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Original Article

Analysis of finite buffer queue with state dependent service and correlated customer arrivals

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Abstract This paper deals with a finite capacity queue with workload dependent service. The arrival of a customer follows Markovian arrival process (MAP). MAP is very effective arrival process to model massage-flow in the modern telecommunication networks, as these messages are very bursty and correlated in nature. The service time, which depends on the queue length at service initiation epoch, is considered to be generally distributed. Queue length distribution at various epoch and key performance measures have been obtained. Finally, some numerical results have been discussed to illustrate the numerical compatibility of the analytic analysis of the queueing model under consideration.

2000 Mathematics Subject Classification: 60K25; 68M20; 90B22

1. Introduction

In many practical queueing situations, a long queue of customers waiting for service is quite common which in turn causes poor system performance (which is measured in terms of loss/rejection/blocking probability). To avoid this inconvenient situation, the decision maker often decide to control arrival rates or service rates to reduce blocking probabilities or congestion. Various queueing models have been studied for overload control to prevent congestion in telecommunication networks, in particular, ATM (asynchronous transfer mode) networks (Jain [1]). Overload control by controlling the service rate is discussed earlier by Choi and Choi [2], Choi et al. [3], Sriram and Lucantoni [4], Banerjee and Gupta [5], Banerjee et al. [6], etc. In particular, Choi and Choi [2] analyzed a finite buffer queue, where customers arrive according to the Markov modulated Poisson process ($MMPP$), with queue-length-dependent service. They considered that if the queue length at the service initiation epoch is less than or equal to a threshold limit (say, L), the service time follows G_1 distribution, and if the queue length at the service initiation-epoch exceeds the threshold limit, the service time distribution of the customers is G_2. They obtained queue length distribution at departure- and arbitrary-epoch. They also obtained performance measures, viz. loss probability, mean waiting time, etc.

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In this paper, we consider a finite capacity queueing model where arrival follows MAP. $MMPP$, with matrix representation $(\mathbf{Q}, \mathbf{\Lambda})$, is a special case of MAP, with matrix representation $(\mathbf{C}, \mathbf{D})$. That is, if one consider $\mathbf{Q} = \mathbf{C} + \mathbf{D}$ and $\mathbf{\Lambda} = \mathbf{D}$, then the arrival process MAP will reduces to $MMPP$. Also in this paper the service time distribution of a customer is considered to be G_n, i.e., service time of a customer depends on the number of customers (n) present in the system at service initiation epoch of a customer and also generally distributed. Hence, for $G_n = G_1$ $(1 \le n \le L)$, $G_n = G_2$ $(n > L)$, $\mathbf{D} = \mathbf{\Lambda}$ and $\mathbf{Q} = \mathbf{C} + \mathbf{D}$ the model presented in this paper will be reduced to the model discussed by Choi and Choi [2]. Therefore, the model considered in this paper is more general and complex than the one considered by Choi and Choi [2]. One may note here that the results obtained in this paper cannot be deduced from [2], whereas the results obtained in [2] can be deduced from the present study as a special case.

The analysis of this paper is carried out as follows: first departure-epoch probabilities have been obtained by using the embedded Markov chain technique. Then using the supplementary variable technique and considering the supplementary variable as remaining service time of a customer, relations between arbitrary- and departure-epoch probabilities have been obtained. Distribution of the number of customers in the queue at arrival-epoch has been also obtained. Performance measures such as average number of customers in the queue, probability of blocking and average waiting time of a customer in the queue have been obtained. Then computational procedure when service time distribution is phase type (PH-distribution) has been discussed. It should be noted here that many distribution (viz. exponential, hyperexponential, hypoexponential, Erlang, Coxian, etc.) in continuous time set up can be approximated by PH-distribution. Finally, comparative studies of queue-length-dependent service with the one when service time of the customers are independent of the queue length have been carried out by using self explanatory graphs as the effect of buffer-size on performance measures. These comparative studies establish the fact that our model is more effective than the one when service time is independent of the queue length. For the sake of notational convenience we denote this model by $MAP/G_n/1/N$ for future reference. It may be remarked here that in a special case when $G_n = G$, i.e., service time of the customers is independent of the queue size, the model reduces to simple $MAP/G/1/N$ queue (Gupta and Laxmi [7]). Gupta and Laxmi [7] analyzed $MAP/G/1$ with finite/infinite buffer and obtained relations among the queue size distributions at departure-, arbitrary- and arrival-epoch using the supplementary variable technique. However, they did not provide the computational procedure and numerical illustrations. One can easily obtain numerical illustrations of $MAP/G/1/N$ queue from the present study.

ATM networks support diverse traffics with different service characteristics such as voice, data and video. In B-ISDN/ATM network, IP packets or cells of voice, video, data are sent over a common transmission channel on statistical multiplexing basis. These traffics are statistically multiplexed and transmitted in superhigh speed. Also, it is seen that the traffic in modern communication networks is highly irregular (e.g., bursty and correlated). A good representation of such traffic is a Markovian arrival process (MAP). Hence, the model discussed in this paper can be used to control congestion in the telecommunication networks by controlling the transmission rate (service rate) depending on the number of the packet waiting in the queue (queue length). In recent years there has been a growing interest to analyze queues with input process as MAP which is a rich class of point processes containing many familiar arrival processes, such as, Poisson process, interrupted Poisson process (IPP), PH-renewal process, Markov modulated Poisson process ($MMPP$), etc. Lucantoni et al. [8]. Later, queueing models with MAP have been studied extensively, in past, see e.g., [9–15] and many others. For recent development in MAP, see, [16–21] and the references therein.

