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ORIGINAL ARTICLE

Solitary wave solutions for a certain class of nonlinear differential equations

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Abstract Generalized forms of exact solitary wave solutions of the class (1.1) are investigated. The analysis rests mainly on the standard a direct algebraic method. The most general solutions are obtained, possibly having a constant term in their expansion into real exponentials. These solutions of the class (1.1) are performed under certain conditions for the relationship between the coefficients of the nonlinear, dispersive and dissipative terms. The analytical solutions of this class are of *pulse-type* and of *kink-type* solitary wave solutions and they are obtained with an arbitrary constant phase shift.

1. Introduction

A vast variety of the powerful and direct methods to find all kinds of analytic solutions of partial differential equations (PDEs) have been developed. Among these are Hirota's bilinear technique [1], inverse scattering transform [2], Painlevè expansions [3], direct algebraic method (the direct real-exponential method) [4], Bäcklund transformation method [1] and so on.

Among these, a direct algebraic method [4], it can be used to investigate nonlinear dispersive and dissipative problems for obtaining the solutions of solitary waves. This method represents the solutions as infinite series in real exponentials that satisfy the linearized equations. The coefficients of these series satisfy the nonlinear recursion relations and the series is then summed in closed form and the exact solitary wave solutions are obtained.

In this paper, we consider the class of third order nonlinear dispersive dissipative PDEs of the form

$$u_t + u_x + a_1 u^n u_x + a_2 u^n u_t + a_3 u_{xx} + a_4 u_{xt} + a_5 u_{tt} + a_6 u_{xxx} + a_7 u_{xxt} + a_8 u_{xtt} + a_9 u_{ttt} = 0, \quad (1.1)$$

in which $a_i (i = 1, 2, \ldots, 9)$ are real constants and $u(x, t)$ is a real scalar function defined for all $(x, t) \in \mathbf{R} \times I$, where t denotes a real variable in the interval $I = (0, \infty)$. We shall here and henceforth, assume that $u(x, t)$ is continuous for all values of its respective arguments and that the various partial derivatives of u with respect to x and t exist and are continuous.

For $n = 1$; the class (1.1) incorporates the Korteweg-de Vries (KdV) equations without dissipative terms [2,5,6]:

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$$u_t + uu_x + u_{xxx} = 0, \tag{1.2}$$

$$u_t + puu_x + u_{xxx} = 0, \tag{1.3}$$

$$u_t + puu_x - \mu u_{xxx} = 0, \tag{1.4}$$

and their alternatives. Also this class contains the *KdV* equation with a dissipative term or the KdV-Burgers (*KdVB*) equation [7]:

$$u_t + puu_x + qu_{xx} - \mu u_{xxx} = 0, \tag{1.5}$$

and its alternatives, where p, q and μ are real constants.

For $n = 2$; this class, also incorporates the modified $KdV(mKdV)$ equations [5,8]:

$$u_t + u^2 u_x + u_{xxx} = 0, \tag{1.6}$$

$$u_t + pu^2 u_x + u_{xxx} = 0, \tag{1.7}$$

and their alternatives.

Also, the class (1.1) contains *mKdVB* equation [7]:

$$u_t + pu^2 u_x + qu_{xx} - \mu u_{xxx} = 0, \tag{1.8}$$

and its alternatives, where p, q and μ are defined as before. These Eqs. (1.2) and (1.8) and their alternatives are widely used to describe complex phenomena in various fields of science, especially in physics, as solid-states physics [9], fluid dynamics [10] and plasma physics [11].

According to the well-known *KdV* Eq. (1.1) [6], which is special case of the class (1.1), is a nonlinear *PDE* that models the time-dependent wave motion in one space dimension in media with nonlinear wave steepening and dispersion, such as shallow water waves and ion acoustic plasma waves. The pioneering study in [6], showed that when nonlinear wave steepening, from the term uu_x, is balanced by wave dispersion, owing to the term u_{xxx}, their equation predicts a unidirectional solitary wave, that is a *pulse* which moves in one direction with a permanent shape and a constant speed . Also, the Eqs. (1.3), (1.4), (1.6) and (1.7) have $pluse - type$ solitary wave solutions. But the Eqs. (1.5) and (1.8) have *kink-type* solutions of the solitary waves.

The first and well-known alternative to the *KdV* equation, was derived in [12], that is Regularized Long Wave (*RLW*) equation:

$$u_t + u_x + uu_x - \mu u_{xxt} = 0, \tag{1.9}$$

where μ is defined as before. Other alternative to the *KdV* equation was established in [13]:

$$u_t + uu_x - \mu u_{xxt} = 0, \tag{1.10}$$

this equation is called the Equal Width (*EW*) equation and it can be easily transformed into (1.8) by the transformation $u \to u + 1$, and therefor, both (1.9) and (1.10) have very similar analytical solitary wave solutions.

Furthermore, the class (1.1) that contains two general types of *RLW* equations, established in [14]. The first type of *RLW*, is the generalized Equal Width (*gEW*) equation without dissipative term:

$$u_t + pu^n u_x - \mu u_{xxt} = 0, \tag{1.11}$$

in which the solitary wave of the *pulse-type* solution. The second type of *RLW*, is the generalized EW-Burgers (*gEWB*) equation with a dissipative term:

$$u_t + pu^n u_x - qu_{xx} - \mu u_{xxt} = 0, \tag{1.12}$$

and the solution of this case of a *kink-type*. These alternatives to the *KdV* equations are propose on the basis that their dispersive and dissipative properties are physically and mathematically preferable to these of the *KdV* equations.

