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Original Article

On invariant submanifolds of $(LCS)_n$-manifolds

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Abstract The object of the present paper is to study the invariant submanifolds of $(LCS)_n$-manifolds. We study semiparallel and 2-semiparallel invariant submanifolds of $(LCS)_n$-manifolds. Among others we study 3-dimensional invariant submanifolds of $(LCS)_n$-manifolds. It is shown that every 3-dimensional invariant submanifold of a $(LCS)_n$-manifold is totally geodesic.

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1. Introduction

In 2003 the first author [1] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds), with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [2] and also by Mihai and Rosca [3]. Then Shaikh and Baishya [4,5] investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. It is to be noted that the most interesting fact is that $(LCS)_n$-manifold remains invariant under a D-homothetic transformation, which does not hold for an LP-Sasakian manifold [6]. The $(LCS)_n$-manifolds have been also studied by Atceken [7], Narain and Yadav [8], Prakasha [9], Shaikh [10], Shaikh et al. [11,12], Shaikh and Binh [13], Shaikh and Hui [14], Sreenivasa et al. [15], Yadav et al. [16] and others.

In modern analysis the geometry of submanifolds has become a subject of growing interest for its significant application in applied mathematics and theoretical physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system [17]. Also the notion of geodesics plays an important role in the theory of relativity [18]. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Hence, totally geodesic submanifolds are also very much important in

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physical sciences. The study of geometry of invariant submanifolds was initiated by Bejancu and Papaghuic [19]. In general the geometry of an invariant submanifold inherits almost all properties of the ambient manifold. The invariant submanifolds have been studied by many geometers to different extent such as [20–35] and many others.

Motivated by the above studies the present paper deals with the study of invariant submanifolds of odd dimensional $(LCS)_n$-manifolds. The paper is organized as follows. Section 2 is concerned with rudiments of $(LCS)_n$-manifolds. Section 3 deals with the study of some basic properties of invariant submanifolds of $(LCS)_n$-manifolds. It is shown that an invariant submanifold of a $(LCS)_n$-manifold is also a $(LCS)_n$-manifold.

Let N and M be two Riemannian or semi-Riemannian manifolds, $f: N \to M$ be an immersion, h be the second fundamental form and $\overline{\nabla}$ be the Vander–Waerden–Bortolotti connection of N. An immersion is said to be semiparallel if

$$\overline{R}(X, Y) \cdot h = (\overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X - \overline{\nabla}_{[X,Y]})h = 0 \tag{1.1}$$

holds for all vector fields X, Y tangent to N [36], where $\overline{R}$ denotes the curvature tensor of the connection $\overline{\nabla}$. Semiparallel immersions have also been studied in [37,38].

In [39] Arslan et. al defined and studied submanifolds satisfying the condition

$$\overline{R}(X, Y) \cdot \overline{\nabla} h = 0 \tag{1.2}$$

for all vector fields X, Y tangent to N and such submanifolds are called 2-semiparallel. In [30] Özgür and Murathan studied semiparallel and 2-semiparallel invariant submanifolds of LP-Sasakian manifolds. In Section 4 of the paper we study semiparallel and 2-semiparallel invariant submanifolds of $(LCS)_n$-manifolds. It is proved that an invariant submanifold N of a $(LCS)_n$-manifold is semiparallel if and only if N is totally geodesic.

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [40]. The interesting invariant of a concircular transformation is the concircular curvature tensor C, which is defined by Yano [40]

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}\big[g(Y, Z)X - g(X, Z)Y\big], \tag{1.3}$$

where r is the scalar curvature of the manifold. Section 5 deals with the study of invariant submanifolds of $(LCS)_n$-manifolds satisfying $\overline{C}(X, Y) \cdot h = 0$ and $\overline{C}(X, Y) \cdot \overline{\nabla} h = 0$. It is shown that if N is an invariant submanifold of a $(LCS)_n$-manifold with $r \neq n(n-1)(\alpha^2 - \rho)$ then the condition $\overline{C}(X, Y) \cdot \overline{\nabla} h = 0$ holds if and only if N is totally geodesic. Section 6 is devoted to the study of 3-dimensional invariant submanifolds of a $(LCS)_n$-manifold and it is proved that such a submanifold is totally geodesic.

