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Prediction and reconstruction of future and missing unobservable modified Weibull lifetime based on generalized order statistics

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Abstract When a system consisting of independent components of the same type, some appropriate actions may be done as soon as a portion of them have failed. It is, therefore, important to be able to predict later failure times from earlier ones. One of the well-known failure distributions commonly used to model component life, is the modified Weibull distribution (MWD). In this paper, two pivotal quantities are proposed to construct prediction intervals for future unobservable lifetimes based on generalized order statistics (gos) from MWD. Moreover, a pivotal quantity is developed to reconstruct missing observations at the beginning of experiment. Furthermore, Monte Carlo simulation studies are conducted and numerical computations are carried out to investigate the efficiency of presented results. Finally, two illustrative examples for real data sets are analyzed.

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1. Introduction

Prediction of future events (or reconstructing past events which have occurred but were unobservable) on the basis of past and present available information is one of the main problems in statistics. This problem has been extensively studied by many authors, including Lingappaiah [1], Aitchison and Dunsmore [2], Lawless [3,4], Kaminsky and Rhodin [5], Kaminsky and Nelson [6], Patel [7], Raqab et al. [8], Barakat et al. [9], El-Adll [10], El-Adll et al. [11], Barakat et al. [12] and AL-Hussaini et al. [13].

The ordered random variables without any doubt play an important role in such prediction problems. Since Kamps [14] had introduced the concept of gos as a unification of several models of ascendingly ordered random variables, the use of such concept has been steadily growing along the years. This is

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due to the fact that such concept includes important well-known models of ordered random variables that have been treated separately in the statistical literature. Kamps [14] defined gos first by defining uniform gos and then using the quantile transformation to obtain the $X(1, n, \tilde{m}, k), \cdots, X(n, n, \tilde{m}, k)$ based on cumulative distribution function (cdf) F. The joint probability density function (jpdf) of $X(1, n, \tilde{m}, k), \cdots, X(n, n, \tilde{m}, k)$ is given by

$$f^{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1, \ldots, x_n)$$
$$= k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(1 - F(x_i))^{m_i} f(x_i)\right)(1 - f(x_n))^{k-1} f(x_n),$$

on the cone $F^{-1}(0) \le x_1 \le \cdots \le x_n \le F^{-1}(1-)$ of $\mathbb{R}^n$. The model parameters are $n \in \mathbb{N}, n \ge 2$, $k > 0, \tilde{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, \ldots, n-1\}$ and $\gamma_n = k$. Particular choices of the parameters $\gamma_1, \cdots, \gamma_n$ lead to different models, e.g., m-gos ($\gamma_n = k, \gamma_r = k + (n-r)(m+1), r = 1, \cdots, n-1$), oos ($\gamma_n = 1, \gamma_r = n - r + 1, r = 1, \ldots, n-1$, i.e., $k = 1, m_i = 0, i = 1, \cdots, n-1$), sos ($\gamma_n = \alpha_n, \gamma_r = (n - r + 1)\alpha_r, \alpha_r > 0$, $r = 1, \ldots, n-1$), pos with censoring scheme $(R_1, \ldots, R_M)$ ($\gamma_n = R_M + 1, \gamma_r = n - r + 1 + \sum_{j=r}^{M} R_j$, if $r \le M - 1$ and $\gamma_r = n - r + 1 + R_M$, if $r \ge M$) and upper records ($\gamma_r = 1, 1 \le r \le n$, i.e., $k = 1, m_i = -1, i = 1, \ldots, n-1$). Therefore, all the results obtained in the model of gos can be applied to the particular models choosing the respective parameters. For more details in the theory and applications of gos see Kamps [14], Ahsanullah [15], Kamps and Cramer [16], Cramer [17], Barakat et al. [9], El-Adll [18], Barakat [19], Atya [20] and Ahmad et al. [21].

Weibull distribution was originally introduced by the Swedish Waloddi Weibull (see Weibull [22]) which currently can be considered as one of the most important distributions in life testes and reliability engineering. Moreover, for more than 60 years Weibull distribution received increasing attention from several researchers in a wide variety of applications. Because of its various shapes of the probability density function and its convenient representation of the distribution/ survival function, the Weibull distribution has been used very effectively for analyzing lifetime data, particularly when the data are censored, which is very common in most life testing experiments. Moreover, Weibull distribution and its extensions are considered as the most important models in modern statistics because of its ability to fit data from various fields, ranging from life data to weather data or observations made in economics and business administration, in hydrology, in biology, and in the engineering sciences. Also, it has been used in many different areas such as material science, reliability engineering, physics, medicine, pharmacy economics, quality control, biology and other fields (for more details and applications of Weibull distribution see Rinne [23]).