The purpose of this paper, is to find the general solutions of the class (1.1) for $n = 1$ or $n = 2$, by a direct algebraic method. However, these solutions of this class may have a constant term in their expansion into real exponentials. In [15,16], we obtained particular solitary wave solutions of the class (1.1) for $n = 1$ and for $n = 2$; where, we neglected the integration constant to look only for solutions without the constant term in their expansion. Moreover, those solutions which have obtained of *pulse-type* and *kink-type* solutions with an arbitrary constant phase shift.

2. General solutions of the class (1.1)

In this section we demonstrate how to construct solutions of the class (1.1), using a direct algebraic method [4], that contain a constant term in their expansion into real exponentials. To obtain the stationary solutions for the class (1.1), we introduce a traveling frame of reference $\xi = x - ct$ to transform the PDEs of (1.1) into an ODE in $f(\xi) = u(x,t)$:

$$-f\prime + \alpha f^n f' + \beta f'' + \gamma f''' = 0, \tag{2.1}$$

where

$$\alpha = \frac{a_1 - a_2 c}{c - 1}, \quad \beta = \frac{a_3 - a_4 c + a_5 c^2}{c - 1}, \quad \gamma = \frac{a_6 - a_7 c + a_8 c^2 - a_9 c^3}{c - 1}, \tag{2.2}$$

in which $c \neq 1$ is the anticipated traveling wave velocity, and the derivatives are performed with respect to the co-ordinate ξ. Integrating (2.1) once with respect to ξ, to get

$$-f + \frac{\alpha}{n+1} f^{n+1} + \beta f' + \gamma f'' + k_1 K = 0, \tag{2.3}$$

where $k_1 K$ is an integration constant As we shall see below, the purpose of this integration constant is to facilitate exponential solutions for the linear part of the transformed equation in ϕ:

$$f = k_1 + \phi, \tag{2.4}$$

where k_1 is a constant. Indeed, substitution of (2.4) in ((2.3) yields

$$-\phi + \frac{\alpha}{n+1}(k_1 + \phi)^{n+1} + \beta\phi' + \gamma\phi'' + k_1(K - 1) = 0. \tag{2.5}$$

Case 1: $n = 1, \beta = 0$:

For the case $n = 1$ and $\beta = 0$, (2.5) takes the form

$$(\alpha k_1 - 1)\phi + \frac{\alpha}{2}\phi^2 + \gamma\phi'' + k_1\left(\frac{\alpha}{2}k_1 + K - 1\right) = 0, \tag{2.6}$$

if

$$c = \frac{1}{2a_5}\left(a_4 \pm \sqrt{a_4^2 - 4a_3 a_5}\right), \tag{2.7}$$

as a sufficient condition.

The linear part of (2.6) (*i.e.*, for $\alpha = 0$ with $c = a_1/a_2$) has two real exponential solutions of the form $\exp[\pm\lambda(\gamma)\xi]$ with

$$\lambda^2 = \frac{1}{\gamma}(1 - \alpha k_1) = \frac{1}{\gamma}(2K - 1), \tag{2.8}$$

if and only if the ϕ-independent part in (2.6) is set equal to *zero*; *i.e.*,

$$k_1 = \frac{2}{\alpha}(1 - K). \tag{2.9}$$

Using (2.8) and (2.9), the transformed Eq. (2.6) becomes

$$(1-2K)\phi + \frac{\alpha}{2}\phi^2 + \gamma\phi'' = 0. \tag{2.10}$$

We now seek stationary solutions of ϕ in terms of the harmonics of, say, decaying exponentials. To this end, we scale ϕ according to

$$\phi = \frac{2}{\alpha}(2K-1)\psi, \tag{2.11}$$

and expand ψ in terms of the harmonics of the decaying exponential solution to the linear equation:

$$\psi = \sum_{m=1}^{\infty} b_m g^m(\xi), \tag{2.12}$$

$$g(\xi) = exp(-\lambda\xi). \tag{2.13}$$

We next substitute Eqs. (2.11)–(2.13) into (2.10), to get the recursion relation

$$(m^2-1)b_m + \sum_{l=0}^{m-1} b_l b_{m-1} = 0, \quad m \geqslant 2, \tag{2.14}$$

and b_1 arbitrary, where use has been made of Eqs. (2.8) and (2.9) to simplify. The $b'_m s$ are then given by our paper [15]:

$$b_m = 6m(-1)^{m+1}b^m, \quad \left(b = \frac{b_1}{6}, \quad b_1 > 0, m = 1, 2, 3, \ldots\right). \tag{2.15}$$

So that with (2.4), (2.9), (2.11) and (2.12), the closed form solution f may be written as:

$$f = \frac{2}{\alpha}(1-K) + \frac{2}{\alpha}(2K-1)\frac{6bg}{(1+bg)^2}. \tag{2.16}$$

