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$\mathcal{IP}$-separation axioms in ideal bitopological ordered spaces II

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Abstract The main purpose of this paper was to continue the study of separation axioms which is introduced in part I (Kandil et al., 2015). Whereas the part I (Kandil et al., 2015) was devoted to the axioms $\mathcal{IPT}_i$-ordered spaces, $i = 0, 1, 2$, in the part II the axioms $\mathcal{IPT}_i$-ordered spaces, $i = 3, 4, 5$ and $\mathcal{IPR}_j$-ordered spaces, $j = 2, 3, 4$ are introduced and studied. Clearly, if $\mathcal{I} = \{\phi\}$ in these axioms, then the previous axioms (Singal and Singal, 1971; Abo Elhamayel Abo Elwafa, 2009) coincide with the present axioms. Therefore, the current work is a generalization of the previous one. In addition, the relationships between these axioms and the previous one axioms have been obtained. Some examples are given to illustrate the concepts. Moreover, some important results related to these separations have been obtained.

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1. Introduction

A bitopological space (X, τ_1, τ_2) was introduced by Kelly [4] in 1963, as a method of generalizes topological spaces (X, τ). Every bitopological space (X, τ_1, τ_2) can be regarded as a topological space (X, τ) if $\tau_1 = \tau_2 = \tau$. Furthermore, he extended some of the standard results of separation axioms of topological spaces to bitopological spaces. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting.

In 1971 Singal and Singal [2] presented and studied the bitopological ordered space (X, τ_1, τ_2, R). It was a generalization of the study of general topological space, bitopological space and topological ordered space. Every bitopological ordered space (X, τ_1, τ_2, R) can be regarded as a bitopological space (X, τ_1, τ_2) if R is the equality relation "Δ".

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Singal and Singal studied separation axioms PT_i-ordered spaces, $i = 0, 1, 2, 3, 4$ and PR_j-ordered spaces, $j = 2, 3$ in bitopological ordered spaces. After that time many authors have already been studied the bitopological ordered spaces [3,5–8].

Abo Elhamayel Abo Elwafa [3] introduced separation axioms P-completely normal ordered spaces, PT_5-ordered spaces and PR_j-ordered spaces, $j = 0, 1$ on the bitopological ordered spaces. Kandil et al. [7] studied the bitopological ordered spaces by using the supra-topological ordered spaces. They introduced new separations axioms P^*T_i-ordered spaces, $i = 0, 1, 2$ which was a generalization of previous one [2].

In 2015 Kandil et al. [1] used the concept of ideal $\mathcal{I}$ to introduce and study the ideal bitopological ordered spaces $(X, \tau_1, \tau_2, R, \mathcal{I})$. Clearly, if $\mathcal{I} = \{\phi\}$, then every ideal bitopological ordered space is bitopological ordered space. Therefore, these spaces are generalization of the bitopological ordered spaces and bitopological spaces. They used the notion of $\mathcal{I}$-increasing (decreasing) [9] sets and introduced separation axioms $\mathcal{I}PT_i$-ordered spaces, $(i = 0, 1, 2)$ in ideal bitopological ordered spaces.

The present paper is a continuation of [1]. So, the aim of the present paper was to study the separation axioms $\mathcal{I}PT_i$-ordered spaces, $i = 3, 4, 5$ and $\mathcal{I}PR_j$-ordered spaces, $j = 2, 3, 4$ on ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$. The current separation axioms are based on the notion of $\mathcal{I}$-increasing (decreasing) sets. Comparisons between these axioms and the axioms in [2,3] have been obtained. The importance of the current study is that the new spaces are more general because the old one can be obtained from the current spaces when $\mathcal{I} = \{\phi\}$. Finally, we show that the properties of being $\mathcal{I}PT_i$-ordered spaces, $i = 3, 4, 5$ and $\mathcal{I}PR_j$-ordered spaces, $j = 2, 3, 4$ are preserved under a bijective, P-open and order (reverse) embedding mappings.

2. Preliminaries

Definition 2.1 [10,11] **.** A relation R on a non-empty set X is said to be:

1. reflexive if $(x, x) \in R$, for every $x \in X$,
2. symmetric if $(x, y) \in R \Rightarrow (y, x) \in R$, for every $x, y \in X$,
3. transitive if $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$, for every $x, y, z \in X$,
4. antisymmetric if $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$, for every $x, y \in X$,
5. preorder relation if it is reflexive and transitive,
6. partial order relation if it is reflexive, antisymmetric and transitive, and the pair (X, R) is said to be a partially ordered set (or poset, for short).

Definition 2.2 [10] **.** For a non-empty set X and a partially order relation R on X, the pair (X, R) is said to be a partially ordered set (or poset, for short).

Definition 2.3 [12] **.** Let (X, R) be a poset. A set $A \subseteq X$ is said to be:

1. decreasing if for every $a \in A$ and $x \in X$, $xRa \Rightarrow x \in A$,
2. increasing if for every $a \in A$ and $x \in X$, $aRx \Rightarrow x \in A$.