Since 1958, the Weibull distribution has been modified by many researchers to allow for non-monotonic hazard functions. Lai et al. [24] proposed a three-parameter distribution known as MWD by multiplying the Weibull cumulative hazard function, αx^{β}, and $e^{\lambda x}$ which was later generalized to exponentiated form by Carrasco et al. [25]. Recent works of the modified Weibull include Sarhan and Zaindin [26], Sarhan and Apaloo, Atya [27,20] and Almalki and Nadarajah [28].

The pdf of the MWD is given by

$$f(x; \alpha, \lambda, \beta) = \begin{cases} \alpha(\beta + \lambda x)x^{\beta-1}e^{\lambda x}e^{-\alpha x^{\beta}e^{\lambda x}}, & x \ge 0; \\ 0, & x < 0, \end{cases} \tag{1.1}$$

where α, β, λ are positive real numbers. The distribution function (cdf) is

$$F(x; \alpha, \lambda, \beta) = \begin{cases} 0, & x < 0; \\ 1 - e^{-\alpha x^{\beta}e^{\lambda x}}, & x \ge 0. \end{cases} \tag{1.2}$$

The rest of this paper is organized as follows. In Section 2, the predictive pivotal quantities and their exact distributions are obtained. Section 3, includes simulation studies. Some applications for real data are presented in Section 4.

2. Pivotal quantities and their distributions

In this section, three pivotal quantities are proposed, two of them are used to construct prediction intervals for future observations from MWD based on gos, while the third is used to reconstruct missing observations. The cdf for each of the pivotal quantities is derived and then the limits of the predictive confidence interval are obtained. Furthermore, an approximate value of the expected upper limit for each predictive confidence interval is derived.

2.1. Prediction intervals of future observations

Suppose that $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are gos based on MWD with cdf given by (1.2). Define the following two pivotal quantities

$$P_1 := P_1(r, s, n, \tilde{m}, k) = \frac{Y(s, n, \tilde{m}, k) - Y(r, n, \tilde{m}, k)}{Y(r, n, \tilde{m}, k)}, \tag{2.1}$$

$$P_2 := P_2(r, s, n, \tilde{m}, k) = \frac{Y(s, n, \tilde{m}, k) - Y(r, n, \tilde{m}, k)}{T_{r,n}}, \tag{2.2}$$

where

$$Y(i, n, \tilde{m}, k) = \alpha(X(i, n, \tilde{m}, k))^{\beta}e^{\lambda X(i,n,\tilde{m},k)}, \quad i = 1, 2, \ldots, n, \tag{2.3}$$

$$T_{r,n} = \sum_{i=1}^{r}\gamma_i(Y(i, n, \tilde{m}, k) - Y(i-1, n, \tilde{m}, k)), \quad \text{with}$$
$$Y(0, n, \tilde{m}, k) = 0. \tag{2.4}$$

The main aim of this subsection was to derive the exact distributions of P_1 and P_2 and to show that their distributions are free of the original distribution parameters, α, β and λ. The results are formulated in the following two theorems.

Theorem 2.1. *Suppose that $X(1, n, \tilde{m}, k), \ldots, X(r, n, \tilde{m}, k)$ are the first observed gos based on MWD with pdf (1.1) . Then the exact cdf of the pivotal quantity P_1, $F_{P_1}(p_1)$, is given by*

$$F_{P_1}(p_1) = 1 - C_{s-1}\sum_{i=r+1}^{s}\sum_{j=1}^{r}\frac{a_i^{(r)}(s)a_j(r)}{\gamma_i}(\gamma_j + \gamma p_1)^{-1}, \quad p_1 \ge 0, \tag{2.5}$$

where,

$$C_{s-1} = \prod_{j=1}^{s}\gamma_j, a_i(r) = \prod_{j=1 j\ne i}^{r}\frac{1}{\gamma_{j,n} - \gamma_{i,n}}, 1 \le i \le r \le n,$$

$$\text{and} a_i^{(r)}(s) = \prod_{j=r+1 j\ne i}^{s}\frac{1}{\gamma_{j,n} - \gamma_{i,n}}, r + 1 \le i \le s \le n.$$