A real life application of the proposed model may be observed in modern telecommunication networks, viz. advanced wireless and mobile internet networks. Demand for high speed wireless internet access, voice and multimedia applications results in the popularity of technologies like 3G and 4G. IEEE 802.16 is a telecommunication standard technology designed to support a wide variety of wireless and wired broadband access of multimedia applications with expectation to provide Quality of Service (QoS). The basic IEEE 802.16 system consists of one Base Station (BS) and one (or more) Subscriber Station (SS). BS acts as a transmitter to transfer all type of data (voice, video, data, etc.). Transmissions take place through two independent channels, downlink channel (from BS to SS) and uplink channel (from SS to BS). In case of internet traffic, downlink gets higher preference over the uplink. This internet traffic flow system may be modeled as a multimedia data transmission system over a wireless channel, where packets are queued at the transmitter (BS). Since the incoming traffic in IEEE 802.16 is irregular and bursty in nature, causing correlation in inter-arrival time, arrival process can be modeled using MAP. Due to rapid increase in the popularity of 2G and 3G system, it has become necessary to develop new schemes for congestion control to reduce queue length, waiting time and probability of rejection (blocking). In this direction if one consider that the BS transmits data, depending on the queue size, to a SS, the system can be modeled and analyzed as queue length dependent service queue with MAP, which has been done (analytically) in this paper. This scheme of transmission of packets (e.g., from BS to SS) enhances the overall efficiency of the system as well as improves the QoS. Fig. 1 illustrates the data transmission system discussed above. The rest of this paper is organized as follows. In Section 2, description of the model and its analysis at various epoch is given. The computational procedure when service time follows phase type distribution are spelled out in Section 5. System performance measures and numerical results are given in Sections 4 and 5, respectively. The paper ends with some concluding remark in Section 6.

2. Model description and solution

Consider a single server queue where customers arrive according to the Markovian arrival process (MAP) with matrix representation $(\mathbf{C}, \mathbf{D})$ of dimension m. The generator $\mathbf{Q}^*$, governing the continuous time Markov chain governing the arrival process, is then given by $\mathbf{Q}^* = \mathbf{C} + \mathbf{D}$. Let $\boldsymbol{\delta}$ denotes the stationary probability vector of the Markov processes with generator $\mathbf{Q}^*$. That is, $\boldsymbol{\delta}$ is the unique (positive) probability vectors satisfying $\boldsymbol{\delta}\mathbf{Q}^* = \mathbf{0}$, $\boldsymbol{\delta}\boldsymbol{e} = 1$. The fundamental arrival rate of the stationary MAP is given by $\lambda^* = \boldsymbol{\delta}\mathbf{D}\mathbf{e}$. Here $\mathbf{e}$ and $\mathbf{0}$ are the $m \times 1$ vectors of ones and zeros, respectively. For more detail on this topic, see, Lucantoni et al. [8].

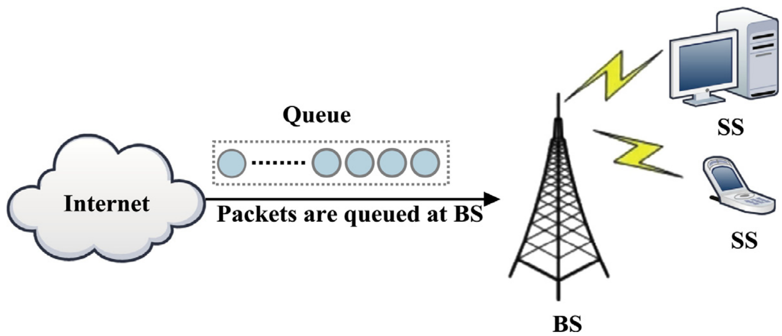

Fig. 1 Framework in IEEE 802.16 network.

Server serves single customer at a time. The queue has finite buffer (capacity) of size $N > 1$, so at any time maximum $N + 1$ customers can be present in the system. The service time of a customer is assumed to be generally distributed and depended on the number of customers present in the queue at service initiation epoch of that customer including that customer. Specifically, let T_n $(1 \leq n \leq N)$ be the service time of a customer, where $n(1 \leq n \leq N)$ is the number of customer in the queue at service initiation epoch of the customer (including him/her), with the distribution function $H_n(\cdot)$, probability density function $h_n(\cdot)$, Laplace-Stieltjes transform $H_n^*(s) = \int_0^\infty e^{-su}h_n(u)du$ $(\mathbb{R}es \geq 0)$ and mean service time of a customer is $h_n = -H_n^{*(1)}(0)$, where $H_n^{*(1)}(0)$ is the derivative of $H_n^*(s)$ evaluated at $s = 0$.

The steady-state analysis of the model under consideration will be carried out using the embedded Markov chain approach and the supplementary variable technique since the service times are assumed to be generally distributed. First, we will look at the semi-Markov process embedded at the points of departure of a customer. Toward this end, let us define the square matrices $\mathbf{A}_{n,k}(x)$ and $\mathbf{B}_{0,k}(x)$ of order m for $x \geq 0$ whose (i, j)th elements $(\mathbf{A}_{n,k}(x))_{ij}$ and $(\mathbf{B}_{0,k}(x))_{ij}$, respectively, are defined as follows

$(\mathbf{A}_{n,k}(x))_{ij}] = Pr\{$Given a departure at time 0, which left $n(1 \leq n \leq N)$ customer in the queue and the arrival process in phase i, the next departure occurs no later than time x with the arrival process in *phase* j, and during that service $k(k \geq 0)$ customers arrive$\}$,

$(\mathbf{B}_{0,k}(x))_{ij} = Pr\{$Given a departure at time 0,which left 0 customer in the queue and the arrival process in phase i,the next departure occurs no later than time x with the arrival process in phase j, and during that service $k(k \geq 0)$ customers arrive$\}$.

Suppose that $\tilde{N}(t)$ and $J(t)$ denote, respectively, the number of customers arriving in $(0, t]$ and the phase of the MAP at time t. The matrices, $\{\tilde{P}(n, t), n \geq 0, t \geq 0\}$, whose (i, j)th entry defined as

$$\tilde{p}_{i,j}(n, t) = Pr\{\tilde{N}(t) = n, J(t) = j|\tilde{N}(0) = 0, J(0) = i\}. \tag{1}$$

These matrices satisfy the following system of difference-differential equations

$$\frac{d}{dt}\tilde{P}(0, t) = \tilde{P}(0, t)C,$$
$$\frac{d}{dt}\tilde{P}(n, t) = \tilde{P}(n, t)C + \tilde{P}(n-1, t)D, n \geq 1,$$

with

$$\tilde{P}(0, 0) = I.$$

These matrices, associated with the counting process $\{\tilde{N}(t), J(t); t \geq 0\}$, have been extensively studied in the literature and an efficient procedure for computing them is given in Neuts and Li [22].

