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SHORT COMMUNICATION

On the (co)homology theory of index category

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Abstract In the present work the different types of regular index categories are given. The related cohomology groups of some of these categories are studied. We give some properties of index category related with monoid and algebra over it.

1. Introduction

Definition 1 6. An index category is a pair (Λ, i), where Λ is category with natural numbers as objects and i is an inclusion $i : \Delta \to \Lambda$, where Δ is a subcategory of Λ. The category Δ is called a simplicial index category, it has objects an ordered set $[n] = \{0,1,\ldots,n\}$, $n = 0,1,\ldots$ and the group morphisms $\delta_i^n : [n] \to [n-1]$, $\sigma_j^n : [n] \to [n+1]$, $o \leqslant i \leqslant n$, $0 \leqslant j \leqslant n$, with the following identities:

$$\begin{aligned} &\delta^j_{n+1}\delta^i_n = \delta^i_{n+1}\delta^{j-1}_n, \quad \text{if } i \prec j \\ &\sigma^j_n\sigma^i_{n+1} = \sigma^i_n\sigma^{j+1}_{n+1}, \quad \text{if } i \leqslant j \\ &\sigma^j_n\delta^i_{n+1} = \begin{cases} \delta^i_{n-1}\sigma^{j-1}_{n-2}, & \text{if } i \prec j \\ Id_{[n]}, & \text{if } i = j \text{ or } i = j+1 \\ \delta^{i-1}_{n+1}\sigma^j_n, & \text{if } i \succ j+1 \end{cases} \end{aligned} \tag{1}$$

Among index categories, there is one, plays an important role in this field. It's called a regular index category.

Definition 2. The index category (Λ, i) is called regular if any morphism $\alpha \in \Lambda(n,m) = Hom_\Lambda(n,m)$ can be uniquely written in the form $\alpha = \varphi \circ \gamma$, where $\varphi \in \Delta\ (m,n) = Hom_\Delta(m,n)$, $\gamma \in \Lambda_n = Aut_\Lambda(n)$. Clearly a regular index category is a subcategory of an index category. A regular index category plays an important role as we will show in the sequel.

Examples of regular index category: [7,9,10].

1. Simplicial index category (Δ). It's generated by morphisms δ^i, σ^j (let $\delta^i_n = \delta^i$, $\sigma^j_n = \sigma^j$).
2. Cyclic index category (C, i). It's generated by morphisms δ^i, σ^j and $t_n : [n] \to [n]$, $t_n^{n+1} = 1$.
3. Reflexive index category (R, i). It's generated by morphisms δ^i, σ^j and $r_n : [n] \to [n]$, $r_n^2 = 1$.
4. Dihedral index category (D, i). It's generated by morphisms δ^i, σ^j, t_n and r_n.

The group of index categories (simplicial, cyclic, reflexive and dihedral) can be extended to the following class of categories [1,6].

1. Symmetry index category (S, i). It's generated by morphisms δ^i, σ^j and $\alpha_n : [n] \to [n]$.
2. Bisymmetry index category (B, i). It's generated by morphisms δ^i, σ^j, α_n.

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3. Hyperoctahedral(Weil) index category (H,i). It's generated by morphisms δ^i, σ^j and α_n.

We shall explain these categories in detail:

1.1. The symmetry index category (S,i) [2,3]

Consider the family $S. = \{S_n\}_{n\geqslant 0}$, where S_n is a permutation group of the set $[n] = \{0,1,\ldots,n\}$. It's generated by the permutations $\bar{1}, \bar{2}, \ldots, \bar{n}$, such that $\bar{k} = (k-1, k)$. S is a symmetry index category with object the natural number and morphisms:

$$\delta^i_n : [n] \to [n-1],\ \sigma^j_n : [n] \to [n+1],\ 0 \leqslant i,\ j \leqslant n, \tag{2}$$

$$\alpha_n : [n] \to [n],\ \alpha_n \in \Lambda_n,\ n \in \aleph, \quad \text{such that}$$

$$\alpha_n \circ \delta^i_n = \delta^{\alpha(i)}_n \circ \partial_i(\alpha_n),$$

$$\alpha_n \circ \sigma^j_n = \sigma^{\alpha(j)}_n \circ S_i(\alpha_n),$$

$$\alpha_n \circ \beta_n = \beta_n \circ \alpha_n.$$

Clearly (S, i_s) is regular index category, where $i_s : \Delta \to S$ is inclusion functor.