Consequently, an observed $100(1-\delta)\%$ predictive confidence interval (PCI) for $X(s, n, \tilde{m}, k), s > r$ is (ℓ, u_1), where $\ell = x_r$, and u_1 can be computed numerically from the relation

$$u_1^{\beta}e^{\lambda u_1} = (1+p_{1,\delta})x_r^{\beta}e^{\lambda x_r}. \tag{2.6}$$

Moreover, the expected value of the upper limit of a $100(1-\delta)\%$ *PCI of* $X(s,n,\tilde{m},k)$ *can be approximated by solving the nonlinear equation*

$$(E[U_1])^{\beta}e^{\lambda E[U_1]} = \frac{1}{\alpha}(1+p_{1,\delta})\sum_{i=1}^{r}\gamma_i^{-1}, \tag{2.7}$$

where x_r *is an observed value of* $X(r,n,\tilde{m},k)$ *and* $p_{1,\delta}$ *satisfies the nonlinear equation* $F_{P_1}(p_{1,\delta}) = 1-\delta$.

Proof. The joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $f_{r,s}(x_r,x_s)$, was derived in [16]. Namely,

$$f_{r,s}(x_r,x_s) = C_{s-1,n}\sum_{i=r+1}^{s}\sum_{j=1}^{r}a_i^{(r)}(s)a_j(r)\left(\frac{\overline{F}(x_s)}{\overline{F}(x_r)}\right)^{\gamma_{i,n}} \times \left(\overline{F}(x_r)\right)^{\gamma_{j,n}}\frac{f(x_r)}{\overline{F}(x_r)}\frac{f(x_s)}{\overline{F}(x_s)}, \quad r<s\le n,\ x_r<x_s. \tag{2.8}$$

For simplicity, we write X_i instead of $X(i,n,\tilde{m},k)$ and Y_i instead of $Y(i,n,\tilde{m},k)$. Since the transformations, $y_r = \alpha x_r^{\beta}e^{\lambda x_r}$ and $p_1 = (\frac{x_s}{x_r})^{\beta}e^{\lambda(x_s-x_r)}-1$, are monotone increasing from $(0,\infty)\times(0,\infty)$ into $(0,\infty)\times(0,\infty)$, the joint pdf of P_1 and Y_r can be obtained by a standard method of transformations of random variables. That is,

$$\begin{aligned} f_{P_1,y_r}(p_1,y_r) &= |J|f_{X_r,X_s}(x_r(p_1,y_r),x_s(p_1,y_r)) \\ &= C_{s-1}\sum_{i=r+1}^{s}\sum_{j=1}^{r}a_i^{(r)}(s)a_j(r)y_r e^{-(\gamma_{j,n}+\gamma_{i,n}p_1)y_r}, \\ &\quad p_1>0,\ y_r>0, \end{aligned}$$

where

$$|J| = \left|\frac{\partial x_r}{\partial y_r}\frac{\partial x_s}{\partial p_1}\right| = \frac{x_r x_s^{1-\beta}}{\alpha(\beta+\lambda x_r)(\beta+\lambda x_s)e^{\lambda x_s}}.$$

By noting that $f_{P_1}(p_1) = \int_0^{\infty} f_{P_1,y_r}(p_1,y_r)dy_r$, we have

$$f_{P_1(p_1)} = C_{s-1}\sum_{i=r+1}^{s}\sum_{j=1}^{r}a_i^{(r)}(s)a_j(r)[\gamma_{j,n}+\gamma_{i,n}p_1]^{-2}, \quad p_1>0.$$

Hence (2.5) follows directly by evaluating the integration $\int_0^{p_1} f_{P_1}(u)du$. The limits of a $100(1-\delta)\%$ PCI of X_s can be obtained by noting that $F_{P_1}(p_{1,\delta}) = Pr(P_1\le p_{1,\delta}) = 1-\delta$. Which can be rewritten as

$$Pr(X_r^{\beta}e^{\lambda X_r} \le X_s^{\beta}e^{\lambda X_s} \le (1+p_{1,\delta})X_r^{\beta}e^{\lambda X_r}) = 1-\delta. \tag{2.9}$$