2. $(LCS)_n$-manifolds

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0,2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \cdots, +)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp., ≤ 0, $= 0$, > 0) [41].

Definition 2.1. [40] In a Lorentzian manifold (M, g) a vector field P defined by

$$g(X, P) = A(X),$$

for any $X \in \Gamma(TM)$, is said to be a concircular vector field if

$$(\widetilde{\nabla}_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\}$$

where α is a non-zero scalar and ω is a closed 1-form and $\widetilde{\nabla}$ denotes the operator of covariant differentiation of M with respect to the Lorentzian metric g.

Let M be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\widetilde{\nabla}_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0 \tag{2.3}$$

for all vector fields X, Y, where $\widetilde{\nabla}$ denotes the operator of covariant differentiation of M with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\widetilde{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. Let us take

$$\phi X = \frac{1}{\alpha}\widetilde{\nabla}_X \xi, \tag{2.5}$$

then from (2.3) and (2.5) we have

$$\phi X = X + \eta(X)\xi, \tag{2.6}$$

from which it follows that ϕ is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and an (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold), [1]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [2]. In a $(LCS)_n$-manifold $(n > 2)$, the following relations hold [1,10]:

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.7}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.8}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.9}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi,Y)Z = (\alpha^2 - \rho)[g(Y,Z)\xi - \eta(Z)Y], \tag{2.11}$$

$$R(\xi,X)\xi = (\alpha^2 - \rho)[\eta(X)\xi + X], \tag{2.12}$$

$$(\widetilde{\nabla}_X \phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.13}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.14}$$

$$R(X,Y)Z = \phi R(X,Y)Z + (\alpha^2 - \rho) \times \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \tag{2.15}$$

for all $X, Y, Z \in \Gamma(TM)$ and $\beta = -(\xi\rho)$ is a scalar function, where R is the curvature tensor and S is the Ricci tensor of the manifold. The ξ-sectional curvature $K(\xi, X) = g(R(\xi, X)\xi, X)$ for a unit vector field X orthogonal to ξ play an important role in the study of an almost contact metric manifold.

By virtue of (2.11) we have from (1.3) that

$$C(\xi,Y)Z = \left[\alpha^2 - \rho - \frac{r}{n(n-1)}\right][g(Y,Z)\xi - \eta(Z)Y], \tag{2.16}$$

$$C(\xi,Y)\xi = \left[\alpha^2 - \rho - \frac{r}{n(n-1)}\right][\eta(Y)\xi + Y]. \tag{2.17}$$

3. Some basic properties of invariant submanifolds of $(LCS)_n$-manifolds

Let N be a submanifold of a $(LCS)_n$-manifold M with induced metric g. Also let ∇ and $\nabla^\perp$ be the induced connection on the tangent bundle TN and the normal bundle $T^\perp N$ of N respectively. Then the Gauss and Weingarten formulae are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{3.1}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{3.2}$$

for all $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^\perp N)$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of N into M. The second fundamental form h and the shape operator A_V are related by [42]

$$g(h(X,Y),V) = g(A_V X, Y) \tag{3.3}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^\perp N)$. We note that $h(X, Y)$ is bilinear and since $\nabla_{fX} Y = f\nabla_X Y$ for any smooth function f on a manifold, we have

$$h(fX,Y) = fh(X,Y). \tag{3.4}$$

Definition 3.1. [19] A submanifold N of a $(LCS)_n$-manifold M is said to be invariant if the structure vector field ξ is tangent to N at every point of N and ϕX is tangent to N for any vector field X tangent to N at every point of N, that is $\phi(TN) \subset TN$ at every point of N. The submanifold N of the $(LCS)_n$-manifold M is called totally geodesic if $h(X, Y) = 0$ for any $X, Y \in \Gamma(TN)$.