Note that a cyclic index category, with object $\aleph$ and group morphisms δ^i, σ^j and $t_n = \alpha_n = (n, n-1, \ldots, 1, 0)$ is an example of symmetry index category.

1.2. Hyperoctahedral(Weil) index category (W,i)

Consider a group $W_n = S_n \times (Z/2)^{n+1}$, where $Z/2 = \{-1,1\}$, $\times$ is a semidirect product and S_n is symmetry group. The elements of W_n is given by $(\alpha; \varepsilon_o, \ldots, \varepsilon_n)$, where $\alpha \in S_n$ and $\varepsilon_i = \pm 1$. The multiplication in W_n defined by:

$$(\alpha;\ \varepsilon_o, \ldots, \varepsilon_n) \cdot (\beta;\ \gamma_o, \ldots, \gamma_n) = (\alpha \cdot \beta;\ \varepsilon_{\beta(o)} \cdot \gamma_o, \ldots, \varepsilon_{\beta(n)} \cdot \gamma_n).$$

Then, clearly that on every class there is a graded group, then the family $\{W_n\}_{n\geq 0}$ is a graded group.

1.3. Bisymmetry index category (T,i)

This group lies between S and W, $S. \subset T. \subset W.$, where, $T_n = S_n \times \aleph/2 \subset T_n = S_n \times (\aleph/2)^{n+1}$, where $\times$ is a semi-direct product. The elements $(\alpha; \varepsilon)$ of T_n through the inclusion of $T_n$in W_n is given by $(\alpha; \varepsilon) \to (\alpha; \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$. The family $\{T_n\}_{n\geqslant 0}$ is a graded group;

Definition 3. Following [6]:

(1) The regular index category can be considered as crossed simplicial group.
(2) There is an isomorphism between the regular index category and the category of crossed simplicial group, that is, the study of regular index category equivalence to the use of crossed simplicial group.

For more informations about crossed simplicial group and its properties see [1,4,5].

It is known that the simplical object, for an arbitrary category $\wp$, is a functor $F : \Delta^{op} \to \wp$ where Δ^{op} is the inverse image of Δ. If $\wp$ is a category of sets (group, module), we can define the simplicial set (group, module) by the following relations:

$$F([n]) = F_n, \quad F([\delta^i]) = \partial_i, \quad F([\sigma^j]) = S_j$$

where the morphisms ∂_i, S_j are satisfies the relation (1).

Example 1. Let $\Delta[n]_n = \Delta(m,n)$ be the set of all morphisms in category Δ from m to n. The set $\{\Delta[n], \partial_i, S_j\}$ is simplicial set, where $\partial_i : \Delta\,[n]_m \to \Delta[n]_{m-1}$, $S_j : \Delta[n]_m \to \Delta[n]_{m+1}$.

Note that for a simplicial set $X.$ we have the following isomorphism $\Delta(X.)_m \approxeq \coprod_{n=0}^{\infty} X_n \times \Delta(m,n)/\backsim\, = X_p \times Aut_\Delta[m]$, where the equivalent relation $\backsim$ satisfies the following identities

$$(\partial_i, \alpha) \backsim (x, \delta^i \circ \alpha), \quad (S_j, \alpha) \backsim (x, \sigma^j \circ \alpha), \quad \alpha \in \Delta(m, n). \tag{3}$$

Definition 4. Let Λ be an index category, for an arbitrary category $\wp$, the Λ-categorical object is functor $F : \Lambda^{op} \to \wp$, such that the following diagram is commutative:

$$\begin{array}{ccc} & & \Lambda^{op} \\ & // & \downarrow \\ \Delta^{op} & \to & \wp \end{array} \tag{4}$$

We can show that index categories S and W are simplicial set by defining the face and degeneracy maps.