Definition 2.4. A mapping $f : (X, R) \to (Y, R^*)$ is said to be:

1. increasing (decreasing) if for every $x_1, x_2 \in X, x_1Rx_2 \Rightarrow f(x_1)R^*f(x_2)(f(x_2)R^*f(x_1))$ [12],
2. order embedding if for every $x_1, x_2 \in X, x_1Rx_2 \Leftrightarrow f(x_1)R^*f(x_2)$ [13],
3. order reverse embedding if for every $x_1, x_2 \in X, x_1Rx_2 \Leftrightarrow f(x_2)R^*f(x_1)$ [3].

Definition 2.5 [14] **.** Let X be a non-empty set. A class τ of subsets of X is called a topology on X iff τ satisfies the following axioms:

1. $X, \phi \in \tau$,
2. arbitrary union of members of τ is in τ,
3. the intersection of any two sets in τ is in τ.

The members of τ are then called τ-open sets, or simply open sets. The pair (X, τ) is called a topological space. A subset A of a topological space (X, τ) is called a closed set if its complement A' is an open set.

Definition 2.6 [10] **.** Let (X, τ) be a topological space and $A \subseteq X$. Then $\tau\text{-}cl(A) = \cap\{F \subseteq X : A \subseteq F$ and F is closed$\}$ is called the τ-closure of a subset $A \subseteq X$

Definition 2.7 [4] **.** A bitopological space (bts, for short) is a triple (X, τ_1, τ_2), where τ_1 and τ_2 are arbitrary topologies for a set X.

Definition 2.8 [15,16] **.** A function $f : (X_1, \tau_1, \tau_2) \to (X_2, \eta_1, \eta_2)$ is said to be:

1. P.continuous (respectively P.open, P.closed) if $f : (X_1, \tau_i) \to (X_2, \tau_i), i = 1, 2$ are continuous (respectively open, closed).
2. P.homeomorphism if $f : (X_1, \tau_i) \to (X_2, \tau_i)$, $i = 1, 2$ are homeomorphism.

Definition 2.9 [2] **.** A bitopological ordered space (bto-space, for short) has the form (X, τ_1, τ_2, R), where (X, R) is a poset and (X, τ_1, τ_2) is a bts.

The notion $a\overline{R}b$ means that a not related to b, i.e., $a\overline{R}b \Leftrightarrow (a, b) \notin R$.

Definition 2.10 [2] **.** A bto-space (X, τ_1, τ_2, R) is said to be:

1. Lower pairwise T_1 (LPT_1, for short)-ordered space if for every $a, b \in X$ such that $a\overline{R}b$, there exists an increasing τ_i-open set U contains a such that $b \notin U, i = 1$ or 2.
2. Upper pairwise T_1 (UPT_1, for short)-ordered space if for every $a, b \in X$ such that $a\overline{R}b$, there exists a decreasing τ_i-open set V contains b such that $a \notin V, i = 1$ or 2.
3. Pairwise T_1 (PT_1, for short), if it is LPT_1 and UPT_1-ordered space.

Definition 2.11 [2] **.** A bto-space (X, τ_1, τ_2, R) is said to be:

1. Lower pairwise regular (LPR_2, for short) ordered space iff for all decreasing τ_i-closed set F and for all $a \notin F$, there exist increasing τ_i-open set U and decreasing τ_j-open set V such that $a \in U, F \subseteq V$ and $U \cap V = \phi$.
2. Upper pairwise regular (PUR_2, for short) ordered space iff for all increasing τ_i-closed set F and for all $a \notin F$, there exist decreasing τ_i-open set U and increasing τ_j-open set V such that $a \in U, F \subseteq V$ and $U \cap V = \phi$.
3. Pairwise regular (PR_2, for short) ordered space iff it is LPR_2 and UPR_2.

Definition 2.12 [2] **.** A PR_2-ordered space which is also PT_1-ordered space is said to be PT_3-ordered space.

Definition 2.13 [2] **.** A bto-space (X, τ_1, τ_2, R) is said to be PR_3-ordered space iff for all increasing τ_i-closed set F_1 and decreasing τ_j-closed set F_2 such that $F_1 \cap F_2 = \phi$ there exist an increasing τ_j-open set U and a decreasing τ_i-open set V such that $F_1 \subseteq U, F_2 \subseteq V$ and $U \cap V = \phi$.

Definition 2.14 [2] **.** A PR_3-ordered space which is also a PT_1-ordered space is said to be a PT_4-ordered space.

Definition 2.15 [17,18] **.** Two sets A and B in (X, τ_1, τ_2) are said to be P-separated sets iff $A \cap \tau_j\text{-}cl(B) = \phi$ and $\tau_i\text{-}cl(A) \cap B = \phi, i, j = 1, 2, i \neq j$.