For the second fundamental form h, the covariant derivative of h is defined by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^\perp_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{3.5}$$

for any vector fields X, Y, Z tangent to N. Then $\overline{\nabla}h$ is a normal bundle valued tensor of type (0,3) and is called the third fundamental form of N, $\overline{\nabla}$ is called the Vander–Waerden–Bortolotti connection of M, i.e. $\overline{\nabla}$ is the connection in $TN \oplus T^\perp N$ built with ∇ and $\nabla^\perp$. If $\overline{\nabla}h = 0$, then N is said to have parallel second fundamental form [42]. Throughout the paper each object K produced by the connection $\overline{\nabla}$ (respectively $\widetilde{\nabla}$) will be denoted by $\overline{K}$ (respectively $\widetilde{K}$). From the Gauss and Weingarten formulae we obtain

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X, \tag{3.6}$$

where $\widetilde{R}(X, Y)Z$ denotes the tangential part of the curvature tensor of the submanifold.

From (1.1), we get

$$(\overline{R}(X,Y)\cdot h)(Z,U) = R^\perp(X,Y)h(Z,U) - h(R(X,Y)Z,U) - h(Z,R(X,Y)U) \tag{3.7}$$

for all vector fields X, Y, Z and U, where

$$R^\perp(X,Y) = [\nabla^\perp_X, \nabla^\perp_Y] - \nabla^\perp_{[X,Y]}$$

and $\overline{R}$ denotes the curvature tensor of $\overline{\nabla}$. In the similar manner we can write

$$(\overline{R}(X,Y)\cdot\overline{\nabla}h)(Z,U,W) = R^\perp(X,Y)(\overline{\nabla}h)(Z,U,W) - (\overline{\nabla}h)(R(X,Y)Z,U,W) - (\overline{\nabla}h)(Z,R(X,Y)U,W) - (\overline{\nabla}h)(Z,U,R(X,Y)W) \tag{3.8}$$

for all vector fields X, Y, Z, U and W tangent to N and $(\overline{\nabla}h)(Z, U, W) = (\overline{\nabla}_Z h)(U, W)$ [39]. Again for the concircular curvature tensor C we have [30]

$$(\overline{C}(X,Y)\cdot h)(Z,U) = R^\perp(X,Y)h(Z,U) - h(C(X,Y)Z,U) - h(Z,C(X,Y)U) \tag{3.9}$$

and

$$(\overline{C}(X,Y)\cdot\overline{\nabla}h)(Z,U,W) = R^\perp(X,Y)(\overline{\nabla}h)(Z,U,W) - (\overline{\nabla}h)(C(X,Y)Z,U,W) - (\overline{\nabla}h)(Z,C(X,Y)U,W) - (\overline{\nabla}h)(Z,U,C(X,Y)W). \tag{3.10}$$

In an invariant submanifold of a $(LCS)_n$-manifold, we have

$$h(X,\xi) = 0. \tag{3.11}$$

Now we have

Proposition 3.1. *Let N be an invariant submanifold of a $(LCS)_n$-manifold M. Then the following relations hold:*

$$\nabla_X \xi = \alpha\phi X, \tag{3.12}$$

$$R(X,Y)\xi = (\alpha^2-\rho)[\eta(Y)X-\eta(X)Y], \tag{3.13}$$

$$S(X,\xi) = (n-1)(\alpha^2-\rho)\eta(X), \text{ i.e., } Q\xi = (n-1)(\alpha^2-\rho)\xi, \tag{3.14}$$

$$(\nabla_X\phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{3.15}$$

$$h(X,\phi Y) = \phi h(X,Y). \tag{3.16}$$

Proof. Since N is an invariant submanifold of a $(LCS)_n$-manifold M, we have

$$\widetilde{\nabla}_X\xi = \alpha\phi X. \tag{3.17}$$

Using Gauss formula (3.1) and (3.17), we get

$$\alpha\phi X = \nabla_X\xi + h(X,\xi). \tag{3.18}$$

By virtue of (3.11) it follows from (3.18) that the relation (3.12) holds. Again since M is a $(LCS)_n$-manifold, we get from (2.13) that

$$(\widetilde{\nabla}_X\phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}. \tag{3.19}$$

From (3.1), we have

$$(\widetilde{\nabla}_X\phi)(Y) = (\nabla_X\phi)(Y) + h(X,\phi Y) - \phi h(X,Y). \tag{3.20}$$

Comparing the tangential and normal parts of (3.19) and (3.20), we get the relation (3.15) and (3.16). Again from (3.6), we have

$$\widetilde{R}(X,Y)\xi = R(X,Y)\xi + A_{h(X,\xi)}Y - A_{h(Y,\xi)}X. \tag{3.21}$$

Using (2.10) and (3.11) in (3.21), we get the relation (3.13) and consequently follows (3.14).