It is easy to verify that

$$\mathbf{A}_{n,k}(x) = \int_0^x \tilde{P}(k, t)dH_n(t), 1 \leq n \leq N, 0 \leq k \leq N - n,$$
$$\overline{\mathbf{A}}_{n,k}(x) = \sum_{l=k}^{\infty} \mathbf{A}_{n,l}(x), 1 \leq n \leq N, k = N - n + 1,$$
$$\mathbf{B}_{0,k}(x) = \int_0^x \tilde{P}(0, x-u)\mathbf{D}\mathbf{A}_{1,k}(u)du, 0 \leq k \leq N - 1,$$
$$\overline{\mathbf{B}}_{0,N}(x) = \sum_{l=N}^{\infty} \mathbf{B}_{0,l}(x),$$

For use in sequel, we define

$$\begin{aligned}
\mathbf{A}_{n,k} &= \mathbf{A}_{n,k}(\infty), 1 \leq n \leq N, 0 \leq k \leq N - n,\\
\overline{\mathbf{A}}_{n,k} &= \overline{\mathbf{A}}_{n,k}(\infty), 1 \leq n \leq N, k = N - n + 1,\\
\mathbf{B}_{0,k} &= \mathbf{B}_{0,k}(\infty) = \tilde{\mathbf{D}}\,\mathbf{A}_{1,k}, 0 \leq k \leq N - 1,\\
\overline{\mathbf{B}}_{0,N} &= \overline{\mathbf{B}}_{0,N}(\infty) = \tilde{\mathbf{D}}\,\overline{\mathbf{A}}_{1,N},
\end{aligned} \tag{2}$$

where $\tilde{\mathbf{D}} = (-\mathbf{C})^{-1}\mathbf{D}$ (Lucantoni et al. [8]).

2.1. *Distribution of the queue length at departure-epoch*

In this section the joint distribution of the number of customers in the queue and phase of the arrival process at departure-epoch has been obtained using the embedded Markov chain technique. Toward this end, let N_n^+ and J_n^+ denote, respectively, the number of customers in the queue and the phase of the arrival process immediately after the n-th departure of a customer. Then the discrete-time process $\{(N_n^+, J_n^+);n \geq 0\}$ constitute a two dimensional Markov chain with state space $\{(i, j);0 \leq i \leq N, 1 \leq j \leq m\}$. Now observing the system immediately after a departure, the transition probability matrix (TPM) $\mathbf{P}$ of the above Markov process is obtained as follows

$$\mathbf{P} = \begin{bmatrix} \mathbf{B}_{0,0} & \mathbf{B}_{0,1} & \mathbf{B}_{0,2} & \mathbf{B}_{0,3} & \ldots & \mathbf{B}_{0,N-1} & \overline{\mathbf{B}}_{0,N} \\ \mathbf{A}_{1,0} & \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \mathbf{A}_{1,3} & \ldots & \mathbf{A}_{1,N-1} & \overline{\mathbf{A}}_{1,N} \\ \mathbf{0} & \mathbf{A}_{2,0} & \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \ldots & \mathbf{A}_{2,N-2} & \overline{\mathbf{A}}_{2,N-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{3,0} & \mathbf{A}_{3,1} & \ldots & \mathbf{A}_{3,N-3} & \overline{\mathbf{A}}_{3,N-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{A}_{N,0} & \overline{\mathbf{A}}_{N,1} \end{bmatrix} \quad (3)$$

Let $\pi_i^+(n), 0 \leq n \leq N$,be the joint probability that there are n customers in the queue and the state of the arrival process is i $(1 \leq i \leq m)$ immediately after the departure of a customer. Further, define $\widetilde{\boldsymbol{\pi}}^+(n) = (\pi_1^+(n), \pi_2^+(n), \ldots, \pi_m^+(n)), 0 \leq n \leq N$. The unknown quantities $\widetilde{\boldsymbol{\pi}}^+(n)$ can be obtained by solving the system of equations $\widetilde{\boldsymbol{\pi}}^+\mathbf{P} = \widetilde{\boldsymbol{\pi}}^+$ with $\widetilde{\boldsymbol{\pi}}^+\mathbf{e} = 1$, where $\widetilde{\boldsymbol{\pi}}^+ = (\widetilde{\boldsymbol{\pi}}^+(0), \widetilde{\boldsymbol{\pi}}^+(1), \cdots, \widetilde{\boldsymbol{\pi}}^+(N))$ is a vector of dimension $(N+1)m$.

2.2. *Distribution of the queue length at an arbitrary-epoch*

In this section the joint distribution of the number of customers in the queue and phase of the arrival process at arbitrary-epoch has been obtained. Toward this end we define the state of the system at time t as

- $N_q(t)$ be the number of customers in the queue waiting for service,
- $J(t)$ be the state of the underlying Markov chain of the MAP, and
- $U(t)$ be the remaining service time of a customer in service (if any).
- $\xi(t)$ be the state of the server, that is,

$$\xi(t) = \begin{cases} 1, & \text{if server is busy,} \\ 0, & \text{if server is idle.} \end{cases}$$

Let us define for $1 \leq i \leq m$,

$$p_i(t) = Pr\{N_q(t) = 0, J(t) = i, \xi(t) = 0\},$$
$$\pi_i(n, u;t)du = Pr\{N_q(t) = n,$$
$$J(t) = i, u < U(t) \leq u + du, \xi(t) = 1\}, 0 \leq n \leq N, u \geq 0.$$

Define the steady-state probabilities as

$$p_i = \lim_{t\to\infty} p_i(t), 1 \leq i \leq m,$$
$$\pi_i(n, u) = \lim_{t\to\infty} \pi_i(n, u;t), 1 \leq i \leq m, 0 \leq n \leq N.$$

