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SHORT COMMUNICATION

Λ-Generalized closed sets in bitopological spaces

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Abstract In this paper, we introduce the notion of Λ-generalized closed sets in bitopological spaces. Also, we give some characterizations and applications of it.

1. Introduction and Preliminaries

Kelly [1] introduced the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. Although [1] is beyond any doubt an original and fundamental work on the theory of bitopological spaces, nevertheless it should be noted that both the notion of a bitopological space and the term itself appeared for the first time in a somewhat narrow sense in [2,3] as auxiliary tool used to characterize Baire spaces. For this use, the topologies τ_1 and τ_2 on a set X, one of which was finer than the other, were connected by certain other relations as well.

Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A is said to be λ-closed [4] if $A = L \cap M$, where L is a Λ-set (i.e., L is equal to its kernel) and M is a closed set. The λ-closure of A is the intersection of all λ-closed sets containing A and is denoted by $Cl_\lambda(A)$. A subset A is called λ-open if $X \setminus A$ is λ-closed. The union of all λ-open subsets of A is called the λ-interior of A and is denoted by $Int_\lambda(A)$. The λ-frontier of A [5], denoted by $Fr_\lambda(A)$, is defined by $Fr_\lambda(A) = Cl_\lambda(A) \setminus Int_\lambda(A)$. A subset A [6] is called Λg-closed if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \lambda O(X,\tau)$.

In 1991, Sundaram [7] introduced the concept of generalized closed sets in bitopological spaces. The notion has been studied extensively in recent years by many topologists. A subset A of a bitopological space (X,τ_1,τ_2) is said to be (i,j)-g-closed [8] if $\tau_j - Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_i$ for $i,j = 1,2$ and $i \neq j$.

In this paper, we introduce and investigate the notion of Λg-closed sets in bitopological spaces. The relationships with other kinds of sets are given. Also, we give some applications of (i,j)-Λ_g-closed sets.

2. (i,j)-Λ_g-closed sets

Definition 2.1. A subset A of a bitopological space (X,τ_1,τ_2) is called an (i,j)-Λ_g-closed (resp. (i,j)-Λg-closed, (i,j)-$g\Lambda$-closed) if $\tau_j - Cl(A) \subseteq U$ (resp. $\tau_j - Cl_\lambda(A) \subseteq U$, $\tau_j - Cl_\lambda(A) \subseteq U$) whenever $A \subseteq U$ and $U \in \lambda O(X,\tau_i)$ (resp. $U \in \lambda O(X,\tau_i)$, $U \in \tau_i$) for $i,j = 1,2$ and $i \neq j$.

Remark 2.2. We have the following implications. None of the implications can be reversed as shown by some examples stated below:

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$$\begin{array}{ccc} (i,j)\text{-}\Lambda_g\text{-}Closed & \Rightarrow & (i,j)\text{-}g\text{-}Closed \\ \Downarrow & & \Downarrow \\ (i,j)\text{-}\Lambda g\text{-}Closed & \Rightarrow & (i,j)\text{-}g\Lambda\text{-}Closed \end{array}$$

Example 2.1.

(1) Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{X, \emptyset, \{a\}\}$and $\tau_2 = \{X, \emptyset, \{b\}\}$. Thus $\lambda O(X, \tau_1) = \{X, \emptyset, \{a\}, \{b, c\}\}$. The subset $A = \{b\}$ is $(1,2)$-Λg-closed but it is not $(1,2)$-Λ_g-closed.
(2) Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{c\}, \{b, c\}\}$. Thus $\lambda O(X, \tau_1) = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $\lambda C(X, \tau_2) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Take $A = \{b, c\}$. Then we obtain that A is $(1,2)$-$g\Lambda$-closed but it is not $(1,2)$-Λg-closed.
(3) By taking $A = \{b\}$in (2), then A is $(1,2)$-$g\Lambda$-closed but it is not $(1,2)$-g-closed.

Theorem 2.3. *The union of two (i,j)-Λ_g-closed sets is (i,j)-Λ_g-closed.*

Proof. Let $A \cup B \subseteq U$, then $A \subseteq U$ and $B \subseteq U$ where $U \in \lambda O(X, \tau_i)$. As A and B are (i,j)-Λ_g-closed, $\tau_j - Cl(A) \subseteq U$ and $\tau_j - Cl(B) \subseteq U$. Hence

$$\tau_j - Cl(A \cup B) = \tau_j - Cl(A) \cup \tau_j - Cl(B) \subseteq U. \qquad \square$$

Remark 2.4. The intersection of two (i,j)-Λ_g-closed sets need not be (i,j)-Λ_g-closed as shown by the following example.