Note that the closed form (2.16) for f has been built up from a convergent power series of decaying exponentials $g(\xi)$ for $bg < 1$ (i.e., in the region $\xi > \xi_0, \xi_0 = \frac{\ln b}{\lambda}$). However, (2.16) is also expressible as a convergent power series in $\frac{1}{bg}$ for $bg > 1$ (i.e., in the region $\xi < \xi_0$), where $\frac{1}{g}$ $(= \exp(\lambda\xi))$ is a (bounded) rising exponential solution to the linear equation. Since (2.16) is continuous at $bg = 1$, it is therefore, a valid solution over the entire region $-\infty < \xi < \infty$. Physically speaking, this means that the solution in the region $\xi \geqslant \xi_0$ (built up from harmonics of decaying exponentials) provides the boundary conditions for the solution in region $\xi \leqslant \xi_0$ (built up from harmonics of rising exponentials) ensuring continuity at $\xi = \xi_0$ The final solution $u(x,t)$ may now be expressed, using (2.8), (2.13) and (2.16) as

$$u(x,t) = \frac{2}{\alpha}(1-K) + \frac{3}{\alpha}(2K-1)\text{sech}^2\left[\frac{1}{2}\left(\frac{2K-1}{\gamma}\right)^{\frac{1}{2}}(x-ct)+\delta\right], \tag{2.17}$$

where $\delta(= \frac{1}{2}\ln\frac{1}{b})$ represents an arbitrary constant phase factor, α and γ are defined in (2.2).

As special cases of (2.17), note that the choice of the constant K as 1 and $\frac{1}{4}$, respectively, lead to the sech^2 solution [15] for the class (1.1):

$$u(x,t) = \frac{3}{\alpha}\text{sech}^2\left[\frac{1}{2\sqrt{\gamma}}(x-ct)+\delta\right], \tag{2.18}$$

and a new-type of tanh^2 solution:

$$u(x,t) = \frac{3}{2\alpha}\tanh^2\left[\frac{1}{2\sqrt{\gamma}}(x-ct)+\delta\right], \tag{2.19}$$

where $\alpha, \gamma > 0$ and δ are defined as before.

Moreover; if $u \to u - \frac{1}{p}, n = 1, a_1 = p$ and $a_i = 0, (i = 2, 3, 4, 5, 7, 8, 9)$, then (1.1) reduced to the *KdV* Eq. (1.3) [17,18]. Hence, from (2.2), we get

$$\alpha = \frac{p}{c}, \quad \beta = 0, \quad \gamma = \frac{1}{c} > 0, \quad (c > 0). \tag{2.20}$$

Using (2.20) in (2.18) and (2.19), respectively, lead to the well-known sech^2 solution of the *KdV* Eq. (1.3) [17,18]

$$u(x,t) = \frac{3c}{p}\text{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct)+\delta\right], \tag{2.21}$$

and a ‘well’-type tanh^2 solution

$$u(x,t) = \frac{3c}{2p}\tanh^2\left[\frac{1}{2}\sqrt{\frac{-c}{2}}(x-ct)+\delta\right]. \tag{2.22}$$

In passing, it is interesting to observe that the solution $u(x,t)$ in (2.17) contains a constant term and a sech^2-type term, both of which are individually solutions of the class (1.1). The general solution u, given by (2.17), may therefore be visualized as the superposition of two particular solutions of (1.1) of appropriate amplitudes which are now locked together with a different velocity $(\frac{2K-1}{\gamma_{as}})$, where $\frac{1}{\gamma_{as}}$ is the velocity of the free (associated) solutions of the class (1.1), for $n = 1$ and $\beta = 0$.

Case 2: $n = 2, \beta = 0$:

Following the above analysis, we derive here the solitary wave solutions of the class (1.1), for $n = 2$ and $\beta = 0$ under the sufficient condition (2.7), thus (2.5) becomes

$$(\alpha k_1^2 - 1)\phi + \alpha k_1\phi^2 + \frac{\alpha}{3}\phi^3 + \gamma\phi'' + k_1\left(\frac{\alpha}{3}k_1^2 + K - 1\right) = 0. \tag{2.23}$$

The linear part of (2.23) has exponential solutions $\exp[\pm\lambda(\gamma)\xi]$, if

$$k_1\left(\frac{\alpha}{3}k_1^2 + K - 1\right) = 0. \tag{2.24}$$

For $k_1 = 0$, (2.23) takes the form [15]

$$-\phi + \frac{\alpha}{3}\phi^3 + \gamma\phi'' = 0. \tag{2.25}$$

In this case, we find

$$\lambda = \frac{1}{\sqrt{\gamma}}, \quad \gamma > 0. \tag{2.26}$$

Hence, we perform the scaling

$$f = \phi = \sqrt{\frac{3}{\alpha}}\psi, \quad \alpha > 0. \tag{2.27}$$

Next, we use (2.27), (2.12), (2.13) and (2.8) into (2.25) to get the recursion relation

$$(m^2-1)b_m + \sum_{r=2}^{m-1}\sum_{l=1}^{r-1} b_l b_{r-l} b_{m-r} = 0, \quad m \geqslant 3, \tag{2.28}$$

where b_1 arbitrary and $b_2 = 0$, from which the general structure of b_m may be calculated as [15]

$$b_{2m} = 0, \tag{2.29}$$

$$b_{2m+1} = \frac{(-1)^m b_1^{2m+1}}{2^{3m}}, \quad m = 1, 2, 3, \ldots \tag{2.30}$$