- For the symmetry index category S:The face and degeneracy maps ∂_i and S_j are given by:

$$\partial_i(k) = \begin{cases} \overline{k+1}, & i \prec k-1, \\ I_S, & i = k, k+1, \\ \bar{k}, & i \succ k, \end{cases} \tag{5}$$

$$S_j(k) = \begin{cases} \overline{k+1}, & i \prec k-1, \\ \bar{k}, & i \succ k, \\ \bar{k}.\overline{k+1}, & i = k-1, \\ \overline{k+1} \cdot \bar{k}, & i = k, \end{cases}$$

The face and degeneracy maps for the composition of maps are given by:

$$\partial_i(\alpha \cdot \beta) = \partial_{\beta(i)}(\alpha) \cdot \partial_i\beta, \quad S_i(\alpha \cdot \beta) = S_{\beta(i)}(\alpha) \cdot S_i\beta,$$
$$\alpha, \beta \in S_n, \quad 0 \leqslant i \leqslant n$$

- For Hyperoctahedral(Weil) index category W:The face and degeneracy maps ∂_i and S_j are given by:

$$\partial_i : W_n \to W_{n-1}, \quad S_j : W_n \to W_{n+1}, \quad 0 \leqslant i \leqslant n$$
$$\partial_i(\alpha; \varepsilon_o, \ldots, \varepsilon_n) = (\partial_i(\alpha),\ \varepsilon_o, \ldots, \widehat{\varepsilon_i}, \ldots, \varepsilon_n),$$
$$S_i(\alpha; \varepsilon_o, \ldots, \varepsilon_n) = ({}_{\varepsilon_0}S_i(\alpha),\ \varepsilon_0, \ldots, \varepsilon_i\varepsilon_i, \ldots, \varepsilon_n)$$
$$\alpha \in S_n, \quad \varepsilon_i \in \{-1, +1\} \in \aleph/2, \quad 0 \leqslant i \leqslant n$$

where

$${}_{\varepsilon_0}S_i(\alpha) = \begin{cases} S_i(\alpha), & \varepsilon_i = +1 \\ (\alpha_i,\ \alpha_{i+1})S_i(\alpha), & \varepsilon_i = -1 \end{cases} \tag{6}$$

Such that

$$(\alpha_i, \alpha_{i+1}) = \begin{cases} 0 \ldots \alpha_i \ldots \alpha_{i+1} \ldots n+1 \\ 0 \ldots \alpha_{i+1} \ldots \alpha_i \ldots n+1 \end{cases} \tag{7}$$

2. Related subject with index category

2.1. The Λ-cohomology of for Λ-index simplicial module

Definition 5. Let Λ be regular index category. For Λ-index simplicial module M_*. The complex $\left(Inv^*_\Lambda\left(M_*, d^n_i\right)\right.$ is invariant complex, such that:

$$Inv^*_\Lambda(M_*) = \{f : M_n \to k | f(m)a = sign(\varepsilon^-_*(\alpha) \cdot f(m), \quad \alpha \in \Lambda_n\} \tag{8}$$

f is linear and the differential $d^n_i : Inv^n_\Lambda(M_*) \to Inv^{n+1}_\Lambda(M_*)$ is given by:

$$d^n(f)(m) = \sum_{i=0}^{m+1} f\left(m\delta^i_n\right) \cdot (-1)^i, \quad f \in Inv^*_\Lambda(M_*),\ m \in M_{n+1}. \tag{9}$$

Theorem 1. *Let k-field with char k = o and M_* is Λ-index simplicial $k-module$. Then the Λ-(co)homology of M_* is isomorphic to the (co)homology of invariant complex, i.e.*

$$H\Lambda_*(M_*) \approxeq H(Inv^*_\Lambda(M_*))$$
$$\times \left(H\Lambda^*\left(Inv^*_\Lambda(M_*)\right) \approxeq H\left(Inv^*_\Lambda(M_*)\right)\right). \tag{10}$$

Consider the following index categories: the simplicial Δ, cyclic C and symmetry index category S.