Example 2.2. Let $X = \{a, b, c\}$with the topologies $\tau_1 = \{X, \emptyset, \{a\}\}$and $\tau_2 = \{X, \emptyset, \{b\}\}$. If $A = \{a, b\}$ and $B = \{a, c\}$ are two $(1,2)$-Λ_g-closed sets but $A \cap B$ is not $(1,2)$-Λ_g-closed.

Theorem 2.5. *If a subset A of (X, τ_1, τ_2) is (i,j)-Λ_g-closed, then $\tau_j - Cl(A) \setminus A$ contains no nonempty τ_i-closed subset of (X, τ_1, τ_2).*

Proof. Let F be a τ_i-closed subset contained in $\tau_j - Cl(A) \setminus A$. Clearly $A \subseteq X \setminus F$ where A is (i,j)-Λ_g-closed and $X \setminus F$ is a τ_i-open subset of X. Thus $\tau_j - Cl(A) \subseteq X \setminus F$ or $F \subseteq X \setminus (\tau_j - Cl(A))$. Then

$$\begin{aligned} F &\subseteq (X \setminus \tau_j - Cl(A)) \cap (\tau_j - Cl(A) \setminus A) \\ &\subseteq (X \setminus \tau_j - Cl(A)) \cap (\tau_j - Cl(A)) \\ &= \emptyset. \qquad \square \end{aligned}$$

Remark 2.6. The converse of the above theorem need not be true in general as shown by the following example.

Example 2.3. Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{c\}, \{b, c\}\}$. If $A = \{a, c\}$, then $\tau_2 - Cl(A) \setminus A = \{b\}$ does not contain nonempty closed set. But A is not $(1,2)$-Λ_g-closed in (X, τ_1, τ_2).

Definition 2.7. The topologies τ_i and τ_j on a nonempty set X are said to have the property $\mathcal{E}$ if $A \in \lambda C(X, \tau_i)$ and $B \in \lambda C(X, \tau_j)$ imply $A \cap B \in \lambda C(X, \tau_i)$.

Theorem 2.8. *Let τ_i and τ_j have the property $\mathcal{E}$. Then, a subset A is (i,j)-Λ_g-closed if and only if $\tau_j - Cl(A) \setminus A$ contains no nonempty $F \in \lambda C(X, \tau_i)$.*

Proof. Suppose that A is (i,j)-Λ_g-closed. Let $F \in \lambda C(X, \tau_i)$ such that $F \subseteq \tau_j - Cl(A) \setminus A$. Then $A \subseteq X \setminus F$. Since A is (i,j)-Λ_g-closed, we have $\tau_j - Cl(A) \subseteq X \setminus F$. Consequently $F \subseteq X \setminus \tau_j - Cl(A)$. Hence $F \subseteq (\tau_j - Cl(A) \setminus A) \cap (X \setminus \tau_j - Cl(A)) = \emptyset$. Therefore, $F = \emptyset$.

Now, suppose that $\tau_j - Cl(A) \setminus A$ contains no nonempty $F \in \lambda C(X, \tau_i)$. Let $A \subseteq G$ and $G \in \lambda O(X, \tau_i)$. If $\tau_j - Cl(A) \subseteq G$, then $\tau_j - Cl(A) \cap (X \setminus G) \neq \emptyset$, $\tau_j - Cl(A) \cap (X \setminus G) \in \lambda C(X, \tau_i)$ and $\tau_j - Cl(A) \cap (X \setminus G) \subseteq \tau_j - Cl(A) \setminus A$. Therefore, A is (i,j)-Λ_g-closed. $\square$

Theorem 2.9. *If A is (i,j)-Λ_g-closed and $A \subseteq B \subseteq \tau_j - Cl(A)$, then B is (i,j)-Λ_g-closed.*

Proof. Let $B \subseteq U$ and $U \in \lambda O(X, \tau_i)$. Then $A \subseteq U$ and A is (i,j)-Λ_g-closed. Hence $\tau_j - Cl(B) \subseteq \tau_j - Cl(A) \subseteq U$ and B is (i,j)-Λ_g-closed. $\square$

Theorem 2.10. *If A is an (i,j)-Λ_g-closed set in (X, τ_1, τ_2) and $A \in \lambda O(X, \tau_i)$, then A is τ_j-closed.*

Proof. Since $A \in \lambda O(X, \tau_i)$ and A is (i,j)-Λ_g-closed, $\tau_j - Cl(A) \subseteq A$ and hence A is τ_j-closed. $\square$

Theorem 2.11. If A is τ_j-closed, then A is (i,j)-Λ_g-closed.