Let us define the vectors $\boldsymbol{p}$ and $\boldsymbol{\pi}(n, u), 0 \leq n \leq N, u \geq 0$, of order m whose jth components are given by p_j and $\pi_j(n, u)$, respectively. Then relating the state of the system at time t and $t + dt$ and using the supplementary variable technique, in steady-state, we have the following (matrix) differential equations.

$$\mathbf{0} = \boldsymbol{p}\mathbf{C} + \boldsymbol{\pi}(0, 0), \quad (4)$$

$$-\frac{d}{du}\boldsymbol{\pi}(0, u) = \boldsymbol{\pi}(0, u)\mathbf{C} + h_1(u)\boldsymbol{\pi}(1, 0) + h_1(u)\boldsymbol{p}\mathbf{D}, \quad (5)$$

$$-\frac{d}{du}\boldsymbol{\pi}(n, u) = \boldsymbol{\pi}(n, u)\mathbf{C} + h_{n+1}(u)\boldsymbol{\pi}(n+1, 0) + \boldsymbol{\pi}(n-1, u)\mathbf{D}, 1 \leq n \leq N-1, \quad (6)$$

$$-\frac{d}{du}\boldsymbol{\pi}(N, u) = \boldsymbol{\pi}(N, u)(\mathbf{C} + \mathbf{D}) + \boldsymbol{\pi}(N-1, u)\mathbf{D}. \quad (7)$$

Define the Laplace transform of $\boldsymbol{\pi}(n, u)$ as

$$\boldsymbol{\pi}^*(n, s) = \int_0^\infty e^{-su}\boldsymbol{\pi}(n, u)du, 0 \leq n \leq N, \mathbb{R}es \geq 0,$$

and observe that

$$\boldsymbol{\pi}(n) \equiv \boldsymbol{\pi}^*(n, 0) = \int_0^\infty \boldsymbol{\pi}(n, u)du, 0 \leq n \leq N. \quad (8)$$

Now Multiplying (5)–(7) by e^{-su} and integrating with respect to u over 0 to ∞, we have

$$-s\boldsymbol{\pi}^*(0, s) + \boldsymbol{\pi}(0, 0) = \boldsymbol{\pi}^*(0, s)\mathbf{C} + H_1^*(s)\boldsymbol{\pi}(1, 0) + H_1^*(s)\boldsymbol{p}\mathbf{D}, \quad (9)$$

$$-s\boldsymbol{\pi}^*(n, s) + \boldsymbol{\pi}(n, 0) = \boldsymbol{\pi}^*(n, s)\mathbf{C} + H_{n+1}^*(s)\boldsymbol{\pi}(n+1, 0) + \boldsymbol{\pi}^*(n-1, s)\mathbf{D}, 1 \leq n \leq N-1, \quad (10)$$

$$-s\boldsymbol{\pi}^*(N, s) + \boldsymbol{\pi}(N, 0) = \boldsymbol{\pi}^*(N, s)(\mathbf{C} + \mathbf{D}) + \boldsymbol{\pi}^*(N-1, s)\mathbf{D}. \quad (11)$$

Post multiplying 4 and (9)–(11) by the vector $\mathbf{e}$, adding them and using $(\mathbf{C} + \mathbf{D})\mathbf{e} = \mathbf{0}$, we obtain

$$\sum_{n=0}^{N} \boldsymbol{\pi}^*(n, s)\mathbf{e} = \frac{1 - H_1^*(s)}{s}\boldsymbol{\pi}(0, 0)\mathbf{e} + \sum_{n=1}^{N} \frac{1 - H_n^*(s)}{s}\boldsymbol{\pi}(n, 0)\mathbf{e}. \quad (12)$$

Taking limit $s \to 0$ in (12) yields

$$\sum_{n=0}^{N} \boldsymbol{\pi}(n)\mathbf{e} = h_1\boldsymbol{\pi}(0, 0)\mathbf{e} + \sum_{n=1}^{N} h_n\boldsymbol{\pi}(n, 0)\mathbf{e},$$

$$1 - \boldsymbol{p}\mathbf{e} = h_1\boldsymbol{\pi}(0, 0)\mathbf{e} + \sum_{n=1}^{N} h_n\boldsymbol{\pi}(n, 0)\mathbf{e}. \quad (13)$$

Using (8) and the normalizing condition $\boldsymbol{p}\mathbf{e} + \sum_{n=0}^{N} \boldsymbol{\pi}(n)\mathbf{e} = 1$, the above relation has been obtained. Before giving relations between $\{\boldsymbol{p}, \boldsymbol{\pi}(n)\}$ and $\{\widetilde{\boldsymbol{\pi}}^+(n)\}$, let us first derive the following results which will be used later.

It may be noted here that, as $\widetilde{\boldsymbol{\pi}}^+(n)$ and $\boldsymbol{\pi}(n, 0)$ are proportional to each other, hence

$$\widetilde{\boldsymbol{\pi}}^+(n) = d\boldsymbol{\pi}(n, 0), 0 \leq n \leq N, \quad (14)$$

where d is a proportionality constant. The following lemma gives an expression for d.

Table 1 Distribution of the number of customers in the queue at various epoch for $MAP/PH_n/1/30$ queue with $m = 2$ and $\lambda^* = 3.999$.