Thus we can state the following: □

Theorem 3.1. *An invariant submanifold N of a $(LCS)_n$-manifold M is a $(LCS)_n$-manifold.*

4. Semiparallel and 2-semiparallel invariant submanifolds of $(LCS)_n$-manifolds

This section deals with semiparallel and 2-semiparallel invariant submanifolds of $(LCS)_n$-manifolds.

Theorem 4.1. *Let N be an invariant submanifold of a $(LCS)_n$-manifold M with $\alpha^2-\rho\neq 0$. Then N is semiparallel if and only if N is totally geodesic.*

Proof. Since N is semiparallel, we have $\overline{R}\cdot h = 0$ and hence from (3.7) we get

$$R^{\perp}(X,Y)h(Z,U) - h(R(X,Y)Z,U) - h(Z,R(X,Y)U) = 0. \tag{4.1}$$

Putting $X = U = \xi$ in (4.1) we obtain

$$R^{\perp}(\xi,Y)h(Z,\xi) - h(R(\xi,Y)Z,\xi) - h(Z,R(\xi,Y)\xi) = 0. \tag{4.2}$$

By virtue of (3.11), (4.2) yields

$$h(Z,R(\xi,Y)\xi) = 0. \tag{4.3}$$

Using (2.10) and (3.11) in (4.3), we get

$$h(Z,Y) = 0,$$

which implies that N is totally geodesic.

The converse is trivial and consequently we get the desired theorem. □

Theorem 4.2. *Let N be an invariant submanifold of a $(LCS)_n$-manifold M. Then the second fundamental form of the submanifold N is parallel if and only if N is totally geodesic.*

Proof. Since N has parallel second fundamental form, it follows from (3.5) that

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z). \tag{4.4}$$

Putting $Z = \xi$ in (4.4) and using (3.11), we have

$$h(Y,\nabla_X\xi) = 0. \tag{4.5}$$

In view of (3.12) we have from (4.5) that

$$h(Y,\phi X) = 0. \tag{4.6}$$

Replacing X by ϕX in (4.6) and using (2.8) and (3.11) we get $h(Y,X) = 0$, which implies that N is totally geodesic.

The converse statement is obvious. This proves the theorem. □

Theorem 4.3. *Let N be an invariant submanifold of a $(LCS)_n$-manifold M with non-vanishing ξ-sectional curvature. Then N is 2-semiparallel if and only if N is totally geodesic.*

Proof. Let N be an invariant submanifold of a $(LCS)_n$-manifold M such that $\alpha^2-\rho\neq 0$, which is 2-semiparallel. Then from (3.8) we get

$$\begin{aligned} &R^{\perp}(X,Y)(\overline{\nabla}h)(Z,U,W) - (\overline{\nabla}h)(R(X,Y)Z,U,W) \\ &-(\overline{\nabla}h)(Z,R(X,Y)U,W) - (\overline{\nabla}h)(Z,U,R(X,Y)W) = 0. \end{aligned} \tag{4.7}$$

Plugging $X = U = \xi$ in (4.7), we obtain

$$\begin{aligned} &R^{\perp}(\xi,Y)(\overline{\nabla}h)(Z,\xi,W) - (\overline{\nabla}h)(R(\xi,Y)Z,\xi,W) \\ &-(\overline{\nabla}h)(Z,R(\xi,Y)\xi,W) - (\overline{\nabla}h)(Z,\xi,R(\xi,Y)W) = 0. \end{aligned} \tag{4.8}$$