Using (2.29) and (2.30) in (2.12), ψ may be expressed in closed form as

$$\psi(\xi) = \frac{2\sqrt{2}\, bg(\xi)}{1 + b^2 g^2(\xi)}, \quad b = \frac{b_1}{2\sqrt{2}} > 0. \tag{2.31}$$

By reasoning as in $n = 1$, it may be readily verified that (2.31) represents a valid solution over the entire region $-\infty < \xi < \infty$. The final solution for ψ and hence for u, after de-normalization using (2.27), we obtain the solitary wave solution

$$u(x,t) = \sqrt{\frac{6}{\alpha}}\operatorname{sech}\left[\frac{(x-ct)}{\sqrt{\gamma}} + \delta\right], \tag{2.32}$$

where $\delta = \ln(\frac{1}{b})$, defines an arbitrary constant phase shift.

We have thus constructed the solitary wave solution (2.32) of the class (1.1) for $n = 2, \beta = 0$ and for $k_1 = 0$. From a physical point of view, it is interesting to realize that a sech solution in (2.32) is only built up of odd harmonics of the fundamental function g, these obviously being the only ones to be generated by a cubic non-linearity.

As special case of (1.1), if $u \to u^2 - \frac{1}{p}$, $n = 2$, $a_1 = p$ and $a_i = 0, (i = 2, 3, 4, 5, 7, 8, 9)$, then (1.1) reduced to the *mKdV* Eq. (1.7) [17,19]. Therefore, from (2.2) we have to obtain on the same of (2.20).

Using (2.20) in (2.32), leads to the well-known sech solution of the *mKdV* Eq. (1.7) [17,19]

$$u(x,t) = \sqrt{\frac{6c}{p}}\operatorname{sech}[\sqrt{c}(x-ct) + \delta], \quad \delta = \ln\left(\frac{1}{b}\right). \tag{2.33}$$

For $k_1^2 = \frac{3}{\alpha}(1-K)$, this implies that

$$k_1 = \pm\sqrt{\frac{3}{\alpha}(1-K)}, \tag{2.34}$$

and (2.23) becomes

$$(2-3K)\phi \pm \alpha\sqrt{\frac{3}{\alpha}(1-K)}\phi^2 + \frac{\alpha}{3}\phi^3 + \gamma\phi'' = 0. \tag{2.35}$$

Next, we obtain

$$\lambda^2 = \frac{1}{\gamma}(3K-2). \tag{2.36}$$

For mathematical convenience, we introduce the scale

$$\phi = \sqrt{\frac{3}{\alpha}(1-K)}\psi. \tag{2.37}$$

We use the scale transformation (2.37), and we substitute the series expansion for ψ into the resulting equation. This yields

$$(m^2-1)b_m \pm \frac{3(1-K)}{3K-2}\sum_{l=1}^{m-1} b_l b_{m-l} + \frac{(1-K)}{3K-2}\sum_{r=2}^{m-1}\sum_{l=1}^{r-1} b_l b_{r-l} b_{m-r} = 0, \quad m \geqslant 3. \tag{2.38}$$

For $K = 0$, (2.38) takes the form

$$(m^2-1)b_m \mp \frac{3}{2}\sum_{l=1}^{m-1} b_l b_{m-l} - \frac{1}{2}\sum_{r=2}^{m-1}\sum_{l=1}^{r-1} b_l b_{r-l} b_{m-r} = 0, \quad m \geqslant 3, \tag{2.39}$$

with b_1 arbitrary and $b_2 = \frac{b_1^2}{2}$. The solution of (2.39) is easily found to be proportional to a constant k, explicitly

$$b_m = \pm 2\left(\frac{b_1}{2}\right)^m \tag{2.40}$$

Using (2.40) in (2.12), ψ may be expressed in closed form as

$$\psi = \frac{2bg}{1 \mp bg}, \quad b = \frac{b_1}{2} > 0. \tag{2.41}$$

Again, it can be shown that the closed form (2.41) is valid over the entire region $-\infty < \xi < \infty$.

For a physical solution we take the plus sign in (2.41); this means we have chosen the minus sign for k_1 in (2.34) and using the scaling in (2.37) with $K = 0$, the final result $u(x,t)$ of the class (1.1) for $n = 2$ and $\beta = 0$, using (2.4), (2.13) and (2.36), takes the form

$$u(x,t) = -\sqrt{\frac{3}{\alpha}}\tanh\left[\sqrt{\frac{-1}{2\gamma}}(x-ct) + \delta\right], \quad \delta = \frac{1}{2}\ln\left(\frac{1}{b}\right). \tag{2.42}$$

Also, if we have taken the plus sign in (2.34) and (2.41), following the above analysis, the final solution of (1.1) becomes

$$u(x,t) = \sqrt{\frac{3}{\alpha}}\left\{2 - \tanh\left[\sqrt{\frac{-1}{2\gamma}}(x-ct) + \delta\right]\right\}. \tag{2.43}$$

We again, take the minus sign in (2.41) and the plus sign for k_1 in (2.34), straightforward, then the final solution $u(x,t)$ of (1.1) takes the form

$$u(x,t) = \sqrt{\frac{3}{\alpha}}\coth\left[\sqrt{\frac{-1}{2\gamma}}(x-ct) + \delta\right], \tag{2.44}$$

Furthermore, we have taken the minus sign in (2.34) and (2.41) and by the same work, the final solitary wave solution of the class (1.1) for $n = 2$ and $\beta = 0$, therefore reads

$$u(x,t) = -\sqrt{\frac{3}{\alpha}}\left\{2 - \coth\left[\sqrt{\frac{-1}{2\gamma}}(x-ct) + \delta\right]\right\}, \tag{2.45}$$

where $\delta = \frac{1}{2}\ln\left(\frac{1}{b}\right)$, α and γ are defined in (2.2).