n	$\widetilde{\boldsymbol{\pi}}^{+}(n)$			$\boldsymbol{\pi}(n)$			$\boldsymbol{\pi}^{-}(n)$		
	$j = 1$	$j = 2$	$\sum_{j=1}^{m} \pi_j^{+}(n)$	$j = 1$	$j = 2$	$\sum_{j=1}^{m} \pi_j(n)$	$j = 1$	$j = 2$	$\sum_{j=1}^{m} \pi_j^{-}(n)$
0	0.00037314	0.00026974	0.00064289	0.00079188	0.00057830	0.00137019	0.00076519	0.00061281	0.00137799
1	0.00079707	0.00058320	0.00138027	0.00142766	0.00104746	0.00247512	0.00138014	0.00110663	0.00248676
2	0.00143637	0.00105451	0.00249088	0.00240720	0.00176993	0.00417713	0.00232754	0.00186732	0.00419487
3	0.00242088	0.00178093	0.00420181	0.00385525	0.00283984	0.00669509	0.00372833	0.00299257	0.00672089
4	0.00387566	0.00285635	0.00673201	0.00588697	0.00434411	0.01023108	0.00569411	0.00457253	0.01026664
5	0.00591594	0.00436769	0.01028362	0.00858789	0.00634813	0.01493603	0.00830792	0.00667450	0.01498242
6	0.00862702	0.00638019	0.01500721	0.01198790	0.00887638	0.02086428	0.01159896	0.00932259	0.02092155
7	0.01203822	0.00891794	0.02095616	0.01603655	0.01189382	0.02793038	0.01551871	0.01247846	0.02799717
8	0.01609827	0.01194521	0.02804349	0.02058734	0.01529369	0.03588104	0.01992564	0.01602881	0.03595445
9	0.02065955	0.01535438	0.03601393	0.02539710	0.01889653	0.04429363	0.02458453	0.01978475	0.04436928
10	0.02547769	0.01896499	0.04444268	0.03014407	0.02246314	0.05260721	0.02918396	0.02349571	0.05267967
11	0.03022988	0.02253693	0.05276682	0.03446372	0.02572090	0.06018462	0.03337087	0.02687720	0.06024806
12	0.03455082	0.02579691	0.06034773	0.03799659	0.02839941	0.06639599	0.03679694	0.02964807	0.06644501
13	0.03808076	0.02847417	0.06655493	0.04043921	0.03026869	0.07070791	0.03916790	0.03157037	0.07073826
14	0.04051646	0.03033883	0.07085529	0.04158777	0.03117231	0.07276008	0.04028584	0.03248350	0.07276934
15	0.04165480	0.03123492	0.07288972	0.04136569	0.03104861	0.07241430	0.04007606	0.03232608	0.07240213
16	0.04142030	0.03110161	0.07252191	0.03983022	0.02993640	0.06976662	0.03859349	0.03114126	0.06973475
17	0.03987143	0.02997868	0.06985011	0.03715776	0.02796460	0.06512235	0.03600861	0.02906560	0.06507422
18	0.03718584	0.02799602	0.06518187	0.03361242	0.02532901	0.05894143	0.03257700	0.02630459	0.05888159
19	0.03362866	0.02535034	0.05897899	0.02950502	0.02226185	0.05176688	0.02859963	0.02310071	0.05170033
20	0.02951140	0.02227446	0.05178586	0.02515098	0.01900007	0.04415104	0.02438211	0.01970051	0.04408263
21	0.02514985	0.01900570	0.04415555	0.02083430	0.01575805	0.03659235	0.02019977	0.01632642	0.03652619
22	0.02082805	0.01575857	0.03658661	0.01678252	0.01270845	0.02949097	0.01627326	0.01315693	0.02943019
23	0.01677329	0.01270559	0.02947888	0.01315425	0.00997246	0.02312671	0.01275652	0.01031680	0.02307332
24	0.01314378	0.00996771	0.02311149	0.01003851	0.00761898	0.01765749	0.00973607	0.00787637	0.01761244
25	0.01002807	0.00761350	0.01764157	0.00746307	0.00567065	0.01313372	0.00723902	0.00585803	0.01309705
26	0.00745343	0.00566529	0.01311871	0.00540809	0.00411397	0.00952206	0.00524632	0.00424678	0.00949311
27	0.00539943	0.00410938	0.00950881	0.00382153	0.00291107	0.00673260	0.00370772	0.00300242	0.00671014
28	0.00381328	0.00290795	0.00672124	0.00263351	0.00201088	0.00464439	0.00255568	0.00207083	0.00462651
29	0.00262352	0.00201064	0.00463416	0.00176748	0.00135874	0.00312622	0.00171639	0.00139327	0.00310966
30	0.00174967	0.00136513	0.00311480	0.00093507	0.00073169	0.00166677	0.00090965	0.00074192	0.00165157
Total	0.57133254	0.42866746	1.00000000	0.57105754	0.42830811	0.99936566	0.55321386	0.44614431	0.99935818

$\boldsymbol{p} = [0.000371030.00026331]$, $\boldsymbol{pe} = .00063434$, $\boldsymbol{p}^{-} = [.00035757.00028426]$, $\boldsymbol{p}^{-}\mathbf{e} = .00064182$
$L_q = 14.71404236$, $W_q = 3.67851059$, $P_{loss} = .00165157$.

Lemma 1. *The value of d, as appeared in (14), is given by*

$$d^{-1} = \sum_{n=0}^{N} \boldsymbol{\pi}(n, 0)\mathbf{e} = \frac{1 - \boldsymbol{pe}}{g}, \tag{15}$$

where $g = h_1 \widetilde{\boldsymbol{\pi}}^{+}(0)\mathbf{e} + \sum_{n=1}^{N} h_n \widetilde{\boldsymbol{\pi}}^{+}(n)\mathbf{e}$.

Proof. Summing both side of (14) over the range of n, the desired result $d^{-1} = \sum_{n=0}^{N} \boldsymbol{\pi}(n, 0)\mathbf{e}$ is obtained. Then dividing (13) by $\sum_{n=0}^{N} \boldsymbol{\pi}(n, 0)\mathbf{e}$ and using (14), the desired result (15) is obtained after little algebraic operation. □