Proof. Straightforward. $\square$

Theorem 2.12. *For each $x \in X$, either $\{x\} \in \lambda C(X, \tau_i)$ or $X \setminus \{x\}$is (i,j)-Λ_g-closed.*

Proof. Suppose that $\{x\}$is not in $\lambda C(X, \tau_i)$. Then $X \setminus \{x\}$is not in $\lambda O(X, \tau_i)$ and the only subset that in $\lambda O(X, \tau_i)$ and contains $X \setminus \{x\}$is X. Therefore $\tau_j - Cl(X \setminus \{x\}) \subseteq X$ and so $X \setminus \{x\}$ is (i,j)-Λ_g-closed. $\square$

Theorem 2.13. *If A is (i,j)-Λ_g-closed and $A \subseteq U \in \lambda O(X, \tau_i)$, then $\tau_i - Fr_\lambda(U) \subset \tau_j - Int(X \setminus A)$.*

Proof. Let A be (i,j)-Λ_g-closed and $A \subseteq U \in \lambda O(X, \tau_i)$. Then $\tau_j - Cl(A) \subseteq U$. Suppose that $x \in \tau_i - Fr_\lambda(U)$. Since $U \in \lambda O(X, \tau_i)$, $\tau_i - Fr_\lambda(U) = \tau_i - Cl_\lambda(U) \setminus U$. Therefore, $x \notin U$ and $x \notin \tau_j - Cl(A)$. This shows that $x \in \tau_j - Int(X \setminus A)$ and hence $\tau_i - Fr_\lambda(U) \subset \tau_j - Int(X \setminus A)$. $\square$

Definition 2.14. A subset A in (X, τ_1, τ_2) is said to be (i,j)-Λ_g-open if $X \setminus A$ is (i,j)-Λ_g-closed.

Theorem 2.15. *The intersection of two (i,j)-Λ_g-open sets is (i,j)-Λ_g-open.*

Proof. Straightforward. $\square$

Theorem 2.16. *A subset A is (i,j)-Λ_g-open if and only if $F \subseteq \tau_j - Int(A)$ whenever $F \in \lambda C(X, \tau_i)$ and $F \subseteq A$.*

Proof. Suppose that $F \subseteq \tau_j - Int(A)$ whenever $F \in \lambda C(X, \tau_i)$ and $F \subseteq A$. Let $X \setminus A \subseteq G$, where $G \in \lambda O(X, \tau_i)$. Hence $X \setminus G \subseteq A$. By assumption $X \setminus G \subseteq \tau_j - Int(A)$. This implies that $X \setminus \tau_j - Int(A) \subseteq G$ and hence $\tau_j - Cl(X \setminus A) \subseteq G$. Hence $X \setminus A$ is (i,j)-Λ_g-closed i.e., A is (i,j)-Λ_g-open.

Now, let A be an (i,j)-Λ_g-open set. Then $X \setminus A$ is (i,j)-Λ_g-closed. Also let $F \in \lambda C(X, \tau_i)$ such that $F \subseteq A$. Then $X \setminus F \in \lambda O(X, \tau_i)$. Therefore whenever $X \setminus A \subseteq X \setminus F$, $\tau_j - Cl(X \setminus A) \subseteq X \setminus F$. This implies that $F \subseteq X \setminus (\tau_j - Cl(X \setminus A)) = \tau_j - Int(A)$. Thus $F \subseteq \tau_j - Int(A)$. □

Theorem 2.17. *If $A \in \tau_j$, then A is (i,j)-Λ_g-open.*

Proof. Straightforward. □

Theorem 2.18. *Let τ_i and τ_j have the property $\mathcal{E}$. Then, a subset A is (i,j)-Λ_g-open if and only if $G = X$ whenever $G \in \lambda O(X, \tau_i)$ and $\tau_j - Int(A) \cup (X \setminus A) \subseteq G$.*

Proof. Let A be (i,j)-Λ_g-open, $G \in \lambda O(X, \tau_i)$ and $\tau_j - Int(A) \cup (X \setminus A) \subseteq G$. Then

$$\begin{aligned} X \setminus G &\subseteq (X \setminus \tau_j - Int(A)) \cap (X \setminus (X \setminus A)) \\ &= (X \setminus \tau_j - Int(A)) \setminus (X \setminus A) \\ &= \tau_j - Cl(X \setminus A) \setminus (X \setminus A). \end{aligned}$$