By virtue of (2.5), (2.11), (2.12), (3.5) and (3.11), we have the following:

$$\begin{aligned} (\overline{\nabla}h)(Z,\xi,W) &= (\overline{\nabla}_Z h)(\xi,W) \\ &= \nabla^{\perp}_Z(h(\xi,W)) - h(\nabla_Z\xi,W) - h(\xi,\nabla_Z W) \\ &= -\alpha h(\phi Z,W), \end{aligned} \tag{4.9}$$

$$\begin{aligned} &(\overline{\nabla}h)(R(\xi,Y)Z,\xi,W) \\ &= (\overline{\nabla}_{R(\xi,Y)Z}h)(\xi,W) \\ &= \nabla^{\perp}_{R(\xi,Y)Z}(h(\xi,W)) - h(\nabla_{R(\xi,Y)Z}\xi,W) - h(\xi,\nabla_{R(\xi,Y)Z}W) \end{aligned}$$

$$= -h(\alpha\phi R(\xi, Y)Z, W)$$
$$= \alpha(\alpha^2 - \rho)\eta(Z)h(\phi Y, W), \tag{4.10}$$

$$\begin{aligned}(\overline{\nabla}h)(Z, R(\xi, Y)\xi, W) &= (\overline{\nabla}_Z h)(R(\xi, Y)\xi, W)\\ &= \nabla_Z^{\perp}(h(R(\xi, Y)\xi, W)) - h(\nabla_Z R(\xi, Y)\xi, W)\\ &\quad -h(R(\xi, Y)\xi, \nabla_Z W)\\ &= (\alpha^2 - \rho)\nabla_Z^{\perp}h(Y, W) - (\alpha^2 - \rho)h(\nabla_Z\{\eta(Y)\xi + Y\}, W)\\ &\quad -(\alpha^2 - \rho)h(Y, \nabla_Z W)\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}(\overline{\nabla}h)(Z, \xi, R(\xi, Y)W) &= (\overline{\nabla}_Z h)(\xi, R(\xi, Y)W)\\ &= \nabla_Z^{\perp}(h(\xi, R(\xi, Y)W)) - h(\nabla_Z\xi, R(\xi, Y)W)\\ &\quad -h(\xi, \nabla_Z R(\xi, Y)W)\\ &= -h(\alpha\phi Z, R(\xi, Y)W)\\ &= \alpha(\alpha^2 - \rho)\eta(W)h(\phi Z, Y).\end{aligned} \tag{4.12}$$

Using (2.5), (4.9)–(4.12) in (4.8), we obtain

$$\begin{aligned}&-\alpha R^{\perp}(\xi, Y)h(\phi Z, W) - (\alpha^2 - \rho)\eta(Z)h(\phi Y, W)\\ &\quad -(\alpha^2 - \rho)\nabla_Z^{\perp}h(Y, W) + (\alpha^2 - \rho)h(\nabla_Z\{\eta(Y)\xi + Y\}, W)\\ &\quad +(\alpha^2 - \rho)h(Y, \nabla_Z W) - \alpha(\alpha^2 - \rho)\eta(W)h(\phi Z, Y) = 0.\end{aligned} \tag{4.13}$$

Putting $W = \xi$ in (4.13) and using (3.11) and (3.12), we get

$$\alpha(\alpha^2 - \rho)h(\phi Z, Y) = 0. \tag{4.14}$$

The ξ-sectional curvature of a $(LCS)_n$-manifold for a unit vector field X orthogonal to ξ is given by $K(\xi, X) = g(R(\xi, X)\xi, X)$. Hence from (2.12), we get

$$K(\xi, X) = (\alpha^2 - \rho). \tag{4.15}$$

Since the manifold under consideration is of non-vanishing ξ-sectional curvature, we have from (4.15) that $\alpha^2 - \rho \neq 0$. Again since $\alpha \neq 0$, (4.14) yields

$$h(\phi Z, Y) = 0. \tag{4.16}$$

Replacing Z by ϕZ in (4.16) and using (2.8) and (3.11), we get $h(Z, Y) = 0$, which implies that N is totally geodesic.