Theorem 1. *The state probabilities* $\{\boldsymbol{p}, \boldsymbol{\pi}(n)\}$ *and* $\{\widetilde{\boldsymbol{\pi}}^{+}(n)\}$ *are related by*

$$\boldsymbol{p} = \mathbf{E}^{*-1} \widetilde{\boldsymbol{\pi}}^{+}(0)(-\mathbf{C})^{-1} \tag{16}$$

$$\begin{aligned} \boldsymbol{\pi}(n) = \mathbf{E}^{*-1}\Bigg[& \widetilde{\boldsymbol{\pi}}^{+}(0)\{(-\mathbf{C})^{-1}\mathbf{D}\}^{n+1}(-\mathbf{C})^{-1} \\ & + \sum_{k=1}^{n+1} \{\widetilde{\boldsymbol{\pi}}^{+}(k) - \widetilde{\boldsymbol{\pi}}^{+}(k-1)\}\{(-\mathbf{C})^{-1}\mathbf{D}\}^{n+1-k}(-\mathbf{C})^{-1}\Bigg], \\ & \times 0 \le n \le N-1, \end{aligned} \tag{17}$$

$$\boldsymbol{\pi}(N) = \boldsymbol{\delta} - \boldsymbol{p} - \sum_{n=0}^{N-1} \boldsymbol{\pi}(n), \tag{18}$$

where $\mathbf{E}^{*} = g + \widetilde{\boldsymbol{\pi}}^{+}(0)(-\mathbf{C})^{-1}\mathbf{e}$ *and g is given in Lemma 1.*

Proof. Dividing (4) by $\sum_{n=0}^{N} \boldsymbol{\pi}(n, 0)\mathbf{e}$ and using (14) and Lemma 1, after little manipulations the desired result (16) is obtained. Now setting $s = 0$ in (9) and (10), using (14) and a recursive procedure, the desired result (17) is obtained with the help of Lemma 1.

The last Eq. (18) follows immediately using normalizing condition $\boldsymbol{p} + \sum_{n=0}^{N} \boldsymbol{\pi}(n) = \boldsymbol{\delta}$. □

Once the probability vectors $\boldsymbol{p}$ and $\boldsymbol{\pi}(n)$ $(0 \le n \le N)$ are obtained, the other distributions of interest such as distribution of the number of customers in the queue at an arbitrary-epoch when server is idle, $\alpha_1(0)$, and busy, $\alpha_2(n)$ $(0 \le n \le N)$, can be easily obtained and are given by

$$\alpha_1(0) = \boldsymbol{pe} + \boldsymbol{\pi}(0)\mathbf{e}, \tag{19}$$

$$\alpha_2(n) = \boldsymbol{\pi}(n)\mathbf{e}, 0 \le n \le N. \tag{20}$$

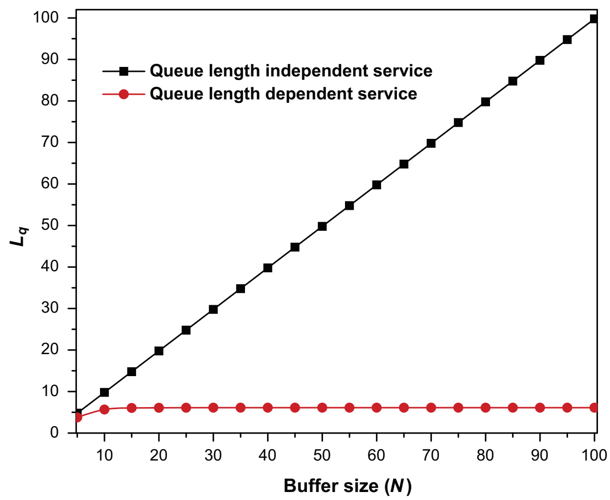

Fig. 2 Effect of N on L_q.

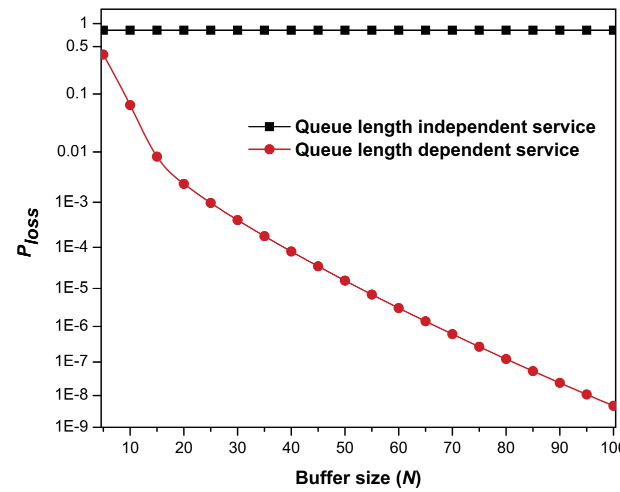

Fig. 4 Effect of N on P_{loss}.

2.3. *Distribution of the queue length at an arrival-epoch*

Let $\boldsymbol{p}^-$ and $\boldsymbol{\pi}^-(n)$ $(0 \le n \le N)$ be the vectors of dimension m whose j-th components are given by p_j^- and $\pi_j^-(n)$, respectively. p_j^- gives the steady-state probability that an arrival finds 0 customer in the queue, server idle, and the state of the arrival process is j. $\pi_j^-(n)$ gives the steady-state probability that an arrival finds n $(0 \le n \le N)$ customer in the queue, server busy, and the state of the arrival process is j. Then the vectors $\boldsymbol{p}^-$ and $\boldsymbol{\pi}^-(n)$ are given by

$$\boldsymbol{p}^- = \frac{\boldsymbol{p}\mathbf{D}}{\lambda^*}, \tag{21}$$

$$\boldsymbol{\pi}^-(n) = \frac{\boldsymbol{\pi}(n)\mathbf{D}}{\lambda^*}, \quad 0 \le n \le N. \tag{22}$$

As $\boldsymbol{p}$ and $\boldsymbol{\pi}(n)$ are known from (16)–(18), one can easily obtain arrival-epoch probabilities using (21) and (22).

3. Computational procedure

In this section, the necessary steps required for the computation of the elements, the matrices $\mathbf{A}_{n,k}$ $(\mathbf{B}_{0,k})$, of *TPM* $\mathbf{P}$ by considering phase type (PH-distribution) service time distribution, are discussed briefly. Computational procedure of $\mathbf{A}_{n,k}$ involves only matrix arithmetic when service time is PH-distribution. It should be noted here that any distribution can be approximated by PH-distribution (viz. exponential, hyperexponential, hypo-exponential, Erlang, Coxian, etc.). PH-distribution can be fully represented by $(\boldsymbol{\beta}, \mathbf{S})$, where $\boldsymbol{\beta}$ and $\mathbf{S}$ are of dimension ν, (i.e., $\boldsymbol{\beta}$ is an initial probability vector and $\mathbf{S}$ is a square matrix governing the transitions to various transition states). For more detail on PH-distribution and their properties authors refer the reader to Nuets [23]. The following theorem gives a procedure for the computation of the matrices $\mathbf{A}_{n,k}$.