Case 3: $n = 1, \beta \neq 0$:

for the case $n = 1$ and $\beta \neq 0$, (2.5) takes the form

$$(\alpha k_1 - 1)\phi + \frac{\alpha}{2}\phi^2 + \beta\phi' + \gamma\phi'' + k_1\left(\frac{\alpha}{2}k_1 + K - 1\right) = 0. \tag{2.46}$$

Clearly, the linear part of (2.46) has exponential solutions $\exp[\pm\lambda(\beta,\gamma)\xi]$ for two different values λ_1 and λ_2, where

$$\lambda_{1,2} = \frac{1}{2\gamma}[\beta \pm \sqrt{\beta^2 + 4\gamma(1-\alpha k_1)}] = \frac{1}{2\gamma}[\beta \pm \sqrt{\beta^2 + 4\gamma(2K-1)}], \tag{2.47}$$

with

$$\gamma > \frac{-\beta^2}{4(2K-1)}, \tag{2.48}$$

as a sufficient condition, if and only if the ϕ-independent part in (2.46) is set equal to zero, *i.e.*,

$$k_1 = \frac{2}{\alpha}(1-K), \tag{2.49}$$

which is the same Eq. (2.9) for the case $n = 1$ and $\beta = 0$.

Using (2.47) and (2.49), the transformed Eq. (2.46) becomes

$$(1-2K)\phi+\frac{\alpha}{2}\phi^2+\beta\phi'+\gamma\phi''=0. \tag{2.50}$$

where use has been made of Cauchy's rule [20], for the double product appearing in the nonlinearity ϕ^2. Since $\lambda \neq \lambda_1 \neq \lambda_2$, it follows from (2.51) with (2.47) that $b_1=0$. [The degenerate case $\lambda_1=\lambda_2, i.e., 4\gamma(2K-1)=-\beta^2$, is still under investigation.] For a nontrivial solution built up of the mixing of the two decaying exponentials $g_{1,2}=\exp(-\lambda_{1,2}\xi)$ as

$$\sum_{m=1}^{\infty}\left[m^2\lambda^2-\frac{\beta}{\gamma}m\lambda-\frac{(2K-1)}{\gamma}\right]b_m g^m + \frac{(2K-1)}{\gamma}\sum_{m=2}^{\infty}\sum_{r=1}^{m-1}b_r b_{m-r}g^m=0, \tag{2.51}$$

where use has been made of Cauchy's rule [20], for the double product appearing in the nonlinearity ϕ^2. Since $\lambda \neq \lambda_1 \neq \lambda_2$, it follows from (2.51) with (2.47) that $b_1=0$. [The degenerate case $\lambda_1=\lambda_2, i.e., 4\gamma(2K-1)=-\beta^2$, is still under investigation.] For a nontrivial solution built up of the mixing of the two decaying exponentials $g_{1,2}=\exp(-\lambda_{1,2}\xi)$ as

$$\phi(\xi)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}b_{m_1m_2}g_1^{m_1}(\xi)g_2^{m2}(\xi). \tag{2.52}$$

We look for two integers $l_{1,2}$, satisfying

$$\lambda=\frac{\lambda_1}{l_1}=\frac{\lambda_2}{l_2}, \tag{2.53}$$

then; we require two coefficients b_m to be arbitrary. An obvious choice is b_2 and b_3, so that, from (2.51), the conditions

$$4\lambda^2-\frac{2\beta}{\gamma}\lambda-\left(\frac{2K-1}{\gamma}\right)=0, \tag{2.54}$$

$$9\lambda^2-\frac{3\beta}{\gamma}\lambda-\left(\frac{2K-1}{\gamma}\right)=0, \tag{2.55}$$

must be fulfilled. Solving for γ and λ in terms of β, we get

$$\gamma=\frac{-6\beta^2}{25(2K-1)}, \tag{2.56}$$

$$\lambda=\frac{-5}{6\beta}(2K-1). \tag{2.57}$$

Hence, using (2.56) and (2.57) in (2.47), it follows from (2.53) that

$$l_1=2, \quad l_2=3, \tag{2.58}$$

as expected on the basis of our choice ($g_1=g^2$ and $g_2=g^3$). Hence,

$$\lambda_1=\frac{5}{3\beta}(1-2K), \quad \lambda_2=\frac{5}{2\beta}(1-2K). \tag{2.59}$$