Since $X \setminus A$ is (i,j)-Λ_g-closed and $X \setminus G \in \lambda C(X, \tau_i)$, by Theorem 2.8, it follows that $X \setminus G = \emptyset$. Therefore $X = G$. Conversely, suppose that $F \in \lambda C(X, \tau_i)$ and $F \subseteq A$. Then $\tau_j - Int(A) \cup (X \setminus A) \subseteq \tau_j - Int(A) \cup (X \setminus F)$. It follows that $\tau_j - Int(A) \cup (X \setminus F) = X$ and hence $F \subseteq \tau_j - Int(A)$. Therefore A is (i,j)-Λ_g-open. □

Theorem 2.19. *If $\tau_j - Int(A) \subseteq B \subseteq A$ and A is (i,j)-Λ_g-open, then B is (i,j)-Λ_g-open.*

Proof. Suppose that $\tau_j - Int(A) \subseteq B \subseteq A$ and A is (i,j)-Λ_g-open. Then $X \setminus A \subseteq X \setminus B \subseteq \tau_j - Cl(X \setminus A)$ and $X \setminus A$ is (i,j)-Λ_g-closed. By using Theorem 2.9, B is (i,j)-Λ_g-open. □

Theorem 2.20. *Let τ_i and τ_j have the property $\mathcal{E}$. Then, a subset A is (i,j)-Λ_g-closed if and only if $\tau_j - Cl(A) \setminus A$ is (i,j)-Λ_g-open.*

Proof. Suppose that A is (i,j)-Λ_g-closed. Let $F \subseteq \tau_j - Cl(A) \setminus A$, where $F \in \lambda C(X, \tau_i)$. By using Theorem 2.8, $F = \emptyset$. Therefore $F \subseteq \tau_j - Int(\tau_j - Cl(A) \setminus A))$ and by using Theorem 2.16, $\tau_j - Cl(A) \setminus A$ is (i,j)-Λ_g-open.
Now, let $A \subseteq G$ where $G \in \lambda O(X, \tau_i)$. Then

$$\tau_j - Cl(A) \cap (X \setminus G) \subseteq \tau_j - Cl(A) \cap (X \setminus A) = \tau_j - Cl(A) \setminus A.$$

Since $\tau_j - Cl(A) \cap (X \setminus G) \in \lambda C(X, \tau_i)$ and $\tau_j - Cl(A) \setminus A$ is (i,j)-Λ_g-open, by Theorem 2.16, we have

$$\tau_j - Cl(A) \cap (X \setminus G) \subseteq \tau_j - Int(\tau_j - Cl(A) \setminus A)) = \emptyset.$$

Hence A is (i,j)-Λ_g-closed. □

Theorem 2.21. *A subset A is (i,j)-Λ_g-closed if and only if $\tau_i - Cl_\lambda(\{x\}) \cap A \neq = \emptyset$ for every $x \in \tau_j - Cl(A)$.*

Proof. Suppose that $\tau_i - Cl_\lambda(\{x\}) \cap A = \emptyset$ for some $x \in \tau_j - Cl(A)$. Then $X \setminus (\tau_i - Cl_\lambda(\{x\})) \in \lambda O(X, \tau_i)$ such that $A \subseteq X \setminus (\tau_i - Cl_\lambda(\{x\}))$. Furthermore, $x \in \tau_j - Cl(A) \setminus (X \setminus \tau_i - Cl_\lambda(\{x\}))$ and hence $\tau_j - Cl(A) \not\subseteq (X \setminus \tau_i - Cl_\lambda(\{x\}))$. This shows that A is not (i,j)-Λ_g-closed.

Now, suppose that A is not (i,j)-Λ_g-closed. There exists $U \in \lambda O(X, \tau_i)$ such that $A \subseteq U$ and $\tau_j - Cl(A) \setminus U \neq = \emptyset$. There exists $x \in \tau_j - Cl(A)$ such that $x \notin U$. Hence $\tau_i - Cl_\lambda(\{x\}) \cap U = \emptyset$. Therefore, $\tau_i - Cl_\lambda(\{x\}) \cap A = \emptyset$ for some $x \in \tau_j - Cl(A)$. □

3. Applications of (i,j)-Λ_g-closed sets

Definition 3.1. A bitopological space (X, τ_1, τ_2) is said to be (i,j)-normal if for disjoint τ_i-closed sets F_1 and F_2, there exist $U_1, U_2 \in \tau_j$ such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

