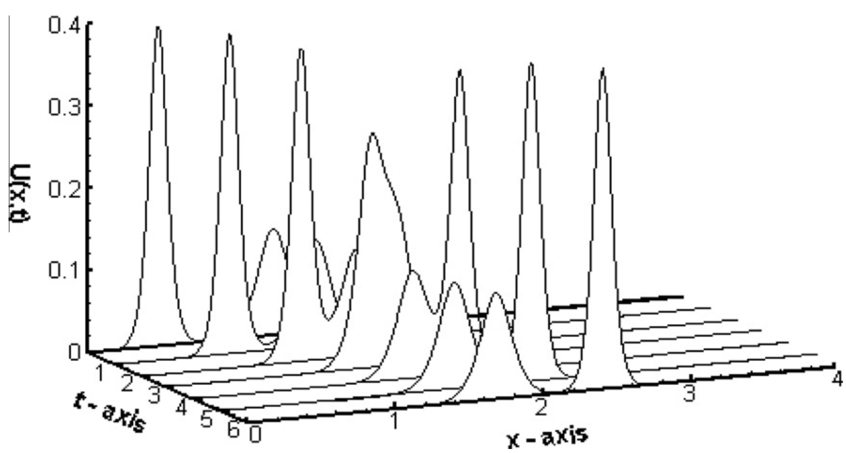

Figure 1 The solution of experiment 2 at $t = 0, 1, \cdots, 6$.

Table 4 Percentage errors of experiment (2) for some selected values of x.

Time	Method	$x = 0.4$	$x = 0.8$	$x = 1.2$	$x = 1.6$	$x = 2.0$
0.005	HBI [2]	2.8928	3.3738	1.1503	3.2403	3.2848
	EFD [3]	1.2138	6.2065	0.3269	4.0980	2.1853
	AN [4]	2.7820	0.0956	0.3860	0.0291	0.0000
	SA [5]	2.0640	0.0030	0.3895	0.0336	0.0000
	MOL [6]	2.0798	0.0093	0.9311	5.9459	0.0000
	QBGFE [7]	2.0295	0.0015	0.3879	0.3333	0.0000
	Present	0.0000	0.0000	0.0000	0.0809	0.0000
0.01	HBI [2]	5.8995	6.6342	2.2426	6.5870	6.6775
	EFD [3]	2.7756	2.1970	0.3489	3.2415	3.9930
	AN [4]	5.7381	0.1874	0.7759	0.0591	0.0000
	SA [5]	4.1864	0.0089	0.7862	0.0684	0.0000
	MOL [6]	4.2541	0.0208	0.7903	0.0728	0.0000
	EQBGF [7]	4.3404	0.0030	0.7850	0.0684	0.0000
	Present	0.0001	0.0000	0.0000	0.0000	0.0000

Table 5 Invariants and error norms for the interaction of two solitary waves, experiment 3, with different amplitudes, at times $t = 0, 1, \cdots, 6$.

t	I_1	I_2	I_3	$L_2 \times 10^8$	$L_\infty \times 10^7$
0	0.228081778	0.107062168	0.0533164255	2.53124242	1.19209291
0.5	0.228082836	0.116139457	0.0629577562	3.02784713	1.19209291
1.0	0.228082955	0.138995498	0.0908666700	5.06542044	2.98023224
1.5	0.228082970	0.169026405	0.1332870720	3.75928941	1.19209291
2.0	0.228083014	0.170222476	0.1350912150	4.99049406	2.38418579
2.5	0.228082836	0.140746310	0.0931761935	4.06871976	2.38418579
3.0	0.228082657	0.117008276	0.0639321804	2.61114312	1.19209291
3.5	0.228080586	0.107343979	0.0535983741	3.22539364	1.19209291