Also, from (2.56) and (2.57), we obtain

$$\lambda=\frac{\beta}{5\gamma}. \tag{2.60}$$

So that, the second and third harmonics of g build up the final general solution of solitary waves of the class (1.1) for $n=1$, through harmonic generation and mixing. The recursion relation following from (2.51) may, in this case, be expressed canonically as

$$(m-2)(m-3)b_m-6\sum_{r=1}^{m-1}b_r b_{m-r}=0, \quad m \geqslant 2, \quad b_1=0, \tag{2.61}$$

from which the general structure of b_m may be calculated as [16]

$$b_m=(-1)^{m+1}(m-1)b^m, \quad b=\frac{-b_3}{2b_2}>0. \tag{2.62}$$

Using (2.62) in (2.12), ψ may be expressed in closed form as

$$\psi=-\left(\frac{bg}{1+bg}\right)^2, \quad -1<bg<1. \tag{2.63}$$

So that with (2.4), (2.49) and (2.63) after de-normalization using (2.11), the closed form solution f may be written as

$$f=\frac{2}{\alpha}(1-K)+\frac{2}{\alpha}(1-2K)\left(\frac{bg}{1+bg}\right)^2. \tag{2.64}$$

By reasoning as in the case $\beta=0$, it may be readily verified that (2.64) represents a valid solution over the entire region $-\infty<\xi<\infty$. The final solution $u(x,t)$ may now be expressed, using 2.47, 2.13 and 2.64, as

$$u(x,t)=f(x-ct) =\frac{2}{\alpha}(1-K)+\frac{1}{2\alpha}(2K-1)\left\{1-\tanh\left[\frac{1}{2}\left(\frac{\beta}{5\gamma}\right)(x-ct)+\delta\right]\right\}^2, \tag{2.65}$$

where $\delta=\frac{1}{2}\ln\left(\frac{1}{b}\right)$ defines an arbitrary constant phase shift, α,β and γ are defined in (2.2).

As special case of (2.65), note that if $K=1$, the solution of the class (1.1) takes the form [16]

$$u(x,t)=\frac{1}{2\alpha}\left\{1-\tanh\left[\frac{1}{2}\left(\frac{\beta}{5\gamma}\right)(x-ct)+\delta\right]\right\}^2, \delta=\frac{1}{2}\ln\left(\frac{1}{b}\right). \tag{2.66}$$

In retrospect, note that the series (2.12), with (2.62), may be re-expressed as

$$\psi(\xi)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}b_{m_1m_2}g_1^{m_1}(\xi)g_2^{m_2}(\xi), \tag{2.67}$$

with

$$g_1=g^2, \quad g_2=g^3, \tag{2.68}$$

and

$$b_{10}=b_2, \quad b_{01}=b_3, \quad b_{20}=b_4=-3b_2^2, \quad b_{02}=3b_2^3,$$
$$b_{11}=b_5=-2b_2b_3, \quad b_{30}=-\frac{b_3^2}{2}, \text{ etc.}, \tag{2.69}$$

which is analogous to (2.52), demonstrating mixing between g_1 and g_2 as in [16], for $n=1$.

Case 4: $n=2, \beta\neq 0$:

We derive here the solutions of solitary waves of the class (1.1) for the case $n=2$ and $\beta\neq 0$, thus (2.5) becomes

$$(\alpha k_1^2-1)\phi+\alpha k_1\phi^2+\frac{\alpha}{3}\phi^3+\beta\phi'+\gamma\phi''+k_1\left(\frac{\alpha}{3}k_1^2+K-1\right)=0, \tag{2.70}$$

which has two real solutions, $\exp[\pm\lambda_{1,2}(\beta,\gamma)\xi]$, for its linear part, if

$$k_1\left(\frac{\alpha}{3}k_1^2+K-1\right)=0. \tag{2.71}$$

For $k_1=0$, (2.70) takes the form [16]

$$-\phi+\frac{\alpha}{3}\phi^3+\beta\phi'+\gamma\phi''=0. \tag{2.72}$$

Real solutions of the linear part of (2.72) may be represented as $\exp(\lambda_{1,2}\xi)$ in which

$$\lambda_{1,2}=\frac{1}{2\gamma}(\beta\pm\sqrt{\beta^2+4\gamma}), \tag{2.73}$$

with

$$\gamma>\frac{-\beta^2}{4}, \tag{2.74}$$

as a sufficient condition.

We next substitute (2.27), (2.12) and (2.13) into (2.72) to get the recursion relation

$$\sum_{m=1}^{\infty}\left(m^2\lambda^2-\frac{\beta}{\gamma}m\lambda-\frac{1}{\gamma}\right)b_mg^m+\frac{1}{\gamma}\sum_{m=3}^{\infty}\sum_{r=2}^{m-1}\sum_{j=1}^{r-1}b_jb_{r-j}b_{m-r}g^m=0. \tag{2.75}$$

It can be shown [16] that the choice

$$\gamma=\frac{2\beta^2}{9}, \tag{2.76}$$

$$\lambda=\frac{-3}{2\beta}, \tag{2.77}$$

ensures the commensurability expressed by (2.53) with

$$\lambda_1=\lambda,\lambda_2=2\lambda, \tag{2.78}$$

so that the second and third harmonics of g, which are legitimate solutions of the linear part of (2.72), build up the final solitary wave solution through harmonic generation and mixing. Also, from (2.76) and (2.77), we get

$$\lambda=\frac{\beta}{3\gamma}. \tag{2.79}$$

The recursion relation following from (2.75) may, in this case, be expressed canonically as

$$(m-1)(m-2)b_m-2\sum_{r=2}^{m-1}\sum_{j=1}^{r-1}b_jb_{r-j}b_{m-r}=0,\quad m\geqslant 3, \tag{2.80}$$