Theorem 3.2. *Let (X, τ_1, τ_2) be a bitopological space. Then the following properties are equivalent:*

1. (X, τ_1, τ_2) is (i,j)-normal;
2. For any disjoint τ_i-closed sets F_1, F_2, there exist (i,j)-Λ_g-open sets V_1, V_2 such that $F_1 \subset V_1$, $F_2 \subset V_2$ and $V_1 \cap V_2 = \emptyset$;
3. For any τ_i-closed set F and any τ_i-open set U containing F, there exists an (i,j)-Λ_g-open set V such that $F \subset V \subset \tau_j - Cl(V) \subset U$;
4. For any τ_i-closed set F and any τ_i-open set U containing F, there exists an τ_j-open set G such that $F \subset G \subset \tau_j - Cl(G) \subset U$;
5. For any disjoint τ_i-closed sets F_1, F_2, there exists an (i,j)-Λ_g-open set V such that $F_1 \subset V$ and $\tau_j - Cl(V) \cap F_2 = \emptyset$;
6. For any disjoint τ_i-closed sets F_1, F_2, there exists an τ_j-open set G such that $F_1 \subset G$ and $\tau_j - Cl(G) \cap F_2 = \emptyset$.

Proof.

(1)⇒ (2): Clear from Theorem 2.17.
(2) ⇒ (3): Let F be a τ_i-closed set and $U \in \tau_i$ such that $F \subset U$. Then F, $X \setminus U$ are disjoint τ_i-closed sets and by (2) there exist (i,j)-Λ_g-open sets V_1 and V_2 such that $F \subset V_1$, $X \setminus U \subset V_2$ and $V_1 \cap V_2 = \emptyset$. Since V_2 is (i,j)-Λ_g-open, by using Theorem 2.16, $X \setminus U \subset \tau_j - Int(V_2)$ and $\tau_j - Int(V_2)$ is τ_j-open. Hence, $\tau_j - Cl(V_1) \cap \tau_j - Int(V_2) = \emptyset$. Therefore, we obtain $F \subset V_1 \subset \tau_j - Cl(V_1) \subset X \setminus (\tau_j - Int(V_2)) \subset U$. Put $V = V_1$, then we obtain $F \subset V \subset \tau_j - Cl(V) \subset U$.
(3)⇒ (4): Let F be a τ_i-closed set and $U \in \tau_i$ such that $F \subset U$. Then by using (3), there exists an (i,j)-Λ_g-open set V such that $F \subset V \subset \tau_j - Cl(V) \subset U$. By using Theorem 2.16, $F \subset \tau_j - Int(V)$. Put $G = \tau_j - Int(V)$. Then $F \subset G \subset \tau_j - Cl(G) \subset \tau_j - Cl(V) \subset U$.
(4)⇒ (5): Let F_1, F_2 be any disjoint τ_i-closed sets. Since $X \setminus F_2$ is a τ_i-open set such that $F_1 \subset X \setminus F_2$, by (4) there exists an τ_j-open set V such that $F_1 \subset V \subset \tau_j - Cl(V) \subset X \setminus F_2$. By using Theorem 2.17, V is (i,j)-Λ_g-open. Furthermore, we have $F_1 \subset V$ and $\tau_j - Cl(V) \cap F_2 = \emptyset$.

(5)⇒ (6): Let F_1, F_2 be any disjoint τ_i-closed sets. Then, there exists an (i,j)-Λ_g-open set V such that $F_1 \subset V$ and $\tau_j - Cl(V) \cap F_2 = \emptyset$. By using Theorem 2.16, $F_1 \subset \tau_j - Int(V)$. Set $G = \tau_j - Int(V)$. Then we have $G \in \tau_j$, $F_1 \subset G$ and $\tau_j - Cl(G) \cap F_2 = \emptyset$.

(6)⇒ (1): Let F_1, F_2 be any disjoint τ_i-closed sets. Then, by (6) there exists $G \in \tau_j$ such that $F_1 \subset G$ and $\tau_j - Cl(G) \cap F_2 = \emptyset$. Now, put $U_1 = G$ and $U_2 = X \setminus (\tau_j - Cl(G))$. Then U_1 and U_2 are disjoint τ_j-open sets, $F_1 \subset U_1$ and $F_2 \subset U_2$. This shows that (X, τ_i, τ_j) is (i,j)-normal. □

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