Clearly, a lower limit of the future observation x_s is the preceding observed value x_r. On the other hand, the actual lower limit of PCI defined by (2.9) is a value ℓ satisfied the equation $x_r^{\beta}e^{\lambda x_r} = \ell^{\beta}e^{\lambda\ell}$. Since the equation $f(x) = x^{\beta}e^{\lambda x}$ is monotone increasing (for all $x>0$), then $f(x_r) = f(\ell)$ has a unique solution, which is $\ell = x_r$. This shows that the actual lower limit of PCI defined by (2.9) is x_r. An approximate upper limit, u_1 can be obtained by solving the nonlinear Eq. (2.6). The expected value of the upper limit can be approximated using (2.9) by the following sequence of inequalities

$$\begin{aligned} (E[X_s])^{\beta}e^{\lambda E[X_s]} &\le E\left[X_s^{\beta}e^{\lambda X_s}\right] \le E\left[(1+p_{1,\delta})X_r^{\beta}e^{\lambda X_r}\right] \\ &= E[(1+p_{1,\delta})Y_r] = (1+p_{1,\delta})\sum_{i=1}^{r}\gamma_i^{-1}, \end{aligned}$$

which completes the proof of the theorem. □

Lemma 2.1. *The random variable* $T_{r,n}$ *defined by (2.4) follows* $\Gamma(r,1)\equiv\text{gamma}(r,1)$ *distribution with shape parameter r and scale parameter 1. Moreover, the random variables* $T_{r,n}$ *and the subrange* $W_{r,s} = Y(s,n,\tilde{m},k)-Y(r,n,\tilde{m},k)$ *are independent.*

Proof. It can be proved that the random variables $Y(i,n,\tilde{m},k)$, $i=1,2,\ldots,n$ are gos based on standard exponential distribution $Exp(1)$ by obtaining their joint pdf. The proof is similar to the proof of Lemma 2.1 of [11] with suitable modifications. Therefore, by Theorem 3.5.5 of [14], $T_{r,n}\sim\Gamma(r,1)$. Furthermore $W_{r,s}$ can be written as

$$\begin{aligned} W_{r,s} &= Y(s,n,\tilde{m},k)-Y(r,n,\tilde{m},k) \\ &= \sum_{i=r+1}^{s}\left(Y(i,n,\tilde{m},k)-Y(i-1,n,\tilde{m},k)\right) \\ &= \sum_{i=r+1}^{s}Z(i,n,\tilde{m},k)/\gamma_{i,n}. \end{aligned}$$

where the normalizing spacings $Z(i,n,\tilde{m},k)$, $i=1,2,\ldots,n$, are independent and identically distributed according to Exp(1) (see [14] Theorem 3.3.5). Therefore, $W_{r,s}$ is independent of $Z(i,n,\tilde{m},k)$, $i=1,2,\ldots,r$. Hence the lemma. □

Theorem 2.2. *Under the same conditions of Theorem 2.1, the cdf of the pivotal quantity,* P_2*, takes the form*

$$F_{P_2}(p_2) = 1-\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i}(1+\gamma_i p_2)^{-r}, \quad p_2\ge 0. \tag{2.10}$$

Therefore, an observed $100(1-\delta)\%$ *PCI for* $X(s,n,\tilde{m},k)$, $s>r$ *is* (ℓ,u_2)*, where* $\ell = x_r$*, and* u_2 *can be obtained numerically from the relation*

$$\alpha u_2^{\beta}e^{\lambda u_2} = t_{r,n}p_{2,\delta}+\alpha x_r^{\beta}e^{\lambda x_r}.$$

Furthermore, the expected value of the upper limit for the PCI of $X(s,n,\tilde{m},k)$ *can be approximated from the nonlinear equation*

$$\alpha(E[U_2])^{\beta}e^{\lambda E[U_2]} = rp_{2,\delta}+\sum_{i=1}^{r}\gamma_i^{-1}, \tag{2.11}$$

where $t_{r,n}$ *is an observed values of* $T_{r,n}$ *and* $p_{2,\delta}$ *satisfies the nonlinear equation,* $F_{P_2}(p_{2,\delta}) = 1-\delta$.

After obtaining the distribution of $W_{r,s}$, the proof of Theorem 2.2 became similar to the proof of Theorem 2.1 with suitable modifications, so we omitted it.

2.2. Reconstructing missing observations

In this subsection a pivotal quantity is introduced to reconstruct missing observations. The proposed pivotal quantity is defined as

$$P_3 := P_3(r, s, n, \tilde{m}, k) = \frac{Y(s, n, \tilde{m}, k) - Y(r, n, \tilde{m}, k)}{Y(s, n, \tilde{m}, k)}, \qquad (2.12)$$

where $Y(i, n, \tilde{m}, k), i = 1, 2, \ldots, n$ are defined by (2.3).