The converse part is trivial. So the theorem is proved. □

5. Invariant submanifolds of $(LCS)_n$-manifolds satisfying $\overline{C}(X, Y) \cdot h = 0$ and $\overline{C}(X, Y) \cdot \overline{\nabla}h = 0$

This section deals with invariant submanifolds of $(LCS)_n$-manifolds satisfying $\overline{C}(X, Y) \cdot h = 0$ and $\overline{C}(X, Y) \cdot \overline{\nabla}h = 0$.

Theorem 5.1. *Let N be an invariant submanifold of a $(LCS)_n$-manifold M such that $r \neq n(n-1)(\alpha^2 - \rho)$. Then $\overline{C}(X, Y) \cdot h = 0$ holds on N if and only if N is totally geodesic.*

Proof. Let N be an invariant submanifold of a $(LCS)_n$-manifold M satisfying $\overline{C}(X, Y) \cdot h = 0$ such that $r \neq n(n-1)(\alpha^2 - \rho)$. Then we have from (3.9) that

$$R^{\perp}(X, Y)h(Z, U) - h(C(X, Y)Z, U) - h(Z, C(X, Y)U) = 0. \tag{5.1}$$

Setting $X = U = \xi$ in (5.1) and using (2.16) and (3.11), we get

$$h(Z, C(\xi, Y)\xi) = 0. \tag{5.2}$$

By virtue of (2.17) it follows from (5.2) that

$$\left[\alpha^2 - \rho - \frac{r}{n(n-1)}\right]h(Z, \eta(Y)\xi + Y) = 0. \tag{5.3}$$

Again in view of (3.11), (5.3) yields

$$\left[\alpha^2 - \rho - \frac{r}{n(n-1)}\right]h(Z, Y) = 0,$$

which gives

$$h(Z, Y) = 0, \quad \text{since } r \neq n(n-1)(\alpha^2 - \rho) \tag{5.4}$$

and hence the submanifold N is totally geodesic. The converse is trivial and hence the theorem. □

Theorem 5.2. *Let N be an invariant submanifold of a $(LCS)_n$-manifold M such that $r \neq n(n-1)(\alpha^2 - \rho)$. Then $\overline{C}(X, Y) \cdot \overline{\nabla}h = 0$ holds on N if and only if N is totally geodesic.*

Proof. Let N be an invariant submanifold of a $(LCS)_n$-manifold M such that $r \neq n(n-1)(\alpha^2 - \rho)$. If N satisfies the condition $\overline{C}(X, Y) \cdot \overline{\nabla}h = 0$, then from (3.10), we get

$$\begin{aligned}&R^{\perp}(X, Y)(\overline{\nabla}h)(Z, U, W) - (\overline{\nabla}h)(C(X, Y)Z, U, W)\\ &\quad -(\overline{\nabla}h)(Z, C(X, Y)U, W) - (\overline{\nabla}h)(Z, U, C(X, Y)W) = 0.\end{aligned} \tag{5.5}$$

Putting $X = U = \xi$ in (5.5), we obtain

$$\begin{aligned}&R^{\perp}(\xi, Y)(\overline{\nabla}h)(Z, \xi, W) - (\overline{\nabla}h)(C(\xi, Y)Z, \xi, W)\\ &\quad -(\overline{\nabla}h)(Z, C(\xi, Y)\xi, W) - (\overline{\nabla}h)(Z, \xi, C(\xi, Y)W) = 0.\end{aligned} \tag{5.6}$$

By virtue of (2.16), (2.17), (3.5) and (3.11), we get

$$\begin{aligned}(\overline{\nabla}h)(C(\xi, Y)Z, \xi, W) &= (\overline{\nabla}_{C(\xi,Y)Z}h)(\xi, W)\\ &= \nabla_{C(\xi,Y)Z}^{\perp}(h(\xi, W)) - h(\nabla_{C(\xi,Y)Z}\xi, W) - h(\xi, \nabla_{C(\xi,Y)Z}W)\\ &= -h(\alpha\,\phi C(\xi, Y)Z, W)\\ &= \alpha\left[\alpha^2 - \rho - \frac{r}{n(n-1)}\right]\eta(Z)h(\phi Y, W),\end{aligned} \tag{5.7}$$