Suppose that (M_*) is symmetry $k-module$, where k is field with characteristic zero. Consider the inclusions $\Delta \to C \to S$. The groups H^*, HC^* and HS^* are respectively simplicial, cyclic and symmetry cohomology groups of $k-modules$.

The aim of this part is to prove the following assertion:

Theorem 2. *Suppose that M_* is symmetric $k-module$ $(chark = 0)$, then the following isomorphism holds.*

$$HC^*(M_*) \approxeq H^*(M_*) \underset{k}{\otimes} H^*(BC, k) \tag{11}$$

Proof. From [10] $BC = BSO(2) = P^\infty$ and hence it is necessary to prove the following isomorphism:

$$HC^n(M_*) \approxeq H^n(M_*) \oplus H^{n-2}(M_*) \oplus H^{n-4}(M_*) \oplus \ldots \tag{12}$$

consider the following sequence of complexes

$$Inv^*_\Delta(M_*) \overset{i_{1_*}}{\longleftarrow} Inv^*_C(M_*) \underset{i_{2_*}}{\overset{P_*}{\rightleftarrows}} Inv^*_S(M_*) \tag{13}$$

where i_{1_*}, i_{2_*} are inclusion and $P_* : Inv^*_C(M_*) \to Inv^*_S(M_*)$ is a projection defined by $P_n(f^n(m)) = \frac{1}{(n+1)!}\sum_{\alpha \in S_n} sign(\alpha) f^n(m(\alpha))$.

Note that the map P_* is covering of the map T_*, such that $T_* : Inv^*_C(M_*) \to Inv^*_S(M_*)$ and the following diagram is commutative:

$$\begin{array}{ccc} & & Inv^*_\Delta(M_*) \\ & \nearrow & i_{1_*} \circ i_{2_*} \uparrow\downarrow T_* \\ Inv^*_C(M_*) & \underset{i_{2_*}}{\overset{P_*}{\rightleftarrows}} & Inv^*_S(M_*) \\ & & P_* \circ i_{2_*} = T_* \circ (i_{1_*} \circ i_{2_*}) = id_{Inv^*_S} \end{array} \tag{14}$$

and

$$\begin{aligned} P_n(f^n(m)) &= \frac{1}{(n+1)!}\sum_{\alpha \in S_n} sign(\alpha) f^n(m(\alpha)) \\ &= \frac{1}{(n+1)!}\sum_{\alpha \in S_n} sgn(\alpha) \cdot sgn f^n(m) \\ &= \frac{(n+1)!}{(n+1)!} f^n(m) = f^n(m), \qquad f^n \in Inv^*_S \end{aligned} \tag{15}$$

From the above consideration we have the following assertions:

- T_* is projective a qnd $Inv^*_\Delta(M_*)$ and $Inv^*_S(M_*)$ have the same homology.
- The morphism $(i_{1_*} \circ i_{2_*})$ and P_* are homological equivalent.
- The maps P_* is also projection and we have the extension $HC^n(M_*) = H^*(M_*) \oplus L^*$.
- The map $i_{1_*} : HC^*(M_*) \to H^*(M_*)$ is projection.
- The following isomorphisms hold $HC^*(M_*) \approxeq HC^*(M_*)$ and $H^*(M_*) \approxeq H_*(M_*)$.