Theorem 2.3. *Based on MWD, assume that the first gos, $X(1, n, \tilde{m}, k), \ldots, X(s-1, n, \tilde{m}, k)$ are missing and that $X(s, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are observed gos. Then the cdf of the pivotal quantity, P_3, is*

$$F_{P_3}(p_3) = P(P_3 \le p_3) = C_{s-1} \sum_{i=r+1}^{s} \sum_{j=1}^{r} a_i^{(r)}(s) a_j(r) \frac{p_3}{\gamma_j[\gamma_j + (\gamma_i - \gamma_j)p_3]}, \quad 0 < p_3 \le 1. \qquad (2.13)$$

Moreover, a $100(1-\delta)\%$ observed reconstructive confidence interval (RCI) for $X(r, n, \tilde{m}, k), r < s$ is (ℓ_3, u), where $u = x_s$, and ℓ_3 can be calculated numerically from the relation $\ell_3^{\beta} e^{\lambda \ell_3} = (1 - p_{3,\delta}) x_s^{\beta} e^{\lambda x_s}$, where x_s is an observed value of $X(s, n, \tilde{m}, k)$. In addition,a $100(1-\delta)\%$ the expected lower limit for the RCI of $X(r, n, \tilde{m}, k)$ based on P_3, can be approximated by solving the nonlinear equation,

$$(E[L_3])^{\beta} e^{\lambda E[L_3]} = \frac{1}{\alpha}(1 - p_{3,\delta}) \sum_{i=1}^{s} \gamma_i^{-1}, \qquad (2.14)$$

where $p_{3,\delta}$ satisfies the nonlinear equation $F_{P_3}(p_{3,\delta}) = 1 - \delta$.

Proof. As in the proof of Theorem 2.1, the joint pdf of P_3 and Y_s can be obtained and written as

Table 1 95% coverage probability, simulated average upper limits, observed average upper limits and expected intervals width based on P_1, and P_2, respectively, for oos model from $MWD(0.3, 1.25, 0.5)$.

r	s	$CP_{P_1}\%$	$CP_{P_2}\%$	$L = \overline{X}_r^*$	$\overline{X}_s^*$	$\overline{X}_{s+1}^*$	$E[U_{P_1}]$	$E[U_{P_2}]$	U_{P_1} $(RMSE_{P_1})$	U_{P_2} $(RMSE_{P_2})$
9	10	94.975	94.983	1.0704	1.1619	1.2539	1.3861	1.3851	1.3587 (0.1711)	1.3582 (0.1705)
	11	94.998	94.982	1.0704	1.2539	1.3476	1.5591	1.5574	1.5284 (0.2491)	1.5273 (0.2478)
	12	95.100	95.049	1.0704	1.3476	1.4433	1.7143	1.7120	1.6808 (0.3123)	1.6791 (0.3103)
	13	94.954	95.007	1.0704	1.4433	1.5436	1.8634	1.8605	1.8273 (0.3663)	1.8251 (0.3638)
	14	94.950	94.952	1.0704	1.5436	1.6496	2.0120	2.0085	1.9733 (0.4146)	1.9708 (0.4116)
	15	94.950	94.961	1.0704	1.6496	1.7638	2.1644	2.1605	2.1233 (0.4590)	2.1204 (0.4555)
	16	94.976	94.990	1.0704	1.7638	1.8921	2.3258	2.3215	2.2822 (0.5009)	2.2789 (0.4970)
	17	94.961	94.955	1.0704	1.8921	2.0422	2.5035	2.4987	2.4574 (0.5403)	2.4537 (0.5359)
	18	95.056	95.088	1.0704	2.0422	2.2357	2.7108	2.7055	2.6618 (0.5743)	2.6578 (0.5696)
	19	95.014	95.021	1.0704	2.2357	2.5417	2.9783	2.9727	2.9259 (0.5976)	2.9216 (0.5928)
12	13	94.991	94.968	1.3476	1.4433	1.5436	1.6673	1.6655	1.6423 (0.1707)	1.6411 (0.1694)
	14	94.952	94.962	1.3476	1.5436	1.6496	1.8491	1.8458	1.8216 (0.2467)	1.8193 (0.2442)
	15	94.999	94.970	1.3476	1.6496	1.7638	2.0199	2.0153	1.9902 (0.3092)	1.9868 (0.3055)
	16	95.058	95.021	1.3476	1.7638	1.8921	2.1938	2.1879	2.1619 (0.3647)	2.1575 (0.3599)
	17	95.022	94.987	1.3476	1.8921	2.0422	2.3810	2.3740	2.3470 (0.4156)	2.3416 (0.4097)
	18	95.083	95.054	1.3476	2.0422	2.2357	2.5965	2.5884	2.5602 (0.4614)	2.5539 (0.4544)
	19	95.122	95.097	1.3476	2.2357	2.5417	2.8726	2.8634	2.8335 (0.5024)	2.8263 (0.4950)
15	16	95.201	95.206	1.6496	1.7638	1.8921	2.0213	2.0173	1.9959 (0.1956)	1.9936 (0.1932)
	17	95.068	95.110	1.6496	1.8921	2.0422	2.2506	2.2433	2.2228 (0.2828)	2.2179 (0.2777)
	18	95.069	95.034	1.6496	2.0422	2.2357	2.4915	2.4811	2.4614 (0.3571)	2.4540 (0.3490)
	19	95.069	95.094	1.6496	2.2357	2.5417	2.7875	2.7741	2.7548 (0.4274)	2.7449 (0.4177)