$$\begin{aligned}(\overline{\nabla}h)(Z, C(\xi, Y)\xi, W) &= (\overline{\nabla}_Z h)(C(\xi, Y)\xi, W)\\ &= \nabla_Z^{\perp}(h(C(\xi, Y)\xi, W)) - h(\nabla_Z C(\xi, Y)\xi, W)\\ &\quad -h(C(\xi, Y)\xi, \nabla_Z W)\\ &= \left[\alpha^2 - \rho - \frac{r}{n(n-1)}\right]\\ &\quad \times\left[\nabla_Z^{\perp}h(Y, W) - h(\nabla_Z\{\eta(Y)\xi + Y\}, W) - h(Y, \nabla_Z W)\right]\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}(\overline{\nabla}h)(Z, \xi, C(\xi, Y)W) &= (\overline{\nabla}_Z h)(\xi, C(\xi, Y)W)\\ &= \nabla_Z^{\perp}(h(\xi, C(\xi, Y)W)) - h(\nabla_Z\xi, C(\xi, Y)W)\\ &\quad -h(\xi, \nabla_Z C(\xi, Y)W)\end{aligned}$$

$$= -h(\alpha\,\phi Z, C(\xi, Y)W)$$
$$= \alpha\left[\alpha^2 - \rho - \frac{r}{n(n-1)}\right]\eta(W)h(\phi Z, Y). \tag{5.9}$$

In view of (2.5), (4.9) and (5.7)–(5.9) we have from (5.6) that

$$-\alpha R^{\perp}(\xi, Y)h(\phi Z, W) - \left[\alpha^2 - \rho - \frac{r}{n(n-1)}\right]$$
$$\times\left[\alpha\eta(Z)h(\phi Y, W) + \nabla_Z^{\perp}h(Y, W) - h(\nabla_Z\{\eta(Y)\xi + Y\}, W)\right.$$
$$\left. - h(Y, \nabla_Z W) + \eta(W)h(\phi Z, Y)\right] = 0. \tag{5.10}$$

Putting $W = \xi$ in (5.10) and using (3.11) and (3.12), we get

$$\alpha\left[\alpha^2 - \rho - \frac{r}{n(n-1)}\right]h(Y, \phi Z) = 0,$$

which implies that

$$h(Y, \phi Z) = 0, \quad \text{since } \alpha \neq 0 \text{ and } r \neq n(n-1)(\alpha^2 - \rho). \tag{5.11}$$

Replacing Z by ϕZ in (5.11) and using (2.8) and (3.11), we get $h(Y, Z) = 0$, which implies that N is totally geodesic. The converse statement is obvious and hence the proof of the theorem is complete. □

By virtue of Theorems 4.1– 4.3, 5.1 and 5.2, we can state the following:

Theorem 5.3. *Let N be an invariant submanifold of a $(LCS)_n$-manifold M. Then the following statements are equivalent:*

(i) *N is semiparallel;*
(ii) *N has parallel second fundamental form;*
(iii) *N is 2-semiparallel;*
(iv) *N satisfies the condition $\overline{C}(X, Y)\cdot h = 0$ with $r \neq n(n-1)(\alpha^2 - \rho)$;*
(v) *N satisfies the condition $\overline{C}(X, Y)\cdot \overline{\nabla} h = 0$ with $r \neq n(n-1)(\alpha^2 - \rho)$;*
(vi) *N is totally geodesic.*