Definition 2.16 [3] **.** (X, τ_1, τ_2, R) is said to be a P-completely normal ordered spaces (PR_4-ordered spaces, for short) iff for any two P-separated subsets A and B of X such that A is an increasing and B is a decreasing there exist an increasing τ_i-open set $U, A \subseteq U$ and a decreasing τ_j-open set $V, B \subseteq V$ such that $U \cap V = \phi$.

Definition 2.17 [3] **.** A PR_4-ordered space which is a PT_1-ordered space is called a PT_5-ordered space.

Definition 2.18 [19] **.** A non-empty collection $\mathcal{I}$ of subsets of a set X is called an ideal on X, if it satisfies the following conditions:

1. $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
2. $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$.

Definition 2.19 [9] **.** Let (X, R) be a poset and $\mathcal{I}$ be an ideal on X. A set $A \subseteq X$ is said to be:

1. $\mathcal{I}$-decreasing iff $Ra \cap A' \in \mathcal{I}$ for every $a \in A$, where $Ra = \{b : (b, a) \in R\}$.
2. $\mathcal{I}$-increasing iff $aR \cap A' \in \mathcal{I}$ for every $a \in A$, where $aR = \{b : (a, b) \in R\}$.

Theorem 2.1 [9] **.** *Let $f : (X, R, \mathcal{I}) \to (Y, R^*, f(\mathcal{I}))$ be a bijective function and order embedding. Then for every $\mathcal{I}$ -increasing (decreasing) subset A of X, $f(A)$ is a $f(\mathcal{I})$-increasing (decreasing) subset of Y.*

Corollary 2.1 [9] **.** *Let $f : (X, R, \mathcal{I}) \to (Y, R^*, f(\mathcal{I}))$ be a bijective function and order embedding. If B is a $f(\mathcal{I})$ -increasing (decreasing) subsets of Y, then $f^{-1}(B)$ is an $\mathcal{I}$ -increasing (decreasing) subset of X.*

Theorem 2.2 [9] **.** *Let $f : (X, R, \mathcal{I}) \to (Y, R^*, f(\mathcal{I}))$ be a bijective function and reverse embedding. Then for every $\mathcal{I}$ -increasing (decreasing) subset A of X, $f(A)$ is a $f(\mathcal{I})$ -decreasing (increasing) subset of Y.*

Corollary 2.2 [9] **.** *Let $f : (X, R, \mathcal{I}) \to (Y, R^*, f(\mathcal{I}))$ be a bijective function and order reverse embedding. If B is a $f(\mathcal{I})$ -increasing (decreasing) subsets of Y, then $f^{-1}(B)$ is an $\mathcal{I}$ -decreasing (increasing) subset of X.*

Definition 2.20 [1] **.** A space $(X, \tau_1, \tau_2, R, \mathcal{I})$ is called an ideal bitopological ordered space if (X, τ_1, τ_2, R) is a bitopological ordered space and $\mathcal{I}$ is an ideal on X.

Definition 2.21 ([1]). An ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$ is said to be:

1. $\mathcal{I}$ lower PT_1($\mathcal{I}LPT_1$, for short) ordered space if for every $a, b \in X$ such that $a\overline{R}b$, there exists an $\mathcal{I}$-increasing τ_i-open set U such that $a \in U$ and $b \notin U$, $i = 1$ or 2.
2. $\mathcal{I}$ upper PT_1($\mathcal{I}UPT_1$, for short) ordered space if for every $a, b \in X$ such that $a\overline{R}b$, there exists an $\mathcal{I}$-decreasing τ_i-open set V such that $b \in V$ and $a \notin V$, $i = 1$ or 2.
3. $\mathcal{I}PT_1$-ordered space if it is $\mathcal{I}LPT_1$ and $\mathcal{I}UPT_1$ ordered space.

3. $\mathcal{I}P$-regularity and $\mathcal{I}P$-normality ordered spaces in ideal bitopological ordered spaces

The aim of this section was to use the notion of $\mathcal{I}$-increasing (decreasing) sets [9] which based on the ideal $\mathcal{I}$, to introduce new separation axioms $\mathcal{I}PT_i$-ordered spaces , $(i = 3, 4, 5)$ and $\mathcal{I}PR_j$-ordered spaces, $j = 2, 3, 4$ on the space $(X, \tau_1, \tau_2, R, \mathcal{I})$. Moreover, the relationship between these axioms and the axioms in [2,3] has been obtained. Some examples are given to illustrate the concepts. Furthermore, some important results related these separations have been studied.