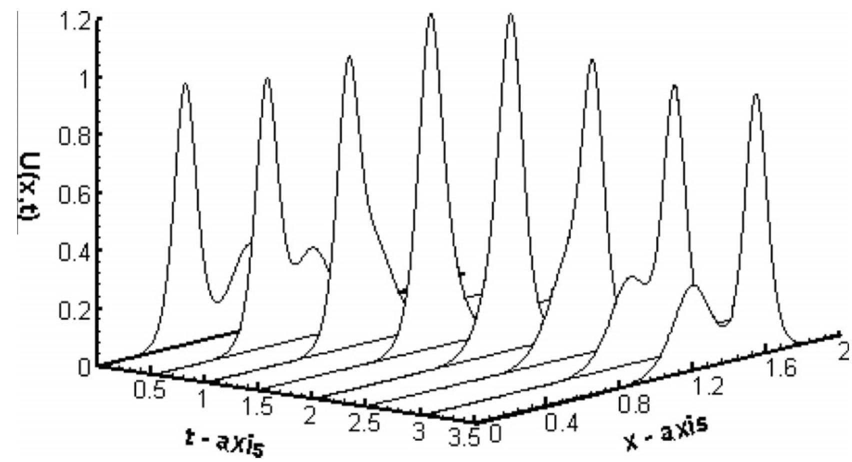

Figure 2 Interaction of two solitary waves, experiment 3, with different amplitudes, at $t = 0, 0.5, \cdots, 3.5$.

simulation. The L_2 and L_∞ error norms are also recorded and the L_2 norm is less than 1.0×10^{-7} while the L_∞ norm is less than 3.0×10^{-7}, while Fig. 1 represents the graphical solution of experiment 2, for times$t = 0, 1, \cdots, 6$, respectively.

To compare the results of the simulations obtained by the present algorithm with their corresponding analytical values and also with the numerical solutions obtained by other numerical methods in the literature [2–7], we display in Table 4 the percentage errors obtained at times $t = 0.005$ and $t = 0.01$, respectively. To measure the difference between the numerical and exact solutions, the percentage error, defined by Eq. (24), is used and displayed in Table 4, together with a comparison with earlier results in the literature [2–7]. It is seen that the present method produces less percentage error than those in the literature.

Experiment 3. *Two soliton simulations*The linear sum of two separated solitons of various amplitudes is considered as the initial condition

$$U(x,t) = 3c_1 sech^2(a_1 x - x_1) + 3c_2 sech^2(a_2 x - x_2), 0 \leqslant x \leqslant 2 \quad (30)$$

and the boundary conditions

$$U(0,t) = U(2,t) = 0, t > 0 \quad (31)$$

Where

$$a_i = \frac{1}{2}\sqrt{(\varepsilon c_i/\mu)}, \text{ and } b_i = a_i \varepsilon c_i, (i = 1, 2)$$

In this experiment, we study the behavior of two solitons with different amplitudes traveling in the same direction. We take the parameters $\mu = 4.84 \times 10^{-4}$, $\varepsilon = 1.0$, $c_1 = 0.3$, $c_2 = 0.1$, $x_1 = x_2 = 6$, $\Delta t = 0.005$ and $\Delta x = 0.005$. Table 5 shows the invariants and error norms for experiment 3. The three invariantsI_1, I_2 and I_3 are monitored and the experiment is stopped when the three invariants retained their original values within acceptable error. The three invariants changed by5.3×10^{-4}%, 2.7×10^{-1}% and 5.3×10^{-1}% of their original values, respectively. The L_2 and L_∞ error norms are also recorded and the L_2 norm is less than 3.5×10^{-8} while the L_∞ norm is less than 1.2×10^{-7}. Fig. 2 displays the interaction of two soliton waves at $t = 0, 0.5, \cdots, 3.5$, respectively. It is observed that the two solitons pass through each other and then emerge unchanged.

6. Conclusions

The KdV equation is a transient nonlinear dispersive equation so that any numerical scheme that simulates this equation must represent faithfully all the features of this equation. To fulfill these requirements, we have constructed a one dimensional

B-spline finite element scheme based on Bubnov Galerkin together with shape and weight functions taken as quintic B-spline functions to cope with the third derivative of Eq. (1). Discretization in time is set up using Crank–Nicholson scheme. This leads to a nonlinear system of equations with 11 diagonal matrices. All calculations were performed in Fortran code under a core 2 duo 2.0 MHz processor using double precession arithmetic.