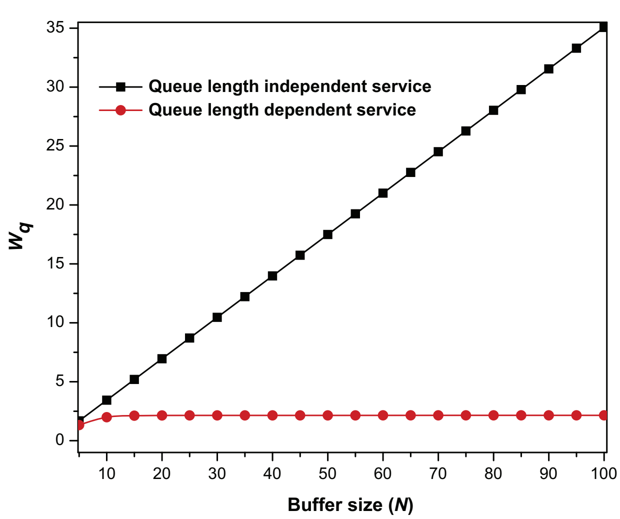

Fig. 3 Effect of N on W_q.

Theorem 2. *Let $H_n(\cdot)$ $(1 \le n \le N)$ follows a PH-distribution with irreducible representation $(\boldsymbol{\beta}_n, \mathbf{S}_n)$, where $\boldsymbol{\beta}_n$ and $\mathbf{S}_n$ are of dimension ν, then the matrices $\mathbf{A}_{n,k}$ $(\mathbf{B}_{0,k})$ appearing in (2) are given by*

$$\mathbf{A}_{n,k} = (\mathbf{I}_m \otimes \boldsymbol{\beta}_n)\mathbf{M}_{n,k}(\mathbf{I}_m \otimes \mathbf{S}_n^0),\ 1 \le n \le N, \quad 0 \le k \le N-n, \tag{23}$$

$$\begin{aligned}\overline{\mathbf{A}}_{n,k} &= (\mathbf{I}_m \otimes \boldsymbol{\beta}_n)[-\mathbf{M}_{n,k-1}(\mathbf{D} \otimes \mathbf{I}_\nu)((\mathbf{C} \oplus \mathbf{S}_n) \\ &\quad + (\mathbf{D} \otimes \mathbf{I}_\nu))^{-1}](\mathbf{I}_m \otimes \mathbf{S}_n^0),\ 1 \\ &\le n \le N, k = N-n+1,\end{aligned} \tag{24}$$

$$\mathbf{B}_{0,k} = \tilde{\mathbf{D}}\ \mathbf{A}_{1,k},\ 0 \le k \le N-1, \tag{25}$$

$$\overline{\mathbf{B}}_{0,N} = \tilde{\mathbf{D}}\ \overline{\mathbf{A}}_{1,N}, \tag{26}$$

where

$$\mathbf{S}_n^0 = -\mathbf{S}_n\mathbf{e},$$
$$\mathbf{M}_{n,k} = \mathbf{M}_{n,k-1}(\mathbf{D} \otimes \mathbf{I}_\nu)\mathbf{M}_{n,0} 1 \le n \le N,\ 1 \le k \le N-n$$
$$\mathbf{M}_{n,0} = -(\mathbf{C} \oplus \mathbf{S_n})^{-1}, \quad 1 \le n \le N-1.$$

Note:- *Here $\mathbf{I}_r$ denotes an identity matrix of dimension r. When there is no need to emphasize the dimension of these vectors we will suppress the suffix. Thus, $\mathbf{I}$ will denote an identity matrix of appropriate dimension. Symbol $\otimes$ and $\oplus$ denotes the Kronecker product and Kronecker sum of two matrices, respectively. If A is* *a matrix of order $m \times n$ and if B is a matrix of order $p \times q$, then $A \otimes B$ will denote a matrix of order $mp \times nq$ whose (i, j)th block matrix is given by $a_{ij}B$. The Kronecker sum of two square matrices, say, G of order g and H of order h, is given by $G \otimes \mathbf{I} + \mathbf{I} \otimes H$, a square matrix of dimension gh. For more details on Kronecker product, we refer the reader to Marcus and Minc [24]. $\widetilde{\mathbf{D}}$ is given in (2).*

Proof. First note that the matrices $\{\tilde{P}(n, t), n \geq 0, t \geq 0\}$ associated with the counting process for MAP satisfy (see, e.g., Neuts and Li [22])

$$\tilde{P}'(n, t) = \tilde{P}(n, t)\mathbf{C} + \tilde{P}(n-1, t)\mathbf{D}, n \geq 0, \tag{27}$$

with $\tilde{P}(-1, t) = \mathbf{0}$ and $\tilde{P}(0, 0) = \mathbf{I}$.

