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Original Article

Classification of conics and Cassini curves in Minkowski space-time plane

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Abstract In this paper we use the Apollonius definition of conics to generate algebraic curves in the Minkowski space-time plane M^2, which turn out to be different from classical conic sections. We extend and classify this sort of "M-conics". We discuss the cases of the singularity points of these M-conics, coming from the transition from timelike world to spacelike world through the lightlike one. Finally, we translate the classical concept of Cassini curves with two foci and that of (multifocal) Cassini curves to Minkowski planes M^2.

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1. Introduction

A Minkowski space-time plane M^2 is pseudo-Euclidean plane, i.e., there are three types of directions, the spacelike, timelike and lightlike directions, and the unit ball in such a plane consists of two conjugate hyperbolas with lightlike asymptotes [1,2], see Fig. 1. Many authors discuss this space from the relativity point of view with some mathematical concepts, e.g., Naber [3–5].

In the following we use the fundamental Apollonian definitions of quadratic conics in Euclidean plane to define "M-conics" in the Minkowski space-time plane M^2.

The elementary geometric Apollonius definition of an ellipse in the Euclidean plane reads as follows:

An ellipse is the set of points P having constant distance sum from two fixed points F_1, F_2, the so-called focal points of the ellipse.

Similar and well-known definitions exist for hyperbolas and parabolas. While the projective geometric point of view distinguishes these "conic sections" (or shortly "conics") by their ideal points, a proper elementary geometric approach has to omit this way to classify conics.

Cassini's modification [6–8], in 1680, of the Apollonius definition replaces the constant sum of distances to two fixed focal points by the constant product of these distances. Cassini believed that the motion of the planets of the solar system revolves in one of these curves. There are many applications of Cassini

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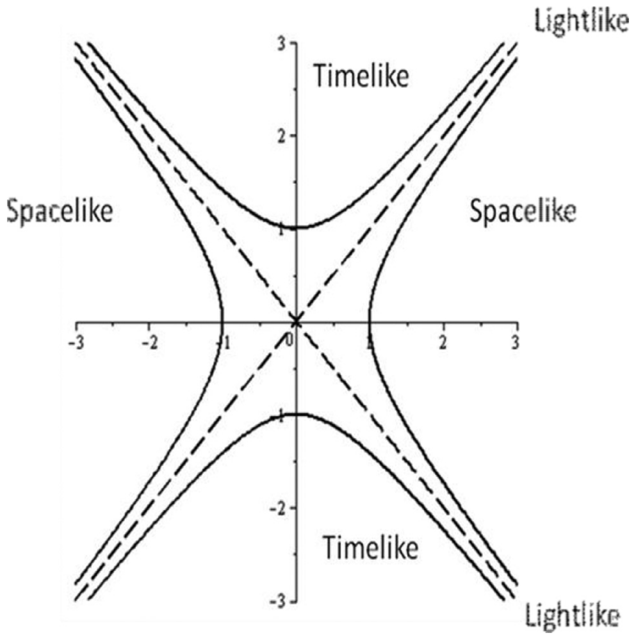

Fig. 1 Unit circle S_M of the Minkowski plane M^2.

curves in BioGeometry, e.g. sectors of onions layers, bacterial colonies and cell shapes. Furthermore, simulation of light scattering [9], by small concave particles is necessary to find an appropriate mathematical description of the particle shape. This can be done easily by the usage of Cassini curves. For example, this approach is used to fit the shape of the human red blood cell [10,11]. For more applications of Cassini curves, see [12–14].

In this paper we aim at changing the place of action from the classical Euclidean plane to the so-called Minkowski space-time plane or, physically spoken, to the two-dimensional space-time world. As expected the topology of the "M-conics" and "M-Cassini curves" depends on the position and distance of the focal points and of the chosen constant distance sum resp. product. Special cases occur, if these foci lie on a lightlike line.

2. Minkowski norm

The Lorentz transformations are designed to preserve the lightlike lines, which is "M-circle true", i.e. they are translations together with pseudo-reflections and pseudo-rotations, which is the set $x^2 - y^2 = 0$. In fact, it preserves each of the hyperbolas $x^2 - y^2 = k$, for all k, i.e. the common asymptotes of these hyperbolas are the lightlike directions.

Definition 1. The Minkowski space-time plane M^2 is a real vector space with usual Minkowski inner product $\langle\ ,\ \rangle_M$ given by

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_M := x_1y_1 - x_2y_2, \tag{1}$$

where $\boldsymbol{x}, \boldsymbol{y} \in M^2$, $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$.

The norm $\|\cdot\|$ is defined by the previous inner product as

$$\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M|}. \tag{2}$$

Any arbitrary vector $\boldsymbol{x} \in M^2$ is classified according to the sign of $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M$ as follows:

i $\boldsymbol{x}$ is timelike if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M < 0$,
ii $\boldsymbol{x}$ is spacelike if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M > 0$,
iii $\boldsymbol{x}$ is lightlike if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M = 0$.