The performance of the method was examined on two test problems (experiments 1 and 2) with known exact solutions. The obtained numerical results indicated that the present method produces more accurate results than the mentioned results in the literature [2–7] compared with the corresponding analytical solutions. The method is then used to study the interaction of two solitons (experiment 3). The results obtained proved the method to be reliable, accurate and efficient through the calculated error norms.

We believe that the scheme presented can be useful for other applications where continuity of derivatives is essential.

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Appendix A. The element matrices A^e, D^e and L^e are calculated algebraically using the computer's program ''Mathematica5'' and the results are as follows,

$$A^e = \frac{h}{2772}\begin{pmatrix} 252 & 9113 & 29558 & 15498 & 1018 & 1 \\ 9113 & 397416 & 1558706 & 1072186 & 121641 & 1018 \\ 29558 & 1558706 & 7464456 & 6602476 & 1072186 & 15498 \\ 15498 & 1072186 & 6602476 & 7464456 & 1558706 & 29558 \\ 1018 & 121641 & 1072186 & 1558706 & 397416 & 9113 \\ 1 & 1018 & 15498 & 29558 & 9113 & 252 \end{pmatrix}$$

$$D^e = \frac{1}{14h^2}\begin{pmatrix} -105 & 145 & 190 & -390 & 155 & 5 \\ -4485 & 3045 & 16710 & -23550 & 7215 & 1065 \\ -16990 & -5650 & 107940 & -120020 & 23770 & 10950 \\ -10950 & -23770 & 120020 & -107940 & 5650 & 16990 \\ -1065 & -7215 & 23550 & -16710 & -3045 & 4485 \\ -5 & -155 & 390 & -190 & -145 & 105 \end{pmatrix}$$

$$L^e_{11} = \frac{1}{18018}(-6006, -75510, -15900, 85800, 11610, 6)u^e$$

$$L^e_{12} = \frac{1}{18018}(-196479, -2640045, -762400, 3123360, 474915, 649)u^e$$

$$L^e_{13} = \frac{1}{18018}(-586644, -8316700, -2956980, 10159260, 1697200, 3864)u^e$$

$$L^e_{14} = \frac{1}{18018}(-277134, -4198800, -1855260, 5330580, 997050, 3564)u^e$$

$$L^e_{15} = \frac{1}{18018}(-14814, -255690, -157760, 348300, 79470, 494)u^e$$

$$L^e_{16} = \frac{1}{18018}(-3, -155, -300, 300, 155, 3)u^e$$

$$L^e_{21} = \frac{1}{18018}(-196479, -2640045, -762400, 3123360, 474915, 649)u^e$$

$$L^e_{22} = \frac{1}{18018}(-6900078, -105555450, -50414400, 135287940, 27405030, 176958)u^e$$

$$L^e_{23} = \frac{1}{18018}(-21839788, -376504110, -256736460, 516367540, 136899000, 1813818)u^e$$

$$L^e_{24} = \frac{1}{18018}(-11104728, -225391560, -223879220, 334775580, 122871300, 2728628)u^e$$

$$L^e_{25} = \frac{1}{18018}(-687693, -19558365, -32766420, 32766420, 19558365, 687693)u^e$$

$$L^e_{26} = \frac{1}{18018}(-494, -79470, -348300, 157760, 255690, 14814)u^e$$

$$L^e_{31} = \frac{1}{18018}(-586644, -8316700, -2956980, 10159260, 1697200, 3864)u^e$$

$$L^e_{32} = \frac{1}{18018}(-21839788, -376504110, -256736460, 516367540, 136899000, 1813818)u^e$$

$$L^e_{33} = \frac{1}{18018}(-72572448, -1514993520, -1621139520, 2278115040, 907749120, 22841328)u^e$$

$$L^e_{34} = \frac{1}{18018}(-39222888, -1065460520, -1757435280, 1757435280, 1065460520, 39222888)u^e$$

$$L^e_{35} = \frac{1}{18018}(-2728628, -122871300, -334775580, 223879220, 225391560, 11104728)u^e$$

$$L^e_{36} = \frac{1}{18018}(-3564, -997050, -5330580, 1855260, 4198800, 277134)u^e$$

$$L^e_{41} = \frac{1}{18018}(-277134, -4198800, -1855260, 5330580, 997050, 3564)u^e$$