Using the properties of Kronecker product and noticing that $\tilde{P}(k, t) \to \mathbf{0}$ as $t \to \infty$, we see that

$$\mathbf{A}_{n,k} = \int_0^\infty \widetilde{P}(k, t)dH_n(t) = (\mathbf{I}_m \otimes \boldsymbol{\beta}_n)\mathbf{M}_{n,k}\left(\mathbf{I}_m \otimes \mathbf{S}_n^0\right), 1 \leq n \leq N, 0 \leq k \leq N-n, \tag{28}$$

where

$$\mathbf{M}_{n,k} = \int_0^\infty \tilde{P}(k, t) \otimes \mathbf{e}^{\mathbf{S}_n t} dt = \begin{cases} -\left(\mathbf{I}_m \otimes \mathbf{S}_n^{-1}\right) - \int_0^\infty \widetilde{P}'(0, t) \otimes \mathbf{e}^{\mathbf{S}_n t}\mathbf{S}_n^{-1} dt, 1 \leq n \leq N, k = 0, \\ -\int_0^\infty \widetilde{P}'(k, t) \otimes \mathbf{e}^{\mathbf{S}_n t}\mathbf{S}_n^{-1} dt, 1 \leq n \leq N, 1 \leq k \leq N-n. \end{cases} \tag{29}$$

Using (27) in (29) and after some manipulations involving Kronecker products and sums, we get

$$\mathbf{M}_{n,0} = -(\mathbf{C} \oplus \mathbf{S}_n)^{-1}, 1 \leq n \leq N,$$
$$\mathbf{M}_{n,k} = \mathbf{M}_{n,k-1}(\mathbf{D} \otimes \mathbf{I}_\nu)\mathbf{M}_{n,0}, 1 \leq n \leq N, 1 \leq k \leq N-n. \tag{30}$$

On noticing that

$$\overline{\mathbf{A}}_{n,k} = \sum_{l=k}^\infty \mathbf{A}_{n,l} = (\mathbf{I}_m \otimes \boldsymbol{\beta}_n)\widetilde{\mathbf{M}}_n(\mathbf{I}_m \otimes \mathbf{S}_n^0), 1 \leq n \leq N, k = N-n+1, \tag{31}$$

where

$$\widetilde{\mathbf{M}}_n = \sum_{j=k}^\infty \int_0^\infty \tilde{P}(j, t) \otimes \mathbf{e}^{\mathbf{S}_n t} dt = -\mathbf{M}_{n,k-1}(\mathbf{D} \otimes \mathbf{I}_\nu)[(\mathbf{C} \oplus \mathbf{S}_n) + (\mathbf{D} \otimes \mathbf{I}_\nu)]^{-1}, \tag{32}$$

stated expression for $\overline{\mathbf{A}}_{n,k}$ in (24) holds good. Using similar approach the expressions for $\mathbf{B}_{0,k}$, $0 \leq k \leq N-1$, and $\overline{\mathbf{B}}_{0,N}$, follow immediately. □

4. Performance measures

The average number of customers in the (L_q), average waiting time of a customer in the queue (W_q) and loss probability of a customer (P_{loss}) is given by $L_q = \sum_{n=0}^{N} n\alpha_2(n)$, $W_q = \frac{L_q}{\lambda^*}$, $P_{loss} = \boldsymbol{\pi}^-(N)\mathbf{e} = \frac{\boldsymbol{\pi}(N)\mathbf{De}}{\lambda^*}$.

5. Numerical results

In this section, the numerical compatibility of the analytical results, as obtained in the previous sections, of the queueing model under consideration is illustrated. Toward this end, in Table 1, all the distributions of $MAP/PH/1/30$ queue, with queue-length-dependent service, has been displayed for the following input parameters:

The MAP representation is taken as $\mathbf{C} = \begin{pmatrix} -6.5 & 1.0 \\ 3.5 & -5.5 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 3.5 & 2.0 \\ 0.5 & 1.5 \end{pmatrix}$. For service time, PH-distribution is taken as $\boldsymbol{\beta}_n = \begin{pmatrix} 1.0 & 0.0 \end{pmatrix}$ $\mathbf{S}_n = \begin{pmatrix} -\mu_n & \mu_n \\ 0 & -\mu_n \end{pmatrix}$for $1 \leq n \leq N$, where $\mu_n = \mu_{n-1} + 0.2$ for $1 \leq n \leq N$ and $\mu_0 = 5.0$.

To bring out some qualitative aspects of the model under consideration, a comparative study of queue-length-dependent service with queue-length-independent service policy has been carried out for the queueing model under consideration. The performance measures of a queueing system usually reflect both the qualitative and quantitative aspects of the concerned model. It provides the system analyst a powerful tool in making decisions and judging the efficacy of the concerned system. The effect of N (buffer size) on the major system performance measures, L_q, W_q and P_{loss}, respectively, has been illustrated in Figs. 2–4 for $MAP/PH_n/1/N$ (N varies from 5 to 100) queue with the following input parameters: The MAP representation is taken as $\mathbf{C} = \begin{pmatrix} -4.657 & 1.761 \\ 1.128 & -3.941 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1.657 & 1.239 \\ 0.872 & 1.941 \end{pmatrix}$. For service time, PH distribution is taken as $\boldsymbol{\beta}_n = \begin{pmatrix} 0.4 & 0.6 \end{pmatrix}$ $\mathbf{S}_n = \begin{pmatrix} -\mu_n & 0 \\ 0 & -\mu_n \end{pmatrix}$. for $1 \leq n \leq N$.

Let us now consider the following two cases:

Case 1: Queue-length-independent service ($\mu_n = \mu_0$ for $1 \leq n \leq N$ and $\mu_0 = 0.5$)
Case 2: Queue-length-dependent service ($\mu_n = n\mu_0$ for $1 \leq n \leq N$ and $\mu_0 = 0.5$)

It is clear from Figs. 2 and 3 that as N increases L_q and W_q increases with N for both the cases, but for Case 1 this increase in the value of L_q (W_q) is very high in comparison with case 2. So from here we can conclude that queue-length-dependent service considerably decreases the queue length (L_q) and waiting time of customers in the queue (W_q), which is always core objective of system analyst. Now observing the Fig. 4 one can conclude the queueing model under consideration also decreases loss probability which is also another important feature of the queueing model. Since it is clear from Fig. 4 that as N increases P_{loss} decreases for both the cases but this decrease is very high for case 2, i.e., for queue-length-dependent service.

6. Concluding remarks

In this paper, a finite capacity queue with Markovian arrival process has been studied with queue length dependent service. The service time of a customer is considered to be generally distributed and dependent on queue length at service initiation epoch. An efficient computational procedure for computing the steady state probabilities at various epoch has been developed by considering phase type service time distribution. The procedure developed in this paper can be used to analyze more complex queuing models involving batch Markovian arrival process ($BMAP$). One can also extend the present work to analyze discrete time $BMAP$ with queue length dependent service.

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