Following [8] from the cyclic module M_* consider the module M^-_* where $M^-_n = M_n \oplus M_{n-2} \oplus M_{n-4} \oplus \ldots$. We have the following exact sequence $0 \to M_* \to M^-_* \to M^-_*[-2] \to 0$ which induces the following isomorphism $HC^n(M_*) \approxeq HC^{n-2}(M_*) \oplus H^n(M_*)$ but $HC^{n-2}(M_*) \approxeq HC^{n-4}(M_*) \oplus HC^{n-2}(M_*)$, then $HC^n(M_*) \approxeq H^n(M_*) \oplus H^{n-2}(M_*) \oplus H^{n-4}(M_*) \oplus \ \ldots$ □

Example 2. Suppose $M_* = kS_*(X)$ be singular simplicial $k-module$ of topological space X $(chark = 0)$, then $H^*(M_*) = H^*(X,k)$. In the complex $kS_*(X)$ there is a subcomplexes $Inv^*_C(kS_*(X))$, $Inv^*_S(kS_*(X))$. The homology of these complexes are given by:

$$H^n(Inv^*_C(kS_*(X))) \approxeq H^n(X,k) \oplus H^{n-2}(X,k) \tag{16}$$

$$H^n(Inv^*_S(kS_*(X))) \approxeq H^n(X,k).$$

By considering the following sequence of injection complexes

$$Inv^*_C(kS_*(X)) \to Inv^*_S(kS_*(X)) \to Inv^*_\Delta(kS_*(X)) \tag{17}$$

where $Inv^*_\Delta(kS_*(X))$ is complex of simplicial cochain of topological space X. The sequence (17) induces the following sequence:

$$\begin{aligned} &H^n(X,k) \overset{i_n}{\to} H^n(X,k) \oplus H^{n-2}(X,k) \\ &\oplus H^{n-4}(X,k) \cdots \overset{P_n}{\to} H^n(X,k) \end{aligned} \tag{18}$$

where i_n is inclusion and P_n is projective.

Note that the same result of Theorem 8 can be got for homology group.

2.2. Monoid and algebra over monoid

In this part we study the relation between monoid and index category. The main result is Theorems 16 and 18.

Firstly we recall definition of monoid and algebra over it.

Definition 6. Monoid $\mathtt{T}$ in category A is the triple (T,μ,υ), when $T: A\rightarrow A$ is covariant functor, $\mu: T\circ T\rightarrow T$ and $\upsilon: id_A\rightarrow A$ are maps with the following commutative diagram.

$$\begin{array}{ccc} \mathtt{T}\circ\mathtt{T}\circ\mathtt{T}(X) & \underset{\rightarrow}{\mu(\mathtt{T}(X))} & \mathtt{T}\circ\mathtt{T} \\ \mathtt{T}(\mu(X))\downarrow & & \downarrow\mu(X), \\ \mathtt{T}\circ\mathtt{T}(X) & \rightarrow & \mathtt{T}(X) \end{array} \quad \begin{array}{ccccc} T(x) & \underset{\rightarrow}{T(\upsilon(x))} & T\circ T(x) & \underset{\rightarrow}{\upsilon(T(x))} & T(x) \\ & \backslash\backslash & \downarrow\mu(x) & // & \\ & & T(x) & & \end{array} \tag{19}$$

where $x\in obA$.

Example 3. For the categories A and B suppose $U: A\rightarrow B$, $Q: B\rightarrow A$ are functors such that an isomorphism $\varphi: Hom_A(-,Q(-))\rightarrow Hom_B(U(-),-)$ is a functor from $A^{0p}\times B$ to category of sets where $Hom_A(-,Q(-))(x,y) = A(x,Q(y))$, $Hom_B(U(-),-)(x,y) = B(U(x),y)$, $x\in obA$, $y\in obB$, then the triple $(Q\circ U,\mu,v)$ is monoid in category A if $v(x) = \varphi^{-1}(id_{v(x)}): x\rightarrow Q\circ U(x)$, $\mu(x) = Q(\varphi^{-1}(id_{\varphi U(x)}): Q\circ U\circ Q\circ U(x)\rightarrow Q\circ U(x)$.