Definition 3.1. An ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$ is said to be:

1. $\mathcal{I}$ lower pairwise regular ($\mathcal{I}LPR_2$, for short) ordered space iff for every $\mathcal{I}$-decreasing τ_i-closed set F and for every $a \notin F$, there exist an $\mathcal{I}$-increasing τ_i-open set U and a $\mathcal{I}$-decreasing τ_j-open set V such that $a \in U, F - V \in \mathcal{I}$ and $U \cap V \in \mathcal{I}$.
2. $\mathcal{I}$ upper pairwise regular ($\mathcal{I}PUR_2$, for short) ordered space iff for every $\mathcal{I}$-increasing τ_i-closed set F and for every $a \notin F$, there exist a $\mathcal{I}$-decreasing τ_i-open set U and an $\mathcal{I}$-increasing τ_j-open set V such that $a \in U, F - V \in \mathcal{I}$ and $U \cap V \in \mathcal{I}$.
3. $\mathcal{I}$ pairwise regular ($\mathcal{I}PR_2$, for short) ordered space iff it is $\mathcal{I}LPR_2$ and $\mathcal{I}UPR_2$.

Definition 3.2. A $\mathcal{I}PR_2$-ordered space which is also $\mathcal{I}PT_1$-ordered space is said to be $\mathcal{I}PT_3$-ordered space.

Remark 3.1. It should be noted that if $\mathcal{I} = \{\phi\}$ in Definitions 3.1 and 3.2, then we get Definitions 2.11 and 2.12 [2], so the current Definitions 3.1 and 3.2 are more general.

Example 3.1. Let $X = \{1,2,3,4\}, R = \Delta \cup \{(1,4),(1,3),(2,3),(4,3)\}, \mathcal{I} = \{\phi,\{1\},\{3\},\{1,3\}\}, \tau_1 = \{X,\phi,\{1\},\{4\},\{1,4\},\{1,3,4\}\}, \tau_2 = \{X,\phi,\{2,3\}\}$.

It is clear that $(X,\tau_1,\tau_2,R,\mathcal{I})$ is $\mathcal{I}LPR_2$-ordered space\text{,} but it is not $\mathcal{I}UPR_2$-ordered space.

Example 3.2. In Example 3.1 take $\mathcal{I} = \{\phi,\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{1,2,4\}\}, \tau_1 = \{X,\phi,\{2\},\{3\},\{2,3\},\{1,2,3\}\}, \tau_2 = \{X,\phi,\{1,4\}\}$.

It is clear that $(X,\tau_1,\tau_2,R,\mathcal{I})$ is $\mathcal{I}UPR_2$-ordered space\text{,} but it is not $\mathcal{I}LPR_2$-ordered space.

The following example shows that $(X,\tau_1,\tau_2,R,\mathcal{I})$ is $\mathcal{I}PR_2$-ordered space, but it is not $\mathcal{I}PT_3$-ordered space.

Example 3.3. In Example 3.1 take $\mathcal{I} = \{\phi,\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{1,2,4\}\}, \tau_1 = \{X,\phi,\{1,2\},\{1,2,3\},\{1,2,4\}\}, \tau_2 = \{X,\phi,\{2,3\}\}$.

It is clear that $(X,\tau_1,\tau_2,R,\mathcal{I})$ is $\mathcal{I}PR_2$-ordered space.

Example 3.4. In Example 3.1 take $\mathcal{I} = \{\phi,\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{1,2,4\}\}, \tau_1 = \{X,\phi,\{1\},\{3\},\{1,2\},\{1,3\},\{1,4\},\{3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\}\}, \tau_2 = \{X,\phi,\{2,3\}\}$.

It is clear that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PT_3$-ordered space.

The following proposition studies the relationship between the current Definitions 3.1 and 3.2 and the previous Definitions 2.11 and 2.12.

Proposition 3.1. *Let $(X, \tau_1, \tau_2, R, \mathcal{I})$ be an ideal bitopological ordered space. Then*

$$\begin{array}{ccc} PR_2 \wedge PT_1 - ordered\ space & \Rightarrow & \mathcal{I}PR_2 \wedge \mathcal{I}PT_1 - ordered\ spaces. \\ \Downarrow & & \Downarrow \\ PR_2 - ordered\ spaces & \Rightarrow & \mathcal{I}PR_2 - ordered\ spaces. \end{array}$$

Proof. The proof follows directly from the definitions of $\mathcal{I}PT_3$-ordered spaces, PT_3-ordered spaces, $\mathcal{I}PR_2$-ordered spaces and PR_2-ordered spaces. □

Example 3.4 shows that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PT_3$-ordered space, but it is PT_3-ordered space.

The following example shows that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_2$-ordered space, but it is PR_2-ordered space.

Example 3.5. In Example 3.1 take $\mathcal{I} = \{\phi,\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{1,2,4\}\}, \tau_1 = \{X,\phi,\{2\},\{1,2\},\{2,3\},\{1,2,3\},\{1,2,4\}\}$, and $\tau_2 = \{X,\phi,\{2,3\}\}$.

It is clear that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_2$-ordered space, but it is not PR_2-ordered space.

The following theorem shows that the property of being $\mathcal{I}PR_2$-ordered space is preserved by a bijective, ordered embedding (order reverse embedding) and P-homeomorphism mapping.

