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SHORT COMMUNICATION

On permutation labeling

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Abstract We determine all permutation graphs of order $\leqslant 9$. We prove that every bipartite graph of order $\leqslant 50$ is a permutation graph. We convert the conjecture stating that "every tree is a permutation graph" to be "every bipartite graph is a permutation graph".

1. Introduction

Hegde and Shetty [1] define a graph G with p vertices to be a permutation graph if there exists an injection f from the set of vertices of G to $\{1, 2, 3, \ldots, p\}$ such that the induced edge function g_f defined by

$$g_f(uv) = \begin{cases} p^{f(u)}_{f(v)}, & \text{if } f(v) < f(u) \\ p^{f(v)}_{f(u)}, & \text{if } f(u) < f(v) \end{cases}$$

is injective, where $p^{f(u)}_{f(v)}$ denotes the number of permutations of $f(u)$ things taken $f(v)$ at a time.

They prove: K_n is a permutation graph if and only if $n \leqslant 5$; They said: "Since the edge values in any permutation labelings are large numbers, investigations of suitable additional constraints to control edge values is a scope for further study". Also they strongly believe that all trees admit permutation labelings. Baskar Babujee and Vishnupriya [2] prove the following graphs are permutation graphs: P_n; C_n; stars; graphs obtained adding a pendent edge to each edge of a star; graphs obtained by joining the centers of two identical stars with an edge or a path of length 2, and complete binary trees with at least three vertices. Here we investigate the permutation labelings for graphs of order $\leqslant 9$, we conjecture that every bipartite graph is a permutation graph, armed with proving this conjecture for bipartite graphs of order $\leqslant 50$. Hence the conjecture that every tree is a permutation graph by Hegde and Shetty is true for trees of order $\leqslant 50$.

2. Permutation labeling

Lemma 2.1. *If G is a permutation graph then $G\backslash e(=$ the graph obtained from G by deleting the edge $e)$ is also a permutation graph.*

Proof. Since there is no repeated edge labels in G then we use the same labeling of G to label $G\backslash e$, so $G\backslash e$ has no repeated edge labels. □

Now suppose i, j, k, l be positive integers and let us verify when $p^j_i = p^l_k$, where $i \neq k, j \neq l$. If $j > l$ and $i > k$ then it is clear that $p^j_i > p^l_k$. So without any loss of generality a necessary condition for p^j_i to be equal to p^l_k is $j > l$ and $i < k$. For example

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$p_1^6 = p_2^3 = 6$. Note that if $p_i^j = p_k^l, j > l \Rightarrow j!/(j-i)! = l!/(l-k)! \Rightarrow j!/l! = (j-i)!/(l-k)! \Rightarrow p_{(j-l)}^j = p_{(j-l-i+k)}^{(j-i)}$.

For obtaining all pairs of permutation of this type we define the following sequences.

$$\{x_i^1\} = \{i; i = 2, 3, 4, \ldots\}$$
$$\{x_i^2\} = \{i(i+1); i = 2, 3, 4, \ldots\}$$
$$\{x_i^3\} = \{i(i+1)(i+2); i = 2, 3, 4, \ldots\}$$
$$\{x_i^4\} = \{i(i+1)(i+2)(i+3); i = 2, 3, 4, \ldots\}$$
$$\vdots$$
$$\{x_i^j\} = \{i(i+1)(i+2)\ldots(i+j-1); i = 2, 3, 4, \ldots\}$$
$$\vdots$$

For each sequence $\{x_i^j\}$ let $Z_i^j = (i + j + 1)$.

Every common member between two sequences gives us two pairs of equal permutations. For example $720 = 8 \times 9 \times 10 = x_7^3 \in \{x_i^3\}$ also $720 = 2 \times 3 \times 4 \times 5 \times 6 = x_1^5 \in \{x_i^5\}$ i.e. $10!/7! = 6!/1! \Rightarrow 10!/6! = 7!/1!$ and that gives us the following permutations: $p_3^{10} = p_5^6 \Rightarrow p_4^{10} = p_6^7$.

Note that $p_4^{10} = 7 \times 8 \times 9 \times 10 \in \{x_i^4\}, p_4^{10} = 2 \times 3 \times 4 \times 5 \times 6 \times 7 \in \{x_i^6\}$. i.e. it is a common member between these two sequences. By using mathematica e.g. we collect those common members between every two sequences, *where Z_i^j is not greater than 50*, we denote the elements of the sequence $\{x_i^j\}$, in which $Z_i^j \leqslant 50$ by $V[j]$ since j is fixed within every sequence.

Then we execute the intersection between $V[j_1]$ and $V[j_2]$, $j_1 < j_2$, if *max* $V[j_1] \geqslant$ *min* $V[j_2]$, since otherwise the intersection will be empty set.