Definition 7. Algebra over monoid (T,μ,v) is defined to be (X,ξ), where $x\in ObA$ and $\xi: T(x)\rightarrow X$ is an isomorphism such that the following diagram commutes.

$$\begin{array}{ccc} \mathtt{T}\circ\mathtt{T}(X) & \underset{\rightarrow}{\mu(\times)} & \mathtt{T}(\times) \\ \mathtt{T}(\xi)\downarrow & & \downarrow\xi, \\ \mathtt{T}(\times) & \rightarrow & \times \end{array} \quad \begin{array}{ccc} \times & \underset{\rightarrow}{v(\times)} & T(\times) \\ & \backslash\backslash & \downarrow\xi \\ & & \times \end{array}. \tag{20}$$

(i) Suppose a category A, for every object x in A, the pair $(\mathtt{T}(X),\mu(\times))$ is algebra over monoid $\mathtt{T}$ and called free $\mathtt{T}$-algebra generated by object x.
(ii) The pair $(Q(X),\xi)$ is algebra over monoid $(Q\circ U,\mu,\upsilon)$ if $\xi = Q_\varphi(id_{\varphi(X)}): Q\circ U\circ Q(X)\rightarrow Q(X)$.

A morphism $\psi: (\mathtt{T},\mu,\upsilon)\rightarrow(\mathtt{T}^\backslash,\mu^\backslash,\upsilon^\backslash)$ between monoids is given by the map $\psi: T\rightarrow T^\backslash$ such that for any object X, the following diagram commutes.

$$\begin{array}{ccc} T\circ T(x) & \underset{\rightarrow}{\psi^2(x)} & T^\backslash\circ T^\backslash(x) \\ \mu(x)\downarrow & & \downarrow\mu^\backslash \\ T(x) & \underset{\rightarrow}{\psi(x)} & T^\backslash(x) \end{array}, \quad \begin{array}{ccc} T(x) & \underset{\rightarrow}{\psi(x)} & T^\backslash(x) \\ \upsilon(x) & \searrow & \downarrow v^\backslash(x) \\ & & X \end{array} \tag{21}$$

Definition 8. Where A morphism $\psi^2(x)$ is orthogonal in the commutative diagram

$$\begin{array}{ccc} \mathtt{T}\circ\mathtt{T}(\times) & \underset{\rightarrow}{\mathtt{T}(\psi(x))} & \mathtt{T}\circ\mathtt{T}^\backslash(\times) \\ \psi(\mathtt{T}(\times))\downarrow & & \downarrow\psi(\mathtt{T}^\backslash(\times)) \\ \mathtt{T}^\backslash\circ\mathtt{T}(\times) & \underset{\rightarrow}{\mathtt{T}^\backslash(\psi(\times))} & \mathtt{T}^\backslash\circ\mathtt{T}^\backslash(\times) \end{array} \tag{22}$$

A morphism $f: (X,\xi)\rightarrow(X^\backslash,\xi^\backslash)$ between algebras over monoid (T,μ,υ) is given by the morphism $\psi: X\rightarrow X^\backslash$ such that the following diagram is commutative.

$$\begin{array}{ccc} \mathtt{T}(\times) & \underset{\rightarrow}{\mathtt{T}(f)} & \mathtt{T}(\times^\backslash) \\ \xi\downarrow & & \downarrow\xi^\backslash \\ \times & \underset{\rightarrow}{f} & \times^\backslash \end{array}, \quad \begin{array}{ccc} \mathtt{T}(\times) & \underset{\rightarrow}{\psi(\times)} & \mathtt{T}^\backslash(\times) \\ v(\times) & \searrow & \downarrow\upsilon^\backslash(\times) \\ & & \times \end{array}. \tag{23}$$

In the following we denote by *Ind Cat* the category of index category and by $Mon(A)$ *cat* the category of monoid in category A and by $T-Alg$ the category of algebra over monoid.

The following theorems give some properties of index category related with monoid and algebra over it.