Theorem 3.1. *If $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_2$ -ordered space, $f : (X, \tau_1, \tau_2, R, \mathcal{I}) \to (Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ is a bijective, ordered embedding (order reverse embedding) and P-homeomorphism mapping. Then $(Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ is $f(\mathcal{I})PR_2$ -ordered space.*

Proof. We prove the theorem in the case of ordered embedding and the other case is similar. Let H be a $\mathcal{I}$-decreasing (increasing) η_i-closed subset of $Y, y \notin H$. Since, f is an onto function, then there exists $x \in X$ such that $x = f^{-1}(y)$. Since, f is P-continuous, $f^{-1}(H)$ is τ_i-closed. By Corollary 2.1, $f^{-1}(H)$ is a $\mathcal{I}$-decreasing (increasing) τ_i-closed subset of X and $x \notin f^{-1}(H)$. As $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_2$-ordered space, there exist an $\mathcal{I}$-increasing (decreasing) τ_i-open set U contains x and a $\mathcal{I}$-decreasing (increasing) τ_j-open set V such that $f^{-1}(H) - V \in \mathcal{I}$ and $U \cap V \in \mathcal{I}$. Since, f is P-open and by Theorem 2.1, $f(U)$ is a $f(\mathcal{I})$-increasing (decreasing) η_i-open set contains $y = f(x)$, $f(V)$ is a $f(\mathcal{I})$-decreasing (increasing) η_j-open set such that $f(f^{-1}(H) - V) = H - f(V) \in f(\mathcal{I})$ and $f(U) \cap f(V) = f(U \cap V) \in f(\mathcal{I})$. Hence, $(Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ is $f(\mathcal{I})PR_2$-ordered space. □

The following corollary shows that the property of being $\mathcal{I}PT_3$-ordered spaces is preserved by a bijective, ordered embedding (order reverse embedding) and P-homeomorphism mapping.

Corollary 3.1. *If $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PT_3$ -ordered space, $f : (X, \tau_1, \tau_2, R, \mathcal{I}) \to (Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ is a bijective, order embedding (order reverse embedding) and P-homeomorphism mapping. Then $(Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ is $f(\mathcal{I})T_3$ -ordered space.*

Definition 3.3. $(X, \tau_1, \tau_2, R, \mathcal{I})$ is said to be $\mathcal{I}P$-normal ordered space ($\mathcal{I}PR_3$-ordered space, for short) iff for all $\mathcal{I}$-increasing τ_i-closed set F_1 and $\mathcal{I}$-decreasing τ_j-closed set F_2 such that $F_1 \cap F_2 \in \mathcal{I}$, there exists an $\mathcal{I}$-increasing τ_j-open set U and a $\mathcal{I}$-decreasing τ_i-open set V such that $F_1 - U \in \mathcal{I}, F_2 - V \in \mathcal{I}$ and $U \cap V \in \mathcal{I}$.

Example 3.6. In Example 3.1 take $\mathcal{I} = \{\phi, \{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{1,2,4\}\}, \tau_1 = \{X,\phi,\{1,4\},\{1,3,4\}\}, \tau_2 = \{X,\phi,\{2\},\{1,2\},\{2,4\},\{1,2,4\},\{2,3,4\}\}$. It is clear that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_3$-ordered space.

Definition 3.4. A $\mathcal{I}PR_3$-ordered space which is also a $\mathcal{I}PT_1$-ordered space is said to be a $\mathcal{I}PT_4$-ordered space.

Example 3.6 shows that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_3$-ordered space, but it is not $\mathcal{I}PT_4$-ordered space.

The following example shows that if $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_3$-ordered space, then it is not necessary to be $\mathcal{I}PT_4$-ordered space.

Example 3.7. In Example 3.1 take $\mathcal{I} = \{\phi, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}\}$, $\tau_1 = \{X, \phi, \{3\}, \{4\}, \{2,4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}$, $\tau_2 = \{X, \phi, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}\}$. It is clear that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PT_4$-ordered space.

Remark 3.2. It should be noted that if $\mathcal{I} = \{\phi\}$ in Definitions 3.3 and 3.4, then we get Definitions 2.13 and 2.14 given by Singal and Singal [2] and so Definitions 2.13 and 2.14 due to Singal and Singal [2] are a special case of the current Definitions 3.3 and 3.4.

The following proposition studies the relationship between Definitions 3.3 and 3.4, and the previous Definitions 2.13 and 2.14 [2].

Proposition 3.2. *Let $(X, \tau_1, \tau_2, R, \mathcal{I})$ be an ideal bitopological ordered space. Then*

$$\begin{array}{ccc} PR_3 \wedge PT_1 - ordered\ spaces & \Rightarrow & \mathcal{I}PR_3 \wedge \mathcal{I}PT_1 - ordered\ spaces. \\ \Downarrow & & \Downarrow \\ PR_3 - ordered\ spaces & \Rightarrow & \mathcal{I}PR_3 - ordered\ spaces. \end{array}$$

Proof. The proof follows directly from the definitions of $\mathcal{I}PT_4$-ordered space, PT_4-ordered space, PR_3-ordered space and $\mathcal{I}PR_3$-ordered spaces. □

Example 3.6 shows that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_3$-ordered space, but it is not PR_3-ordered space.