It is clear to define the "unit circle" as a pair of Euclidean hyperbolas $x^2 - y^2 = \pm 1$ as follows:

$$S_M := \{\boldsymbol{x} \in M^2 : \|\boldsymbol{x}\| = 1\}, \tag{3}$$

and the unit ball is

$$B_M := \{\boldsymbol{x} \in M^2 : \|\boldsymbol{x}\| \le 1\}. \tag{4}$$

The unit circle S_M has four sheets. Two of them come from the equation $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M = 1$ which refers to the spacelike directions. The others from $\langle \boldsymbol{x}, \boldsymbol{x}\rangle_M = -1$, which refers to the timelike directions. Any lightlike vector is parallel to the asymptotes $\boldsymbol{y} = \boldsymbol{x}$ and $\boldsymbol{y} = -\boldsymbol{x}$ of the unit circle S_M, see Fig. 1. The pair of asymptotes of S_M forms the so-called light cone of M^2.

3. Conics in Minkowski space-time planes M^2

3.1. M-ellipse in M^2

We discuss the conics in Minkowski plane M^2 by using the usual definition like in the Euclidean plane. Using the metric definition of M^2, an M-ellipse obtains by distance sum from its foci $\boldsymbol{z}$ and $\boldsymbol{w}$ to the locus point $\boldsymbol{x}$ on it:

$$\|\boldsymbol{x} - \boldsymbol{z}\| + \|\boldsymbol{x} - \boldsymbol{w}\| = 2a,\ a > 0 \tag{5}$$

where $\boldsymbol{x} = (x, y)$ is a position point with given two foci points $\boldsymbol{z} = (z_1, z_2)$ and $\boldsymbol{w} = (w_1, w_2)$.

3.2. Classification of M-ellipse in M^2

Now, we classify all cases of the M-ellipse (5). First of all, we can rewrite (5) as follows:

$$\sqrt{\left|(x - z_1)^2 - (y - z_2)^2\right|} + \sqrt{\left|(x - w_1)^2 - (y - w_2)^2\right|} = 2a. \tag{6}$$

Generally, M-ellipses (6) have at most eight singular points. Furthermore, they are symmetric with respect to the center of their focal segments. Denote the distance between the two foci $\boldsymbol{z} = (z_1, z_2)$ and $\boldsymbol{w} = (w_1, w_2)$ by $2d$. Then we have

$$d = \frac{1}{2}\sqrt{\left|(z_1 - w_1)^2 - (z_2 - w_2)^2\right|}. \tag{7}$$

To analyze the singularity of (6), we need to find the limit points of (6) with the four lines $y = x + z_2 - z_1$, $y = -x + z_2 + z_1$, $y = x + w_2 - w_1$ and $y = -x + w_2 + w_1$.

Case I:

In the region of $|x - z_1| \ge |y - z_2|$ and $|x - w_1| \ge |y - w_2|$ we have two symmetric parts with $\sqrt{(x - z_1)^2 - (y - z_2)^2} + \sqrt{(x - w_1)^2 - (y - w_2)^2} = 2a$. We have the following singular points on the four lines $y = x + z_2 - z_1$, $y = -x + z_2 + z_1$, $y = x + w_2 - w_1$ and $y = -x + w_2 + w_1$.

1- The limit point $P_1 = (x_1, y_1)$ at the line $y = x + z_2 - z_1$; therefore we have

$$x_1 = \frac{4a^2 + w_2^2 - (z_1 - z_2 - w_1)^2}{2(z_1 - z_2 - w_1 + w_2)} + z_1 - z_2, \tag{8}$$

$$y_1 = \frac{4a^2 + w_2^2 - (z_1 - z_2 - w_1)^2}{2(z_1 - z_2 - w_1 + w_2)}. \tag{9}$$

2- The limit point $P_2 = (x_2, y_2)$ at the line $y = -x + z_2 + z_1$; therefore we have

$$x_2 = \frac{4a^2 + w_2^2 - (z_2 + z_1 - w_1)^2}{2(z_2 + z_1 - w_2 - w_1)} + z_2 + z_1, \tag{10}$$

$$y_2 = \frac{4a^2 + w_2^2 - (z_2 + z_1 - w_1)^2}{2(w_2 + w_1 - z_2 - z_1)}. \tag{11}$$

3- The limit point $P_3 = (x_3, y_3)$ at the line $y = x + w_2 - w_1$; similarly, in (8) and (9) we have

$$x_3 = \frac{4a^2 + z_2^2 - (w_1 - w_2 - z_1)^2}{2(w_1 - w_2 - z_1 + z_2)} + w_1 - w_2, \tag{12}$$

$$y_3 = \frac{4a^2 + z_2^2 - (w_1 - w_2 - z_1)^2}{2(w_1 - w_2 - z_1 + z_2)}. \tag{13}$$

4- The limit point $P_4 = (x_4, y_4)$ at the line

1. $y = -x + w_2 + w_1$, like in (10) and (11) we have

$$x_4 = \frac{4a^2 + z_2^2 - (w_1 + w_2 - z_1)^2}{2(w_1 + w_2 - z_1 - z_2)} + w_2 + w_1, \tag{14}$$

$$y_4 = \frac{4a^2 + z_2^2 - (w_1 + w_2 - z_1)^2}{2(z_1 + z_2 - w_1 - w_2)}. \tag{15}$$

Case II:

In the region of $|x - z_1| \leq |y - z_2|$ and $|x - w_1| \leq |y - w_2|$ we have two symmetric parts with $\sqrt{(y - z_2)^2 - (x - z_1)^2} + \sqrt{(y - w_2)^2 - (x - w_1)^2} = 2a$. We have the following singular points intersected with the same four lines as in case I.