There is an isomorphism between the category of index category (*Ind Cat*) and the category of monoid in the category of simplicial set (MON (Δ^{op} set), i.e.

$$\Omega: IndCat\rightarrow Mon(\Delta^{op})$$

Proof. Consider an index category (Λ,i), the monoid (T,μ,υ) and the functors: $i^*: \Lambda^{op}set\rightarrow\Delta^{op}set$, $i^*: \Delta^{op}set\rightarrow\Lambda^{op}set$, where $i^*(X.) = X.\circ i$, $X.$ is simplicial set and $i^*(X.)_m = \prod_{n=0}^{\infty}X_n\times\Lambda(m,n)/\sim$, $(x\tau,\rho)\sim(x,\tau\circ\rho)$, $\tau\in\Delta(m,n)$. From the simplicial set theory the relation between the functors i^* and i_* is given by the following assertion. □

Proposition 3. ([10,11]) Given the simplicial maps Φ and Φ^{-1}, the map is identity map that is; $(\Phi^{-1}\circ\Phi)(f)\ (\times) = \Phi^{-1}(\Phi(f))(\times) = \Phi(f)([\times,1]) = (f)\ (\times)$.

Proof. Consider a simplicial map Φ such that: $\Phi(X.,Y.): \Delta^{op}set(X.,i^*(Y.))\rightarrow\Lambda^{op}set(i_*(X.),Y.)$ where $\Phi(X.,Y.)(F.)([\times,\rho]) = (F_n(\times))\rho$, $F.: X.\rightarrow i^*(Y.), \times\in X_n$ and $\rho\in\Lambda\ (P,n)$. The inverse functor Φ^{-1} is given by:

$$\Phi_{-1}(X.,\ Y.): \Lambda_{op}set(i_*(X.),\ Y.)\rightarrow\Delta_{op}set(X.,\ i_*(Y.))$$

where $\Phi^{-1}(X.,Y.)(g.): X\rightarrow i_*(Y.)$, $\Phi^{-1}g(x) = g([x,1])$, $g.: i_*(X.)\rightarrow Y.$. $(\Phi^{-1}\circ\Phi)(f)(x) = \Phi^{-1}(\Phi(f))(x) = \Phi(f)\ ([x,1]) = f(x)$.

From the above discussion the triple $(i^*\circ i_*,\mu,\upsilon)$ is a monoid and there is a homomorphism Ω between the index category (Λ,i) and $(i^*\circ i_*,\mu,\upsilon)$, since $i^*\circ i_*: \Delta^{op}-Sets\rightarrow\Delta^{op}-Sets$, $\mu(x) = T\circ T(X)\rightarrow T(X)$, $\forall X\in\Delta^{op}-Sets$ such that $\mu(x)(i^*\circ i_*\circ i^*\circ i_*)\ (X)\rightarrow i^*\circ i_*(X)$, where $(i^*\circ i_*)(X) = \prod_{n=0}^{\infty}X_n\times\Lambda[n]/\sim.$, $\mu([x,\alpha,\gamma]) = [x,\alpha\circ\gamma]\in i^*\circ i_*(X)$ and $\upsilon(x): X\rightarrow T(X)$ such that $\upsilon(x) = [x,1]\in(i^*\circ i_*)(X)$. To get the converse proof suppose the functor $\Omega(\Lambda,i) = (i^*\circ i_*,\mu,\upsilon)$, then for an arbitrary monoid $\mathtt{T}$ in the category $\Delta^{op}Set$, the isomorphism Ω^{-1} is given by $\Omega^{-1}(T)(n,m) = T(\Delta(-,n))_m$. □

For an index category (Λ,i), the following isomorphism holds.

$$\Delta^{op}Set\approxeq\Omega(\Lambda)-algebra.$$

Proof. For an arbitrary $\Delta^{op}Set\ Y.$ we consider the following algebra $(i^*(Y),i^*\Phi(id_{i^*(X)}))$ which is Ω-algebra.

Also for every Ω-algebra $(X.,\xi.)$ we can define the action of the operator $\rho\in\Lambda(m,n)$ on the graded set by: $(\times)\rho = \xi_m([\times,\rho])$, where $\times\in X_n$. □

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