Example 3.7 shows that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PT_4$-ordered space, but it is not PT_4-ordered space.

The following theorem shows that the property of being $\mathcal{I}PR_3$-ordered space is preserved by a bijective, ordered embedding (order reverse embedding) and P-homeomorphism mapping.

Theorem 3.2. *If $(X, \tau_1, \tau_2, R, \mathcal{I})$ is a $\mathcal{I}PR_3$ -ordered space, $f : (X, \tau_1, \tau_2, R, \mathcal{I}) \to (Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ is a bijective, order embedding (order reverse embedding) and P-homeomorphism mapping. Then $(Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ is $f(\mathcal{I})PR_3$ -ordered space.*

Proof. Let H_1 be a $f(\mathcal{I})$-decreasing (increasing) η_i-closed subset of Y and H_2 be a $f(\mathcal{I})$-increasing (decreasing) η_j-closed subset of Y such that $H_1 \cap H_2 \in f(\mathcal{I})$. Since, f is P-continuous, $f^{-1}(H_1)$ is τ_i-closed subsets of X and $f^{-1}(H_2)$ is τ_j-closed subsets of X and by Corollary 2.1, $f^{-1}(H_1)$ is a $\mathcal{I}$-decreasing (increasing) τ_i-closed subsets of X and $f^{-1}(H_2)$ is an $\mathcal{I}$-increasing (decreasing) τ_j-closed subsets of X. Now, we have $f^{-1}(H_1) \cap f^{-1}(H_2) = f^{-1}(H_1 \cap H_2) \in \mathcal{I}$. Since, $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_3$-ordered space, then there exist an $\mathcal{I}$-increasing (decreasing) τ_j-open set U, $f^{-1}(H_1) - U \in \mathcal{I}$ and a $\mathcal{I}$-decreasing (increasing) τ_i-open set V, $f^{-1}(H_2) - V \in \mathcal{I}$ such that $U \cap V \in \mathcal{I}$. Since, f is P-open and by Theorem 2.1, $f(U)$ is an $f(\mathcal{I})$-increasing (decreasing) η_j-open set, $f(f^{-1}(H_1) - U) = H_1 - f(U) \in f(\mathcal{I})$ and $f(V)$ is a $f(\mathcal{I})$-decreasing (increasing) η_i-open set, $f(f^{-1}(H_2) - V) = H_2 - f(V) \in f(\mathcal{I})$ such that $f(U) \cap f(V) = f(U \cap V) \in f(\mathcal{I})$. Hence, $(Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ is $f(\mathcal{I})PR_3$-ordered space. □

Corollary 3.2. *If $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PT_4$ ordered space, $f : (X, \tau_1, \tau_2, R, \mathcal{I}) \to (Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ is a bijective, order embedding (order reverse embedding) and P-continuous mapping. Then $(Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ is $f(\mathcal{I})PT_4$ ordered space.*

Definition 3.5. Two sets A and B in $(X, \tau_1, \tau_2, \mathcal{I})$ are said to be $\mathcal{I}P$-separated sets iff $A \cap \tau_j\text{-}cl(B) \in \mathcal{I}$ and $\tau_i\text{-}cl(A) \cap B \in \mathcal{I}$.

Example 3.8. Let $(\mathbb{R}, \tau_l, \tau_{\mathbb{U}}, R, \mathcal{I})$ be an ideal bitopological ordered space in which $\mathbb{R}$ is the real numbers and R is the usual order, $\tau_l = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \phi\}$, $\tau_{\mathbb{U}}$ is the usual topology, and $\mathcal{I} = \{\phi, (1, \infty), (a, \infty), [a, \infty), (a, b), [a, b), (a, b], [a, b], \{c\}\}$, where $1 < a < b, 1 < c < \infty, a, b, c \in \mathbb{R}$ and let $A, B \subseteq \mathbb{R}$ such that $A = (1, \infty)$ and $B = (-\infty, 1)$. It is clear that A and B are $\mathcal{I}P$-separated sets as $\tau_{\mathbb{U}}\text{-}cl(A) = [1, \infty)$, $\tau_l\text{-}cl(B) = \mathbb{R}$ and so $\tau_{\mathbb{U}}\text{-}cl(A) \cap B = \phi \in \mathcal{I}$ and $\tau_l\text{-}cl(B) \cap A = (1, \infty) \in \mathcal{I}$.

Definition 3.6. An ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$ is said to be $\mathcal{I}P$-completely normal ordered space ($\mathcal{I}PR_4$-ordered space, for short) iff for any two $\mathcal{I}P$-separated subsets A and B of X such that A is an $\mathcal{I}$-increasing set and B is a $\mathcal{I}$-decreasing set there exist an $\mathcal{I}$-increasing τ_i-open set U, $A \subseteq U$ and a $\mathcal{I}$-decreasing τ_j-open set V, $B \subseteq V$ such that $U \cap V \in \mathcal{I}$.