1- The limit point $P_5 = (x_5, y_5)$ at the line $y = x + z_2 - z_1$; therefore we have

$$x_5 = \frac{-4a^2 + w_2^2 - (z_1 - z_2 - w_1)^2}{2(z_1 - z_2 - w_1 + w_2)} + z_1 - z_2, \tag{16}$$

$$y_5 = \frac{-4a^2 + w_2^2 - (z_1 - z_2 - w_1)^2}{2(z_1 - z_2 - w_1 + w_2)}. \tag{17}$$

2- The limit point $P_6 = (x_6, y_6)$ at the line $y = -x + z_2 + z_1$; therefore we have

$$x_6 = \frac{-4a^2 + w_2^2 - (z_2 + z_1 - w_1)^2}{2(z_2 + z_1 - w_2 - w_1)} + z_2 + z_1, \tag{18}$$

$$y_6 = \frac{-4a^2 + w_2^2 - (z_2 + z_1 - w_1)^2}{2(w_2 + w_1 - z_2 - z_1)}. \tag{19}$$

3- The limit point $P_7 = (x_7, y_7)$ at the line $y = x + w_2 - w_1$; in the same way, as in (16) and (17), we have

$$x_7 = \frac{-4a^2 + z_2^2 - (w_1 - w_2 - z_1)^2}{2(w_1 - w_2 - z_1 + z_2)} + w_1 - w_2, \tag{20}$$

$$y_7 = \frac{-4a^2 + z_2^2 - (w_1 - w_2 - z_1)^2}{2(w_1 - w_2 - z_1 + z_2)}. \tag{21}$$

4- The limit point $P_8 = (x_8, y_8)$ at the line $y = -x + w_1 + w_2$; in the same way, as in (18) and (19), we have

$$x_8 = \frac{-4a^2 + z_2^2 - (w_1 + w_2 - z_1)^2}{2(w_1 + w_2 - z_1 - z_2)} + w_2 + w_1, \tag{22}$$

$$y_8 = \frac{-4a^2 + z_2^2 - (w_1 + w_2 - z_1)^2}{2(z_1 + z_2 - w_1 - w_2)}. \tag{23}$$

In the same manner, we have other two cases: $|x - z_1| \geq |y - z_2|$ and $|x - w_1| \leq |y - w_2|$, the other case is in the region of $|x - z_1| \leq |y - z_2|$ and $|x - w_1| \geq |y - w_2|$. These cases have four branches with the same critical points as we previously derived.

However, we have eight critical points, and this number of points may be less depending on the factor and the position of the two foci $\boldsymbol{z}$ and $\boldsymbol{w}$. Then we have the following classification:

i. If $a = d$, we have only six singular points on the M-ellipse. Two of them are the foci. In Fig. 2a, we see two foci lying in timelike directions; therefore, the others should lie in the spacelike directions and the converse is true. From Eqs. (16)–(19) we get $P_5 = P_6 = \boldsymbol{z}$. Furthermore, from (20)–(23) we get $P_7 = P_8 = \boldsymbol{w}$. In other words, the two foci lie on the M-ellipse as singular points.

ii. If $a > d$ and the foci are not on the same lightlike line, then the two foci are interior points of the M-ellipse. Furthermore, we have eight singular points on the M-ellipse. Four of them lie in the timelike directions, and the others lie in the spacelike directions. See Fig. 2b.

iii. If $a < d$ with the absence of two foci lying together on the same lightlike line, the two foci should be exterior points of the M-ellipse. Furthermore, we have eight singular points on the M-ellipse with two separated parts. Each of them contains four singular points. The existence of points in timelike or spacelike directions depends on the locus of the foci. See Fig. 2c.

iv. If the foci are on the same lightlike line, we have six singular points: two of them are ideal points (they lie in the projective line), i.e. points at infinity. Also, the two foci are inside the M-ellipse. See Fig. 2d.

3.3. M-hyperbola in M^2

Similarly, like an M-ellipse, we can define an M-hyperbola in M^2 by using its foci $\boldsymbol{z} = (z_1, z_2)$ and $\boldsymbol{w} = (w_1, w_2)$. Using the elementary definition we get

$$|\|\boldsymbol{x} - \boldsymbol{z}\| - \|\boldsymbol{x} - \boldsymbol{w}\|| = 2a, \ a > 0. \tag{24}$$

In the Euclidean case the geometric properties of a hyperbola do not depend on the parameter a. However, in Minkowski space-time plane M^2 the topological properties of an M-hyperbola (24) depend on the parameter a and the position of its foci.

The classification of an M-hyperbola (24) looks like that of an M-ellipse (5). In the same way, we rewrite (24) as follows:

$$\left| \sqrt{|(x - z_1)^2 - (y - z_2)^2|} - \sqrt{|(x - w_1)^2 - (y - w_2)^2|} \right| = 2a. \tag{25}$$