For $j_1 = 2, 3, \ldots, 48$ we get:

$V[1] \cap V[2] = \{6,12,20,30,42\}$ and that means:

$$6 = 2 \times 3 = 6!/5! = 3!/1! \Rightarrow P_1^6 = P_2^3 \Rightarrow P_3^6 = P_4^5$$
$$12 = 3 \times 4 = 12!/11! = 4!/2! \Rightarrow P_1^{12} = P_2^4 \Rightarrow P_8^{12} = P_9^{11}$$
$$20 = 4 \times 5 = 20!/19! = 5!/3! \Rightarrow P_1^{20} = P_2^5 \Rightarrow P_{15}^{20} = P_{16}^{19}$$
$$30 = 5 \times 6 = 30!/29! = 6!/4! \Rightarrow P_1^{30} = P_2^6 \Rightarrow P_{24}^{30} = P_{25}^{29}$$
$$42 = 6 \times 7 = 42!/41! = 7!/5! \Rightarrow P_1^{42} = P_2^7 \Rightarrow P_{35}^{42} = P_{36}^{41}$$
$$V[1] \cap V[3] = \{24\}$$
$$24 = 2 \times 3 \times 4 = 24!/23! = 4!/1! \Rightarrow P_1^{24} = P_3^4 \Rightarrow P_{20}^{24} = P_{22}^{23}$$
$$V[2] \cap V[3] = \{210\}$$
$$210 = 14 \times 15 = 15!/13! = P_2^{15} 210 = 5 \times 6 \times 7 = 7!/4! = P_3^7$$
$$\Rightarrow P_2^{15} = P_3^7 \Rightarrow P_8^{15} = P_9^{13}$$

similarly,

$$V[3] \cap V[4] = \{120\} \Rightarrow P_3^6 = P_4^5 \Rightarrow P_1^6 = P_2^3$$
$$V[3] \cap V[5] = \{720\} \Rightarrow P_3^{10} = P_5^6 \Rightarrow P_4^{10} = P_6^7$$
$$V[4] \cap V[6] = \{5040\} \Rightarrow P_4^{10} = P_6^7 \Rightarrow P_3^{10} = P_5^6$$
$$V[8] \cap V[9] = \{19958400, 259459200\}$$
$$\Rightarrow P_8^{12} = P_9^{11} \Rightarrow P_1^{12} = P_2^4; P_8^{15} = P_9^{13} \Rightarrow P_2^{15} = P_3^7$$
$$V[15] \cap V[16] = \{20274183401472000\}$$
$$\Rightarrow P_{15}^{20} = P_{16}^{19} \Rightarrow P_1^{20} = P_2^5$$
$$V[20] \cap V[22] = \{25852016738884976640000\}$$
$$\Rightarrow P_{20}^{24} = P_{22}^{23} \Rightarrow P_1^{24} = P_3^4$$
$$V[24] \cap V[25] = \{368406749739154248105984000000\}$$
$$\Rightarrow P_{24}^{30} = P_{25}^{29} \Rightarrow P_1^{30} = P_2^6$$
$$V[35] \cap V[36]$$
$$= \{2787710551096983925680838504453395972096000000000\}$$
$$\Rightarrow P_{35}^{42} = P_{36}^{41} \Rightarrow P_1^{42} = P_2^7.$$

Finally, we put these pairs in a table, where $\{j,i\}$ and $\{l,k\}$, $\{j,j-1\}$ and $\{j-i, j-l-i+k\}$ are the indices that cause the pairs of equal permutations.

n	$\{j, i\}$	$\{l, k\}$	$\{j, j-l\}$	$\{j-i, j-l-i+k\}$
< 10	{6,1}	{3,2}	{6,3}	{5,4}
< 12	{10,3}	{6,5}	{10,4}	{7,6}
< 15	{12,1}	{4,2}	{12,8}	{11,9}
< 20	{15,2}	{7,3}	{15,8}	{13,9}
< 24	{20,1}	{5,2}	{20,15}	{19,16}
< 30	{24,1}	{4,3}	{24,20}	{23,22}
< 42	{30,1}	{5,2}	{30,25}	{29,26}
⩽50	{42,1}	{7,2}	{42,35}	{41,36}

Let G be a graph of order $n \leqslant 50$ we can easily see all pairs of permutations $p_i^j = p_k^l$ and $p_{(j-l)}^j = p_{(j-l-i+k)}^{(j-i)}$, where$j \leqslant n$.

Now to check whether this graph is a permutation graph or not, we try to avoid one of the pairs (j, i) or (l, k) to be labels of two adjacent vertices. Also we avoid one of the pairs $(j, j-l)$ or $(j-i, j-l-i+k)$ to be labels of two adjacent vertices.