Example 3.9. Let $(\mathbb{R}, \tau_{\mathbb{U}}, \tau_l, R, \mathcal{I})$ be an ideal bitopological ordered space in which $\mathbb{R}$ is the real numbers and R is the usual order, $\tau_{\mathbb{U}}$ is the usual topology, $\tau_l = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \phi\}$ and $\mathcal{I} = \{\phi, (0, \infty), (a, \infty), [a, \infty), (a, b), [a, b), (a, b], [a, b], \{c\}\}$, where $: 0 \le a < b, 0 \le c < \infty, a, b, c \in \mathbb{R}$. Then, it is clear that $(\mathbb{R}, \tau_{\mathbb{U}}, \tau_l, R, \mathcal{I})$ is $\mathcal{I}PR_4$-ordered space.

Definition 3.7. A $\mathcal{I}PR_4$-ordered space which is $\mathcal{I}PT_1$-ordered space is called a $\mathcal{I}PT_5$-ordered space.

Remark 3.3. It should be noted that if $\mathcal{I} = \{\phi\}$ in Definitions 3.5–3.7, then we get Definitions 2.15–2.17 [3,17,18], so Definitions 2.15–2.17 [3,17,18] are special case of the current Definitions 3.5–3.7.

The following proposition studies the relationship between Definitions 3.5–3.7 and 2.15–2.17 given in [3,17,18].

Proposition 3.3. *Let $(X, \tau_1, \tau_2, R, \mathcal{I})$ be an ideal bitopological ordered space. Then*

$$\begin{array}{ccc} PR_4 \wedge PT_1 - ordered\ spaces & \Rightarrow & \mathcal{I}PR_4 \wedge \mathcal{I}PT_1 - ordered\ spaces. \\ \Downarrow & & \Downarrow \\ PR_4 - ordered\ spaces & \Rightarrow & \mathcal{I}PR_4 - ordered\ spaces. \\ \Downarrow & & \Downarrow \\ P - separated & \Rightarrow & \mathcal{I}P - separated. \end{array}$$

Proof. The proof follows directly from the definitions of $\mathcal{I}PT_5$-ordered space, PT_5-ordered space, PR_4-ordered space and $\mathcal{I}PR_4$-ordered spaces. □

Example 3.8 shows that $A = (1, \infty)$ and $B = (-\infty, 1)$ are $\mathcal{I}P$-separated sets, but it is not P-separated sets as $\tau_l\text{-}cl(B) \cap A = (1, \infty) \neq \phi$.

Example 3.9 shows that $(\mathbb{R}, \tau_{\mathbb{U}}, \tau_l, R, \mathcal{I})$ is $\mathcal{I}PR_4$-ordered space, but it is not $\mathcal{I}PT_5$-ordered space, as it is not $\mathcal{I}PT_1$-ordered space, for $3\overline{R}2$ all $\mathcal{I}$-decreasing sets which are containing 2, also, contain 3.

The following example shows that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_4$-ordered space, but it is not PR_4-ordered space.

Example 3.10. Let $(\mathbb{R}, \tau_{\mathbb{U}}, \tau_u, R, \mathcal{I})$ be an ideal bitopological ordered space in which $\mathbb{R}$ is the real numbers and R is the usual order, $\tau_{\mathbb{U}}$ is the usual topology, $\tau_u = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \phi\}$ and $\mathcal{I} = \{\phi, (0, \infty), (a, \infty), [a, \infty), (a, b), [a, b), (a, b], [a, b], \{c\}\}$, where $: 0 \leq a < b, 0 \leq c < \infty, a, b, c \in \mathbb{R}$. It is clear that $(\mathbb{R}, \tau_{\mathbb{U}}, \tau_u, R, \mathcal{I})$ is $\mathcal{I}PR_4$-ordered space, but it is not PR_4-ordered space, as there exist $A, B \subseteq \mathbb{R}$, where $A = (1, \infty)$ and $B = (-\infty, 0)$ which are two P-separated sets, A is an increasing set and B is a decreasing set, but the increasing $\tau_{\mathbb{U}}$-open superset of A is $U = (d, \infty)$, $d \leq 1$, the only decreasing τ_u-open superset of B is $V = \mathbb{R}$ and $U \cap V = (d, \infty) \cap \mathbb{R} = (d, \infty) \neq \phi, d \leq 1$.