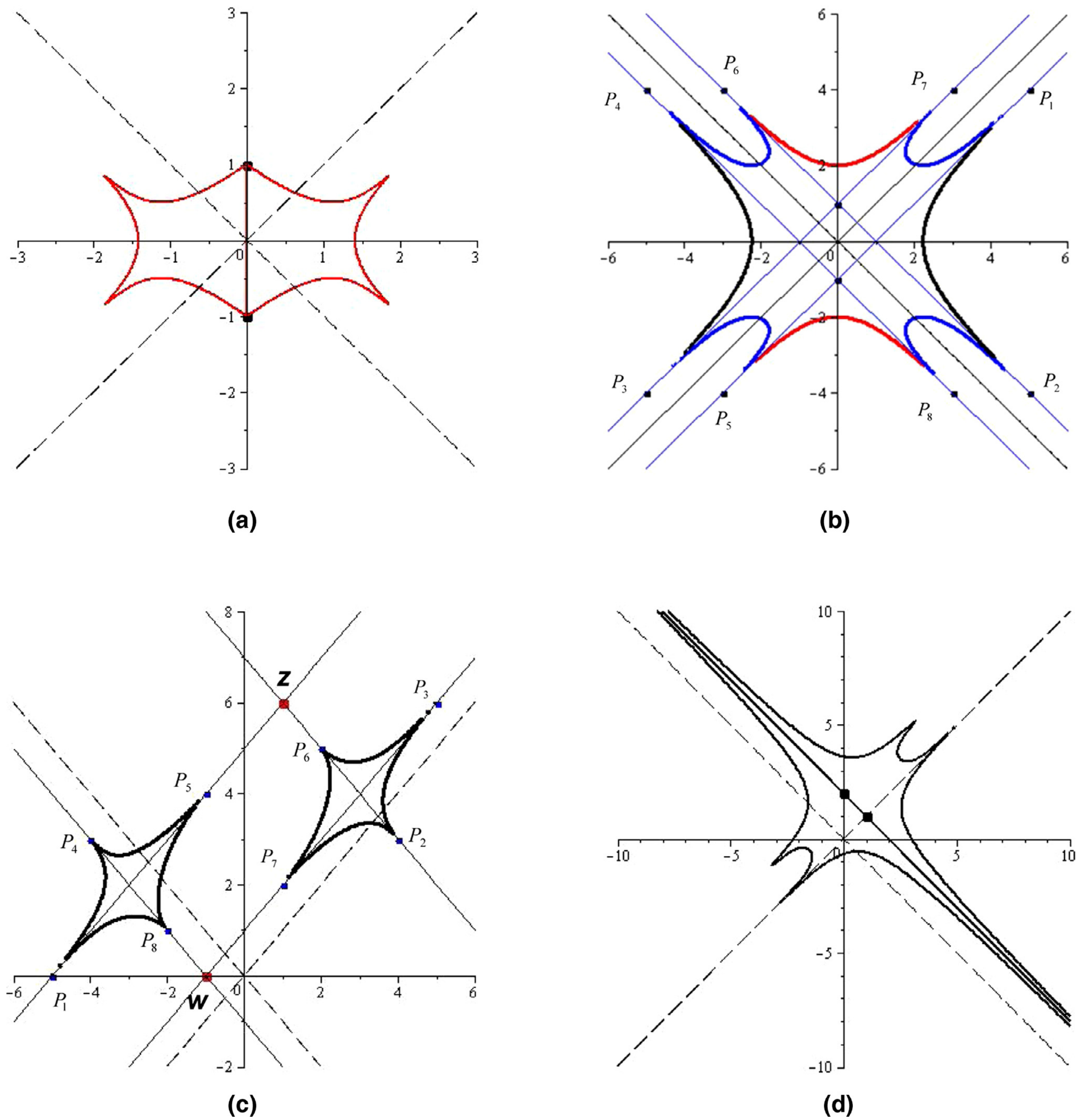

Fig. 2 (a) M-ellipse in M^2 with $a = d = 1$, where the two foci (0,1), (0,–1) lie on the M-ellipse. (b) M-ellipse in M^2 with $a = 2$ and the two foci (0,1), (0,–1) inside the M-ellipse. (c) M-ellipse in M^2 with $a = 2$ and the two foci (1,6), (–1,0) outside the M-ellipse (Two separated Astroids). (d) M-ellipse in M^2 with$a = 2$ and the two foci (1,1), (0,2) with foci line parallel to the Light-like line.

Therefore, we have the following:

i. If $a = d$, we have six singular points with two singular foci. See Fig. 3a.
ii. If $a > d$, we have eight singular points distributed evenly over the spacelike and timelike directions. See Fig. 3b.
iii. If $a < d$, also we have eight singular points evenly distributed over the spacelike and timelike directions. Furthermore, we get two combined sheets in both spacelike and timelike directions. See Fig. 3c.
iv. If the straight line connecting the two focal points is the lightlike line, then M-hyperbolas consist of only four branches with four singular points. See Fig. 3d, while all former cases consist of eight branches.

3.4. M-parabola in M^2

The elementary geometric Apollonius definition of a parabola in the Euclidean plane reads as follows:

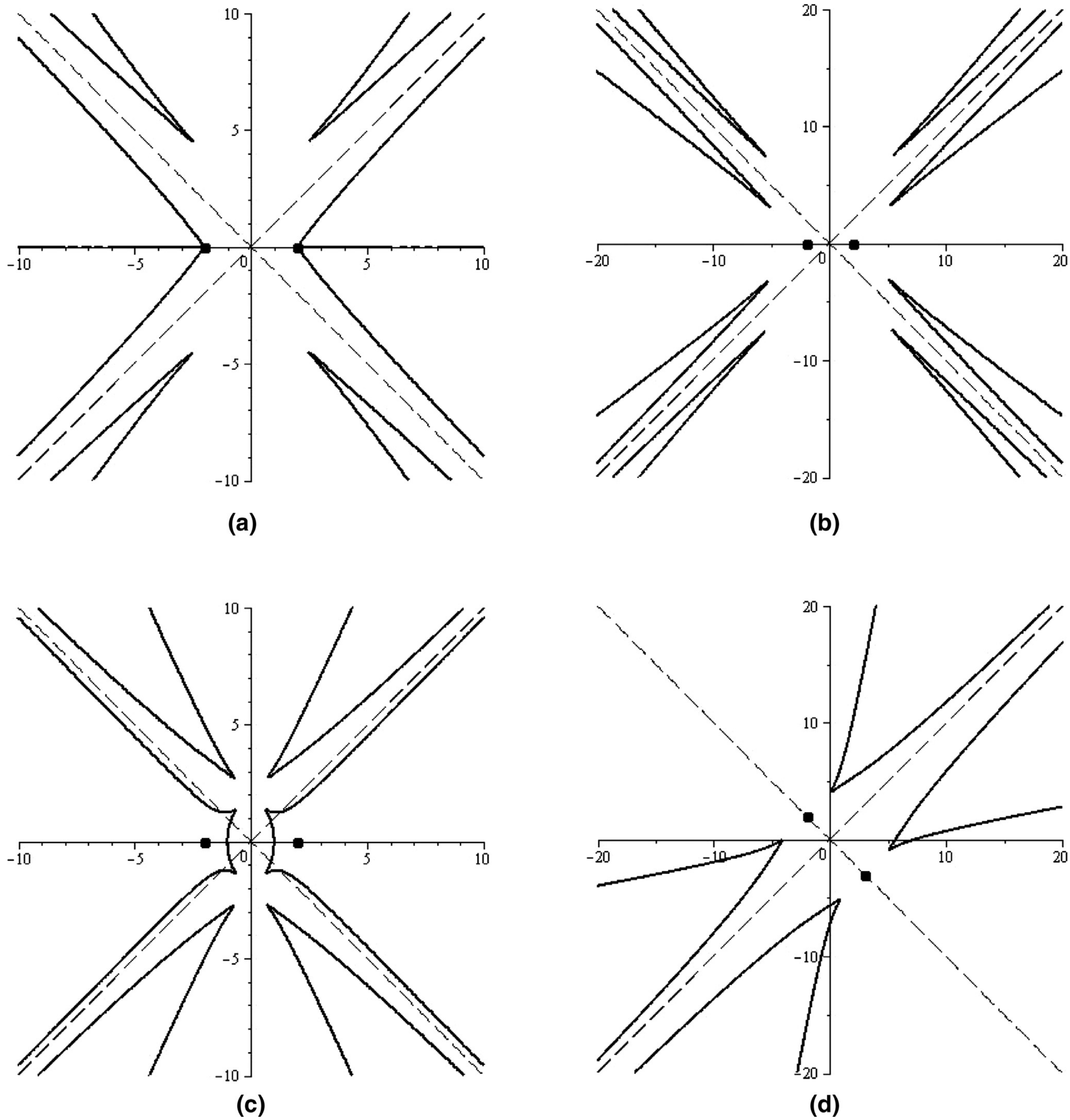

Fig. 3 (a) M-hyperbola in M^2 with foci (2,0), (–2,0), a = 2. (b) M-hyperbola in M^2 with foci (2,0), (–2,0), a = 3. (c) M-hyperbola in M^2 with foci (2,0), (–2,0), a = 1. (d) M-hyperbola in M^2 with a = 3 and foci (–2,2), (3,–3) and the line connecting the foci is the Light-like line.

A parabola is the locus of points P having constant segment distance from a fixed point F, the so-called "focus" of the parabola, and a straight line, called the "directrix" L.

Definition 2. In Minkowski space-time plane M^2 a segment or straight line is said to be of the *first (second) kind* if it is parallel to a line through the origin located in the sectors containing the spacelike (timelike) equilateral hyperbola, i.e. the slope of the line m satisfies $|m| < 1$ ($|m| > 1$). The first (second) kind is called *spacelike (timelike) lines*.

Therefore the straight line equation of the first (second) kind passing through a point (x_o, y_o) can be represented, respectively, as follows:

$$\frac{y - y_o}{x - x_o} = m,\ |m| < 1 \tag{26}$$

$$\frac{y - y_o}{x - x_o} = m,\ |m| > 1. \tag{27}$$

The previous two equations reflect the topological characterizations of the M^2 plane.

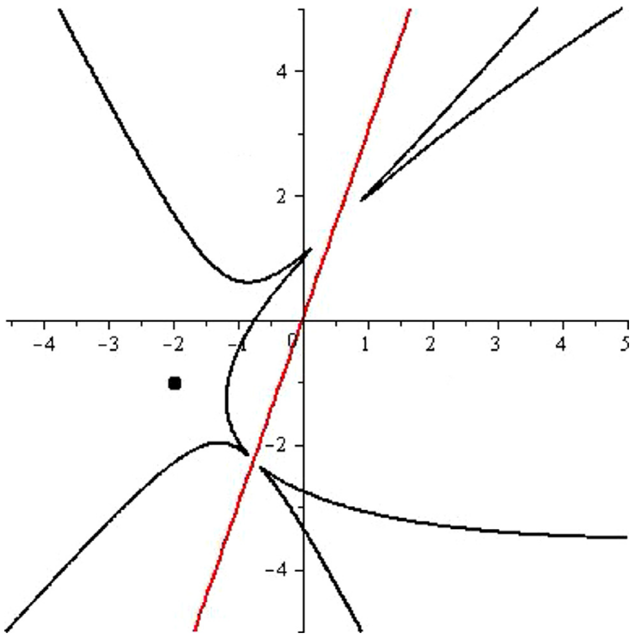

Fig. 4 M-parabola in M^2 with focal point (–2,–1) and line $y = 3x$.