6. 3-dimensional invariant submanifolds of $(LCS)_n$-manifolds

Proposition 6.1. *Let N be an invariant submanifold of a $(LCS)_n$-manifold M. Then there exist two differentiable orthogonal distributions D and $D^{\perp}$ on N such that*

$$TN = D \oplus D^{\perp} \oplus \{\xi\}$$

and

$$\phi(D) \subset D^{\perp}, \quad \phi(D^{\perp}) \subset D.$$

Proof. For an invariant submanifold N, ξ is tangent to N. Hence we can write $TN = D^1 \oplus \{\xi\}$. Since $g(X_1, \phi X_1) = 0$ and $g(\xi, \phi X_1) = 0$ for $X_1 \in D^1$. So ϕX_1 is orthogonal to X_1 and ξ. Consequently, we can write $D^1 = D \oplus D^{\perp}$, where $X_1 \in D \subset D^1$ and $\phi X_1 \in D^{\perp} \subset D^1$. For $\phi X_1 \in D^{\perp}$, we have $\phi(\phi X_1) = \phi^2 X_1 = X_1 + \eta(X_1)\xi = X_1 \in D$. Let $\phi X_1 = X_2 \in D^{\perp}$. Hence for $X_1 \in D$, $\phi X_1 \in D^{\perp}$ and for $X_2 \in D^{\perp}$, $\phi X_2 \in D$. This proves the proposition. □

Proposition 6.2. *For an invariant submanifold N of a $(LCS)_n$-manifold M, we have*

$$h(X, \xi) = 0, \tag{6.1}$$

$$h(X, \phi Y) = \phi h(X, Y) = h(\phi X, Y) \tag{6.2}$$

for two differentiable vector fields $X, Y \in \Gamma(TN)$.

Proof. From (3.4) and (3.11) it can be easily checked that the relations (6.1) and (6.2) hold.

Now we prove the following: □

Theorem 6.1. *Every 3-dimensional invariant submanifold of a $(LCS)_n$-manifold is totally geodesic.*

Proof. Let N be a 3-dimensional invariant submanifold of a $(LCS)_n$-manifold M. Then for $X_1, Y_1 \in D$, we have

$$h(X_1, \phi Y_1) = \phi h(X_1, Y_1). \tag{6.3}$$

By virtue of (2.8) it follows from (6.3) that

$$\phi h(X_1, \phi Y_1) = \phi^2 h(X_1, Y_1) = h(X_1, Y_1) + \eta(h(X_1, Y_1))\xi. \tag{6.4}$$

Since $h(X_1, Y_1)$ is a vector field normal to N. So $h(X_1, Y_1)$ and ξ are orthogonal. Consequently we get by virtue of (2.2) that $\eta(h(X_1, Y_1)) = 0$. Thus in view of (6.2) and (6.4) we have

$$h(\phi X_1, \phi Y_1) = h(X_1, Y_1). \tag{6.5}$$

Let us take $\phi X_1 = X_2 \in D^{\perp}$ and $\phi Y_1 = Y_2 \in D^{\perp}$. Then from (6.5), we get

$$h(X_2, Y_2) = h(X_1, Y_1). \tag{6.6}$$

As $h(X, Y)$ is bilinear, for $X_1, Y_1 \in D$ and $X_2, Y_2 \in D^{\perp}$ we obtain

$$h(X_1 + X_2 + \xi, Y_1) = h(X_1, Y_1) + h(X_2, Y_1) + h(\xi, Y_1), \tag{6.7}$$

$$h(X_1 + X_2 + \xi, -Y_2) = -h(X_1, Y_2) - h(X_2, Y_2) - h(\xi, Y_2), \tag{6.8}$$

$$h(X_1 + X_2 + \xi, \xi) = h(X_1, \xi) + h(X_2, \xi) + h(\xi, \xi). \tag{6.9}$$

Adding (6.7)–(6.9) and using (6.1) and (6.6) we get

$$h(X_1 + X_2 + \xi, Y_1 - Y_2 + \xi) = h(X_2, Y_1) - h(X_1, Y_2). \tag{6.10}$$

As $TN = D \oplus D^{\perp} \oplus \{\xi\}$, we can write $U = X_1 + X_2 + \xi \in TN$ and $W = Y_1 - Y_2 + \xi \in TN$. Hence from (6.10) we have

$$h(U, W) = h(X_2, Y_1) - h(X_1, Y_2). \tag{6.11}$$

From (6.11) it follows that

$$\phi h(U, W) = h(X_2, \phi Y_1) - h(\phi X_1, Y_2)$$
$$= h(X_2, Y_2) - h(X_2, Y_2)$$
$$= 0.$$

This proves the theorem. □

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