$$L^e_{42} = \frac{1}{18018}(-11104728, -225391560, -223879220, 334775580, 122871300, 2728628)u^e$$

$$L^e_{43} = \frac{1}{18018}(-39222888, -1065460520, -1757435280, 1757435280, 1065460520, 39222888)u^e$$

$$L^e_{44} = \frac{1}{18018}(-22841328, -907749120, -2278115040, 1621139520, 1514993520, 72572448)u^e$$

$$L^e_{45} = \frac{1}{18018}(-1813818, -136899000, -516367540, 256736460, 376504110, 21839788)u^e$$

$$L^e_{46} = \frac{1}{18018}(-3864, -1697200, -10159260, 2956980, 8316700, 586644)u^e$$

$$L^e_{51} = \frac{1}{18018}(-14814, -255690, -157760, 348300, 79470, 494)u^e$$

$$L^e_{52} = \frac{1}{18018}(-687693, -19558365, -32766420, 32766420, 19558365, 687693)u^e$$

$$L^e_{53} = \frac{1}{18018}(-2728628, -122871300, -334775580, 223879220, 225391560, 11104728)u^e$$

$$L^e_{54} = \frac{1}{18018}(-1813818, -136899000, -516367540, 256736460, 376504110, 21839788)u^e$$

$$L^e_{55} = \frac{1}{18018}(-176958, -27405030, -135287940, 50414400, 105555450, 6900078)u^e$$

$$L^e_{56} = \frac{1}{18018}(-649, -474915, -3123360, 762400, 2640045, 196479)u^e$$

$$L^e_{61} = \frac{1}{18018}(-3, -155, -300, 300, 155, 3)u^e$$

$$L^e_{62} = \frac{1}{18018}(-494, -79470, -348300, 157760, 255690, 14814)u^e$$

$$L^e_{63} = \frac{1}{18018}(-3564, -997050, -5330580, 1855260, 4198800, 277134)u^e$$

$$L^e_{64} = \frac{1}{18018}(-3864, -1697200, -10159260, 2956980, 8316700, 586644)u^e$$

$$L^e_{65} = \frac{1}{18018}(-649, -474915, -3123360, 762400, 2640045, 196479)u^e$$

$$L^e_{66} = \frac{1}{18018}(-6, -11610, -85800, 15900, 75510, 6006)u^e$$

References

[1] P.G. Drazin, R.S. Johnson, Solitons: an introduction, Cambridge University press, 1989.

[2] S. Kutluay, A.R. Bahadir, A. Ozdes, A small time solutions for the Korteweg-de Vries equation, Appl. Math. Comput 107 (2000) 203–210.

[3] A.R. Bahadir, Exponential finite-difference method applied to Korteweg-de Vries equation for small times, Appl. Math. Comput 160 (2005) 675–682.

[4] S. Ozer, S. Kutluay, An analytical–numerical method for solving the Korteweg-de Vries equation, Appl. Math. Comput 164 (2005) 789–797.

[5] D. Irk, I. Dag, B. Saka, A small time solutions for the Korteweg-de Vries equation using spline approximation, Appl. Math. Comput 173 (2006) 834–846.

[6] A. Ozdes, E.N. Aksan, The method of lines solution of the Korteweg-de Vries equation for small times, Int. J. Contemp. Math. Sci. 1 (13) (2006) 639–650.

[7] A. Ozdes, E.N. Aksan, Numerical solution of Korteweg-de Vries equation by Galerkin B-spline finite element metho, Appl. Math. Comput 175 (2006) 1256–1265.

[8] L.R.T. Gardner, G.A. Gardner, F.A. Ayoub, N.K. Amein, Approximation of solitary waves of the MRLW equation by B-spline finite elements, Arab J. Sci. Eng. Sect. A Sci. A 2 (1997) 183–193.

[9] L.R.T. Gardner, G.A. Gardner, F.A. Ayoub, N.K. Amein, Modelling an undular bore with B-splines, Comput. Method Appl. Mech. Eng 147 (1-2) (1997) 147–152.

[10] L.R.T. Gardner, G.A. Gardner, F.A. Ayoub, N.K. Amein, Simulations of the EW undular bore, Commun. Numer. Method Eng. 13 (7) (1997) 583–592.