Theorem 3. The M-orthogonal of the line of the first (second) kind in M^2 is of the second (first) kind. Furthermore, the product of their slopes, is equal to one.

Theorem 4. The distance from a point $P(x_1, y_1)$ to a straight line L with equation $y = mx + q$, $m \neq \pm 1$ is equal to $\frac{|y_1 - m x_1 - q|}{\sqrt{|m^2-1|}}$ which is independent on the kind of the line L. This distance corresponds to a maximum as is well known from special relativity, see [1,3].

Definition 5. Similarly as in elementary geometry, we can define the M-parabola in M^2 using a fixed focal point $\boldsymbol{z} = (z_1, z_2)$ and a straight line $L: \ y = mx + q$ as follows:

$$\left|(x - z_1)^2 - (y - z_2)^2\right| = \frac{|y - mx - q|^2}{|m^2 - 1|}. \tag{28}$$

Of course (28) fails if the straight line is lightlike, because all M-orthogonal lines to such a line are again lightlike of the same direction. An M-parabola is well defined in M^2, if the directrix is of first or second kind only, see Fig. 4. Also, (28) fails if we choose the line L to be the y-axis because the slope is undefined and $m \to \infty$. In spite of the right hand side of (28), we can easily get the squared distance of the position point (x, y) to be x^2. Then (28) becomes $|(x - z_1)^2 - (y - z_2)^2| = x^2$, see [15].

4. M-Cassini curves in M^2

As mentioned in Section 1, we define M-Cassini curves in Minkowski space-time plane M^2 as the set of points having constant distance product from two fixed focal points $\boldsymbol{z} = (z_1, z_2)$ and $\boldsymbol{w} = (w_1, w_2)$. Let b^2 be the value of this constant distance product, then the equation of the M-Cassini curves can be written as follows:

$$\left|(x - z_1)^2 - (y - z_2)^2\right| \cdot \left|(x - w_1)^2 - (y - w_2)^2\right| = b^4. \tag{29}$$

Geometrically, the shapes of the curves depend on the position of the foci and the constant b, see, e.g., Fig. 5a.

Also, if we convert (29) by changing the parameter using the hyperbolic functions $(x, y) \to (r, \alpha)$ where $x = \pm r \cosh\alpha$, $y = \pm r \sinh\alpha$ and $r^2 = x^2 - y^2$, we have:

- i $r^2 > 0$ in spacelike directions,
- ii $r^2 < 0$ in timelike directions,
- iii $r^2 = 0$ in lightlike directions.

Then (29) becomes:

$$\begin{aligned} &\big|\big(r^2 + z_1^2 - z_2^2\big)\big(r^2 + w_1^2 - w_2^2\big) \\ &\quad + 4r^2(z_1\cosh\alpha - z_2\sinh\alpha)(w_1\cosh\alpha - w_2\sinh\alpha) \\ &\quad - 2r\big[(z_1\cosh\alpha - z_2\sinh\alpha)\big(r^2 + w_1^2 - w_2^2\big) \\ &\quad + (w_1\cosh\alpha - w_2\sinh\alpha)\big(r^2 + z_1^2 - z_2^2\big)\big]\big| = b^4. \end{aligned} \tag{30}$$

Special case I:

We may start with a "normal form" of an M-Cassini curve $\boldsymbol{c}$ namely the case, where the two foci lie symmetric on the x-axis. Then we have $z_1 = -\ w_1 = c$ and $z_2 = w_2 = 0$, see Fig. 5a. All the cases, where the line connecting the two foci is timelike, can be transformed into this case by a Minkowski congruence transformation, see Fig. 5a. Then (30) becomes

$$\left|\left(r^2 + c^2\right)^2 - 4r^2c^2\cosh^2\alpha\right| = b^4, \tag{31}$$

and for the normal form c we may take $c = 1$. Then (31) reduces to

$$\left|r^4 - 2r^2\cosh 2\alpha + 1\right| = b^4. \tag{32}$$

There are still two subcases to distinguish:

i For $r^4 + c^4 > 2r^2c^2\cosh 2\alpha$ we get $r = \pm c\sqrt{\cosh 2\alpha \pm \sqrt{\sinh^2 2\alpha + (\frac{b}{c})^4}}$.

1. Hence,

$$x = \pm c\sqrt{\cosh 2\alpha \pm \sqrt{\sinh^2 2\alpha + \left(\frac{b}{c}\right)^4}}\,\cosh\alpha, \tag{33}$$

$$y = \pm c\sqrt{\cosh 2\alpha \pm \sqrt{\sinh^2 2\alpha + \left(\frac{b}{c}\right)^4}}\,\sinh\alpha. \tag{34}$$

ii For $r^4 + c^4 < 2r^2c^2\cosh 2\alpha$ we have $r = \pm c\sqrt{\cosh 2\alpha \pm \sqrt{\sinh^2 2\alpha - (\frac{b}{c})^4}}$

1. Hence,

$$x = \pm c\sqrt{\cosh 2\alpha \pm \sqrt{\sinh^2 2\alpha - \left(\frac{b}{c}\right)^4}}\,\cosh\alpha, \tag{35}$$

$$y = \pm c\sqrt{\cosh 2\alpha \pm \sqrt{\sinh^2 2\alpha - \left(\frac{b}{c}\right)^4}}\,\sinh\alpha. \tag{36}$$

In Fig. 6, we show M-Cassini curves for different values of b and d which are given by (7).