Theorem 3.3. *Every $\mathcal{I}PR_4$ -ordered space is $\mathcal{I}PR_3$ -ordered space.*

Proof. Let $(X, \tau_1, \tau_2, R, \mathcal{I})$ be $\mathcal{I}PR_4$-ordered space, A and B be subsets of X such that $A \cap B \in \mathcal{I}$, A is an $\mathcal{I}$-increasing τ_i-closed set and B is a $\mathcal{I}$-decreasing τ_j-closed set. Then $\tau_i\text{-}cl(A) \cap B \in \mathcal{I}$ and $A \cap \tau_j\text{-}cl(B) \in \mathcal{I}$. Consequently, A and B being two $\mathcal{I}P$-separated subsets of the $\mathcal{I}PR_4$-ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$. Therefore, there exist an $\mathcal{I}$-increasing τ_i-open set $U, A \subseteq U$ and a $\mathcal{I}$-decreasing τ_j-open set $V, B \subseteq V$ such that $U \cap V \in \mathcal{I}$. Hence, $(X, \tau_1, \tau_2, R, \mathcal{I})$ be a $\mathcal{I}PR_3$-ordered space. □

The following example shows that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_3$-ordered space, but it is not $\mathcal{I}PR_4$-ordered space.

Example 3.11. In Example 3.1 take $\mathcal{I} = \{\phi, \{3\}, \{4\}, \{3, 4\}, \}$, $\tau_1 = \{X, \phi, \{1, 4\}, \{1, 3, 4\}\}$, $\tau_2 = \{X, \phi, \{2\}, \{1, 2\}, \{2, 4\}, \{1, 2, 4\}, \{2, 3, 4\}\}$. It is clear that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PR_3$-ordered space, but it is not $\mathcal{I}PR_4$-ordered space as $A = \{2, 3\}, B = \{1\}$ are $\mathcal{I}P$-separated sets, A is an $\mathcal{I}$-increasing set and B is a $\mathcal{I}$-decreasing set, the only $\mathcal{I}$-increasing τ_1-open superset of A is X and the $\mathcal{I}$-decreasing τ_2-open supersets of B are $X, \{1, 2\}, \{1, 2, 4\}$ and their intersection is $X, \{1, 2\}, \{1, 2, 4\} \notin \mathcal{I}$.

Corollary 3.3. *Every $\mathcal{I}PT_5$ -ordered space is $\mathcal{I}PT_4$ -ordered space.*

Example 3.7 shows that $(X, \tau_1, \tau_2, R, \mathcal{I})$ is $\mathcal{I}PT_4$-ordered space, but it is not $\mathcal{I}PT_5$-ordered space as it is not $\mathcal{I}PR_4$-ordered space, $A = \{3, 4\}, B = \{1, 2, 4\}$ are $\mathcal{I}P$-separated sets, A is an $\mathcal{I}$-increasing set and B is a $\mathcal{I}$-decreasing set, the only $\mathcal{I}$-increasing τ_2-open superset of A is X and the only $\mathcal{I}$-decreasing τ_1-open superset of B is X and their intersection is $X \notin \mathcal{I}$.

Theorem 3.4. *The property of being $\mathcal{I}PR_4$ -ordered space is preserved by a bijective, order embedding and P-homeomorphism mapping.*

Proof. Let $f : (X, \tau_1, \tau_2, R, \mathcal{I}) \to (Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ be a bijective, order embedding and P-homeomorphism mapping. Let A and B be two $f(\mathcal{I})P$-separated subsets of Y such that A is a $f(\mathcal{I})$-increasing set, B is a $f(\mathcal{I})$-decreasing set. Then $A \cap \eta_j\text{-}cl(B) \in f(\mathcal{I})$ and $\eta_i\text{-}cl(A) \cap B \in f(\mathcal{I})$. Now, $f^{-1}(A)$ is an $\mathcal{I}$-increasing and $f^{-1}(B)$ is a $\mathcal{I}$-decreasing, $f^{-1}(A) \cap \tau_j\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(A) \cap f^{-1}(\eta_j\text{-}cl(B)) = f^{-1}(A \cap \eta_j\text{-}cl(B)) \in \mathcal{I}$ and $\tau_i\text{-}cl(f^{-1}(A)) \cap f^{-1}(B) \subseteq f^{-1}(\eta_i\text{-}cl(A)) \cap f^{-1}(B) = f^{-1}(\eta_i\text{-}cl(A) \cap B) \in \mathcal{I}$. Thus, $f^{-1}(A)$ and $f^{-1}(B)$ are $\mathcal{I}P$-separated subsets of X. So, there exist an $\mathcal{I}$-increasing τ_j-open set $U, f^{-1}(A) \subseteq U$ and a $\mathcal{I}$-decreasing τ_i-open set $V, f^{-1}(B) \subseteq V$ such that $U \cap V \in \mathcal{I}$. Therefore, $f(f^{-1}(A)) \subseteq f(U)$ and $f(f^{-1}(B)) \subseteq f(V)$. Thus, $A \subseteq f(U)$ and $B \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) \in f(\mathcal{I})$. Consequently, $(Y, \eta_1, \eta_2, R^*, f(\mathcal{I}))$ is $f(\mathcal{I})PR_4$-ordered space. □

Corollary 3.4. *The property of being $\mathcal{I}PT_5$ -ordered space is preserved by a bijective, order embedding and P-homeomorphism mapping.*

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