from which the general structure of b_m, may be calculated as [16]

$$b_m=\pm(-1)^{m+1}b^m,\quad b=\frac{-b_2}{b_1}>0. \tag{2.81}$$

Using (2.81) in (2.12), ψ may be expressed in closed form as

$$\psi=\pm\frac{bg}{1+bg},\quad -1<bg<1. \tag{2.82}$$

For the same reasons as in the preceding case ($n=1$), it may be readily verified that (2.82) represents a valid solution over the entire region $-\infty<\xi<\infty$. The final solution for f, and hence for u, after de-normalization using (2.27) therefore reads

$$u(x,t)=f(x-ct)=\pm\frac{1}{2}\sqrt{\frac{3}{\alpha}}\left\{1-\tanh\left[\frac{1}{2}\left(\frac{\beta}{3\gamma}\right)(x-ct)+\delta\right]\right\}, \tag{2.83}$$

where $\delta=\frac{1}{2}\ln\left(\frac{1}{b}\right)$, which is defined as before, α, β and γ are defined in (2.2).

For $k_1^2=\frac{3}{\alpha}(1-K)$, following the analysis for the case $n=1$ and $\beta\neq 0$, the linear part of (2.70) has two real exponential solutions $\exp(\pm\lambda\xi)$ for two different values λ_1 and λ_2, where

$$\lambda_{1,2}=\frac{1}{2\gamma}\left[\beta\pm\sqrt{\beta^2+4\gamma(1-\alpha k_1^2)}\right]=\frac{1}{2\gamma}\left[\beta\pm\sqrt{\beta^2+4\gamma(3K-2)}\right], \tag{2.84}$$

with

$$\gamma>\frac{\beta^2}{4(2-3K)},\quad K\neq\frac{2}{3}, \tag{2.85}$$

as sufficient conditions, and (2.70) becomes

$$(2-3K)\phi\pm\alpha\sqrt{\frac{3}{\alpha}(1-K)}\phi^2+\frac{\alpha}{3}\phi^3+\beta\phi'+\gamma\phi''=0. \tag{2.86}$$

We perform the scaling

$$\phi=\pm\sqrt{\frac{3}{\alpha(1-K)}}\psi, \tag{2.87}$$

and, therefore, substitute the expansion (2.12) into the re-scaled non-linear equation. This yields

$$\sum_{m=1}^{\infty}\left[m^2\lambda^2-\frac{\beta}{\gamma}m\lambda+\frac{(2-3K)}{\gamma}\right]b_mg^m+\frac{3(1-K)}{\gamma}\sum_{m=2}^{\infty}\sum_{r=1}^{m-1}b_rb_{m-r}g^m+\frac{(1-K)}{\gamma}\sum_{m=3}^{\infty}\sum_{r=2}^{m-1}\sum_{j=1}^{r-1}b_jb_{r-j}b_{m-r}g^m=0, \tag{2.88}$$

where use has been made of Cauchy's rule [20], for the double product and for the triple product appearing in ψ^2 and ψ^3, respectively.

It follows from (2.88) with (2.84) that $b_1=0$, where $\lambda_1\neq\lambda_2$. [The degenerate case $\lambda_1=\lambda_2$, i.e., $4\gamma(2-3K)=\beta^2$ still under investigation.] For same reasons as in the preceding case ($n=1$), we require two coefficients b_m to be arbitrary. An obvious choice is b_2 and b_3, so that, from (2.88), the conditions

$$4\lambda^2-\frac{2\beta}{\gamma}\lambda+\frac{(2-3K)}{\gamma}=0, \tag{2.89}$$

$$9\lambda^2-\frac{3\beta}{\gamma}\lambda+\frac{(2-3K)}{\gamma}=0, \tag{2.90}$$

must be fulfilled. Hence, it follows that

$$\gamma=\frac{6\beta^2}{25(2-3K)}, \tag{2.91}$$

$$\lambda=\frac{5(2-3K)}{6\beta}. \tag{2.92}$$

From (2.91) and (2.92), we get

$$\lambda=\frac{\beta}{5\gamma}. \tag{2.93}$$

It follows from (2.53), and by using (2.91) and (2.92) in (2.84), we have

$$l_1=2,\quad l_2=3. \tag{2.94}$$

Therefore, we get

$$\lambda_1 = 2\lambda, \quad \lambda_2 = 3\lambda. \tag{2.95}$$

The recursion relation following from (2.88) may in this case, be expressed canonically as

$$(m-2)(m-3)b_m + \frac{18(K-1)}{3K-2}\sum_{r=1}^{m-1} b_r b_{m-r} + \frac{6(K-1)}{3K-2}\sum_{r=2}^{m-1}\sum_{j=1}^{r-1} b_j b_{r-j} b_{m-r} = 0, \quad m \geqslant 3. \tag{2.96}$$

For $K = 0$, (2.96) takes the form

$$(m-2)(m-3)b_m + 9\sum_{r=1}^{m-1} b_r b_{m-r} + 3\sum_{r=2}^{m-1}\sum_{j=1}^{r-1} b_j b_{r-j} b_{m-r} = 0, \quad m \geqslant 3, b_1 = 0. \tag{2.97}$$

From (2.97), we can calculate the first few coefficients b_m, leading to

$$b_4 = \frac{-9}{2}b_2^2, \tag{2.98}$$

$$b_5 = -3b_2 b_3, \tag{2.99}$$

$$b_6 = \frac{13}{2}b_2^3 - \frac{3}{4}b_3^2, \tag{2.100}$$

etc., but it is very hard to speculate on that the explicit form of b_m will be. We remark that regarding (2.98)–(2.100), one may expect an alternation sign in successive b_m if b_2 and b_3 have opposite signs. Furthermore, since $b_1 = 0$, then $m - 1$ is a factor in b_m.