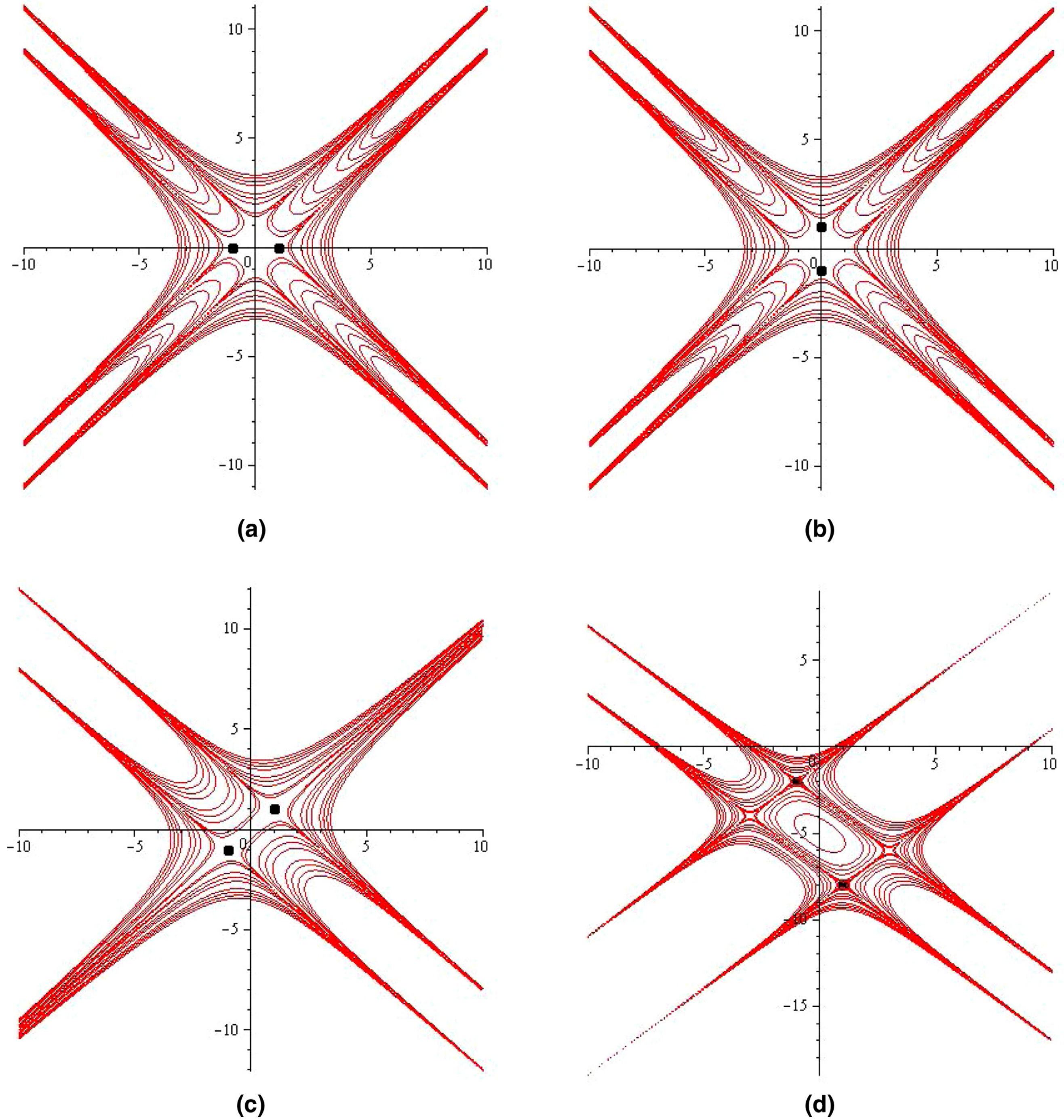

Fig. 5 (a) M-Cassini curves in M^2 with foci (1,0), (–1,0). (b) M-Cassini curves in M^2 with foci (0,–1), (0,1). (c) M-Cassini curves in M^2 with foci (1,1), (–1,–1). (d) M-Cassini curves in $M^2$with foci (1,–8), (–1,2).

Special case II:

Also in this case we may start with a "normal form" of an M-Cassini curve ***c***, namely the case, where the two foci lie symmetric on the y-axis, then we have $z_1 = w_1 = 0$, $z_2 = -w_2 = c$. Again, all the cases, where the line connecting the two foci is timelike, can be transformed into this case by a Minkowski congruence transformation, see Fig. 5b. Then we have

$$r = \pm c\sqrt{\cosh 2\alpha \pm \sqrt{\sinh^2 2\alpha + \left(\frac{b}{c}\right)^4}}$$

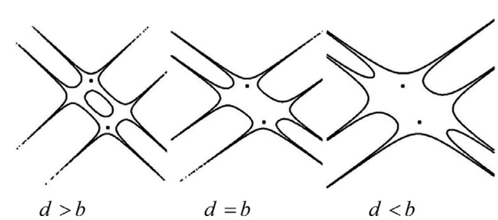

Fig. 6 M-Cassini curves for a given d and varying values of $b$with foci (1,–8), (–1,–2).