If this is possible for all these pairs, then the graph G is a permutation graph.

3. Results

Theorem 3.1. *Among all graphs of order ⩽9 the only graphs, which are permutation graphs, are $K_n, K_n \backslash e$, where $n = 6, 7, 8, 9$.*

Proof. In $K_n, n = 6$ each of the pairs: {6,1}, {3,2}, {6,3} and {5,4} will be found as labels of adjacent vertices, in $K_n \backslash e$ we can avoid just one of them, but it remains the others, which cause repeated edge labels.

All graphs of order ⩽5 are permutation graphs (there are no pairs of equal permutation). For the remaining graphs we prove first that the graphs $\{K_6 \backslash e_1\} \backslash e_2$ are permutation graphs as follows: we have two possibilities to remove e_1 and e_2.

(1) They are adjacent, so we leave two vertices of degree 4 and one of degree 3. (2) they are not adjacent, so we leave four vertices of degree 4. In (1) we give the vertex of degree 3 the label 6 and give the two other vertices of degree 4 the labels 1 and 3. So each of the permutations P_1^6 and P_4^5 appears just once. In (2) we give the four vertices of degree 4 the labels 3, 2, 4 and 5.

Hence we avoid repeated edge labels in both cases. The same argument can be applied for the graphs $\{K_n \backslash e_1\} \backslash e_2$, $n = 7, 8, 9$.

Here *Lemma 2.1* completes our proof. □

Theorem 3.2. *All bipartite graphs of order $n \leqslant 50$ are permutation graphs.*

Proof. We prove this theorem for complete bipartite graphs $K_{r,s}$, then *Lemma 2.1* completes the proof. First if $n = r + s \leqslant 5$ then $K_{r,s}$ is a permutation graph. Now it is always possible for us to put the vertex labels that cause repeated edge labels in one partite set as follows: when $6 \leqslant n < 10$, this case is excluded by *Theorem 3.1* and *Lemma 2.1*.

When $10 \leqslant n < 12$, we put the vertex labels 1, 3, 4, 6, 10 in one partite set, hence each of the permutations P_1^6, P_3^6, P_3^{10} and P_4^{10} might appears once, so there is not repeated edge labels.

And so on when $12 \leqslant n < 15$, if $n = 12$, $r = s = 6$, we put the vertex labels 1, 3, 4, 6, 10, 12 in one partite set, and we put the vertex labels 9, 11 in the other, otherwise we put 1, 3, 4, 6, 10, 12, 8 in the partite set of the greater order, so there is not repeated edge labels in both cases.

When $15 \leqslant n < 20$, we put the vertex labels 2, 3, 4, 6, 8, 10, 12, 15 in one partite set, so there is not repeated edge labels.

When $20 \leqslant n < 24$, we put the vertex labels 2, 3, 4, 6, 8, 10, 12, 15, 5, 20 in one partite set, so there is not repeated edge labels.

When $24 \leqslant n < 30$, we put the vertex labels 2, 3, 4, 6, 8, 10, 12, 15, 5, 20, 24 in one partite set, so there is not repeated edge labels.

When $30 \leqslant n < 42$, we put the vertex labels 2, 3, 4, 6, 8, 10, 12, 15, 5, 20, 24, 25, 30 in one partite set, so there is not repeated edge labels.

When $42 \leqslant n \leqslant 50$, we put the vertex labels 2, 3, 4, 6, 8, 10, 12, 15, 5, 20, 24, 25, 30, 7, 35, 42 in one partite set, so there is not repeated edge labels.

By *Lemma 2.1* we deduce that every bipartite graph is a permutation graph. (since it is a complete bipartite graph after a sequence of removing edges). □

Remark 3.3. We see by mathematica that, when the order of a bipartite graph increases exceeding 50,then the frequency of the appearance of the pairs $\{j, i\}$, $\{l, k\}$, $\{j, j-l\}$ and $\{j-i, j-l-i+k\}$, which cause equal permutations, is low compared with the widen of the partite set of the greater order, that allows us to stand the following open problem:

Open Problem 3.4. Prove that every bipartite graph has a permutation labeling for $n \geqslant 51$.

Corollary 3.5. *Every tree of order ⩽50 is a permutation graph.*

Proof. Straight forward since every tree is a bipartite graph. □

References

[1] S.M. Hegde, S. Shetty, Combinatorial labelings of graphs, Appl. Math. E-Notes 6 (2006) 251–258.

[2] J. Baskar Babujee, V. Vishnupriya, Permutation labelings for some trees, Int. J. Math. Comput. Sci. 3 (2008) 31–38.