Finally, note that if b_m is a solution of (2.97) then $b_m b^m$, with $b > 0$ and constant, is also a solution of the same recursion relation. Taking all this into account, the form of b_m must be

$$b_m = L(-1)^{m+1}(m-1)b^m. \tag{2.101}$$

Now, the constants L and b which both may depend on b_2 and b_3, must be determined. In order to calculate b and L, it follows readily from (2.101) expressed for $m = 2$ and $m = 3$, provided

$$b = \frac{-b_3}{2b_2} > 0, \quad L = \frac{-4b_2^3}{b_3^2}. \tag{2.102}$$

Next, we obtain $L = \frac{2}{3}$ for $m = 4$ after substituting (2.101) into (2.97). Hence, we have

$$b_2^3 = \frac{-b_3^2}{6}, \tag{2.103}$$

and

$$b_m = \frac{2}{3}(-1)^{m+1}(m-1)b^m. \tag{2.104}$$

After substituting (2.104) into (2.12) and application of the formula for the binomial series

$$\sum_{m=2}^{\infty}\frac{2}{3}(-1)^{m+1}(m-1)y^m = \frac{-2}{3}\left(\frac{y}{1+y}\right)^2, \quad |y| < 1, \tag{2.105}$$

$\psi(\xi)$ can be re-expressed in closed form:

$$\psi = \frac{-2}{3}\left(\frac{bg}{1+bg}\right)^2, \quad -1 < bg < 1. \tag{2.106}$$

So that with (2.4), (2.34) (2.106) for $K = 0$ after de-normalization using (2.87), the closed form solution f may be written as:

$$f = \pm\sqrt{\frac{3}{\alpha}} \mp \frac{2}{3}\sqrt{\frac{3}{\alpha}}\left(\frac{bg}{1+bg}\right)^2. \tag{2.107}$$

By reasoning as before, it may be readily verified that (2.107) represents a valid solution over the entire region $-\infty < \xi < \infty$. The final solution of the class (1.1) (for $n = 2$ and $\beta \neq 0$) for f, and hence for u, using (2.13) and (2.93), therefore reads

$$u(x,t) = f(x-ct) = \pm\sqrt{\frac{3}{\alpha}} \pm \frac{1}{6}\sqrt{\frac{3}{\alpha}} \left\{1 - \tanh\left[\frac{1}{2}\left(\frac{\beta}{5\gamma}\right)(x-ct) + \delta\right]\right\}^2, \tag{2.108}$$

where $\delta = \frac{1}{2}\ln\left(\frac{1}{b}\right)$ defines an arbitrary constant phase shift, α, β and γ are defined in (2.2)

Conclusion

In this work, we have found the general solutions of the class (1.1) for $n = 1$ and for $n = 2$ using a direct algebraic method [4]. These solutions are obtained with the relationship between the coefficients which have given in (2.2) of the present class. When $\beta = 0$ in (2.5) for the case $n = 1$ (using (2.7)), the general solution $u(x,t)$ given in (2.17) contains a constant term and a sech^2-type term, both of which are individually solutions of the class (1.1). This solution, may therefore be visualized as the superposition of two particular solutions given in (2.18) and (2.19) of (1.1), and names the sech^2 and $\tanh^2$ solutions. Furthermore, the solutions (2.18) and (2.19) reduced to the other solutions (2.21) and (2.22) of the *KdV* Eq. (1.3) [17,18], which as the special case of (1.1).

Also, for $n = 2$ with $\beta = 0$ in (2.5), exact solutions of (1.1) given in (2.32) and (2.42)–(2.45). These solutions called the well-known sech and 'well'-type (tanh or coth) solutions. The special case of (2.32) given in (*mKdV*: sech) (2.33) [17,19] of the *mKdV* Eq. (1.7).

Moreover; for the case $n = 1$ and $\beta \neq 0$ in (2.5), the general solution u of (1.1) given in (2.65) contains a constant term and a 'well'-type $(1 - \tanh)^2$ term, and the special case of (2.65) reduced to the solution (2.66) [16] for the class (1.1). Also, for $n = 2$ with $\beta \neq 0$, the solution u of (1.1) given in (2.83) for the constant term $k_1 = 0$ in (2.70) and this solution is nothing else than the $(1 - \tanh)$-type solution of [16]. Finally; for $n = 2$ and $\beta \neq 0$, and using (2.34), the solution of (1.1) given in (2.108).

As a collection, the general solutions which are studied of (1.1) when $n = 1, 2$ for $\beta = 0$ or $\beta \neq 0$ (using a direct algebraic method) are obtained, having a constant term in their expansion into real exponentials. These solutions and others (any particular solution) of the class (1.1) are of *pulse-type* or of *kink-type* solutions with an arbitrary constant phase shift.

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