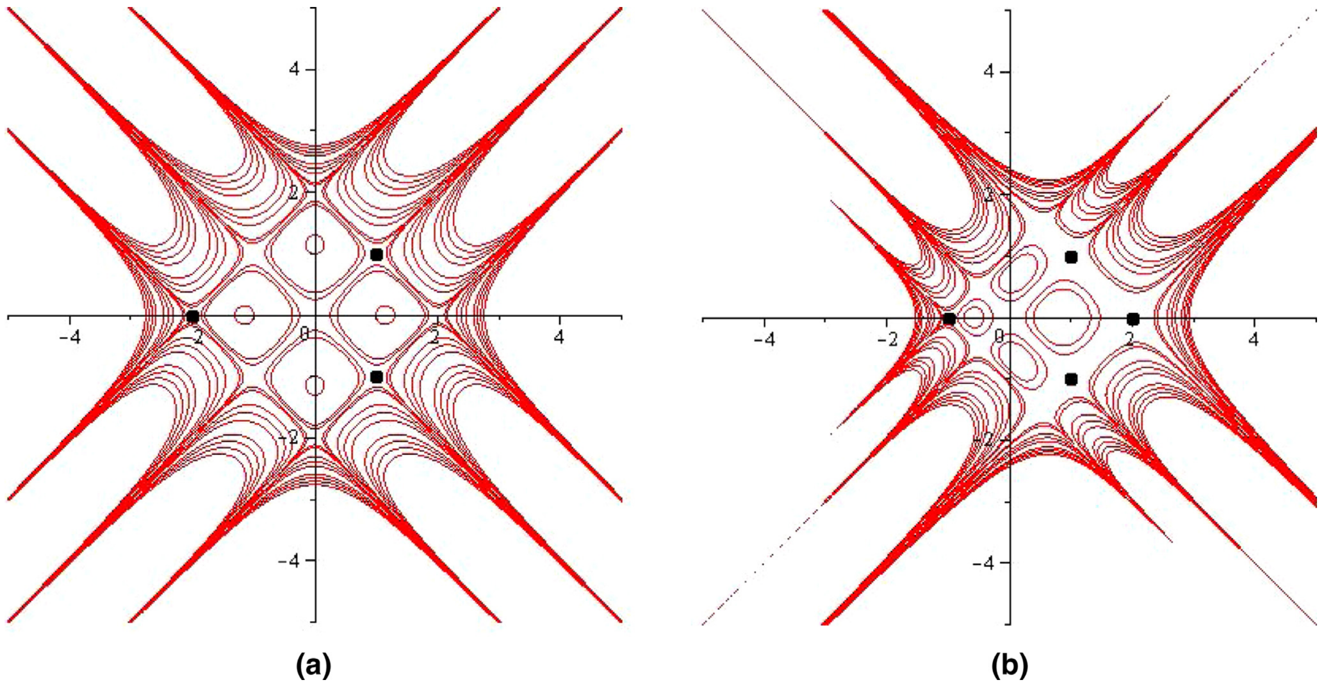

Fig. 7 (a) Multifocal M-Cassini curves in M^2 with three foci. (b) Multifocal M-Cassini curves in M^2 with four foci.

where Eqs. (33) and (34) should be valid, but do not give real values for all parameters α. The interval of α with real x and y values is restricted by $\sinh^2 2\alpha > (\frac{b}{c})^4$.

This means $\frac{1}{2}\cosh^{-1}(\frac{\sqrt{b^4+c^4}}{c^2}) < \alpha$ or $-\frac{1}{2}\cosh^{-1}(\frac{\sqrt{b^4+c^4}}{c^2}) > \alpha$, see Fig. 5b.

Special case III:

If the two foci lie on a lightlike line or a parallel one, then we can again, without loss of generality, choose the focal points to be symmetric to the origin. We assume $z_1 = z_2 = 1$ and $w_1 = w_2 = 1$. Then (30) becomes

$$\left|r^2\right|\left|r^2 - 4r\,e^{-\alpha} + 4e^{-2\alpha}\right| = b^4, \tag{36}$$

and the corresponding M-Cassini curve is shown in Fig. 5c. For a more general case see Fig. 5d.

5. Multifocal M-Cassini curves in M^2

As a generalization already due to Tschirnhaus [16] one can consider curves with constant distance sum or product to more than two focal points. Of course, in the Lorentz–Minkowski plane one gets already for three focal points many topologically different cases. We omit such a discussion and show in Fig. 7a and b images of such multifocal M-Cassini curves with three and four focal points.

6. Conclusion

Conics in the Minkowski space-time plane have some critical points coming from the transition between future and past timelike directions into spacelike directions through asymptotes lines of the equilateral hyperbolas unit circle (lightlike lines). Furthermore, their topological form depends on its foci and some factors in the case of M-ellipse and M-hyperbola, also the position of the directrix in the case of M-hyperbola. It is surprising to find that in some cases the foci lie on the conics as singular points. In Euclidean plane, the conic is a section of the usual quadratic cone. However in Minkowski plane, the shape of analogues of such cones is still unknown.

Almost all M-Cassini curves in Minkowski Space-Time plane are defined using hyperbolic functions. We believe that in pseudo-Minkowski plane it can be clearly introduced by using the functions of a hypercomplex variable since its conformal mappings have the same geometrical properties as the conformal mappings corresponding to the functions of a complex variable. See [1].

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