Table 2 95% coverage probability, average lower limit, $\overline{X}_s^*$, expected values of the upper limits, average upper limits, and estimated root-mean-square errors for pos model from $MWD(0.03, 0.25, 0.1)$, with pos scheme $R_1 = R_2 = R_3 = 0$, $R_4 = R_5 = R_6 = 1 R_7 = 2$, $R_8 = 3$, $R_9 = 4$, $R_{10} = 5$, $R_{11} = 6$, $R_{12} = 7$ based on P_1, and P_2, respectively.

r	s	$CP_{P_1}\%$	$CP_{P_2}\%$	$L = \overline{X}_r^*$	$\overline{X}_s^*$	$E[U_{P_1}]$	$E[U_{P_2}]$	U_{P_1} $(RMSE_{P_1})$	U_{P_2} $(RMSE_{P_2})$
6	7	95.001	95.001	9.9738	11.4717	15.2452	15.2423	14.6051 (2.7028)	14.6036 (2.7016)
	8	95.058	95.059	9.9738	12.8885	17.7208	17.7164	17.0484 (3.7377)	17.0456 (3.7351)
	9	95.056	95.080	9.9738	14.2955	19.8542	19.8486	19.1609 (4.4538)	19.1572 (4.4502)
	10	95.117	95.112	9.9738	15.7804	21.9537	21.9471	21.2442 (4.9690)	21.2397 (4.9645)
	11	95.009	95.010	9.9738	17.5232	24.3194	24.3121	23.5956 (5.2668)	23.5903 (5.2617)
7	8	95.070	95.085	11.4717	12.8885	16.3523	16.3482	15.7856 (2.4692)	15.7838 (2.4676)
	9	95.079	95.087	11.4717	14.2955	18.7858	18.7791	18.1961 (3.4232)	18.1925 (3.4198)
	10	95.106	95.149	11.4717	15.7804	21.0384	21.0298	20.4326 (4.0934)	20.4276 (4.0886)
	11	95.022	95.020	11.4717	17.5232	23.5162	23.5060	22.8966 (4.5499)	22.8902 (4.5437)
8	9	95.094	95.103	12.8885	14.2955	17.6233	17.6166	17.1090 (2.3759)	17.1065 (2.3729)
	10	95.007	95.031	12.8885	15.7804	20.1758	20.1648	19.6438 (3.2861)	19.6383 (3.2806)
	11	94.980	94.961	12.8885	17.5232	22.8179	22.8033	22.2720 (3.9416)	22.2638 (3.9337)
9	10	95.081	95.090	14.2955	15.7804	19.1598	19.1474	18.6850 (2.4210)	18.6802 (2.4156)
	11	95.028	94.978	14.2955	17.5232	22.1252	22.1042	21.6353 (3.3749)	21.6244 (3.3641)
10	11	94.878	94.894	15.7804	17.5232	21.2388	21.2118	20.7889 (2.7696)	20.7775 (2.7578)

$$f_{P_3,y_s}(p_3,y_s) = C_{s-1}\sum_{i=r+1}^{s}\sum_{j=1}^{r} a_i^{(r)}(s)a_j(r)y_s e^{-[\gamma_{j,n}+(\gamma_{i,n}-\gamma_{j,n})p_3]y_s},$$
$$0 < p_3 \le 1, \quad y_s > 0, \tag{2.15}$$

Thus the pdf of P_3 takes the form

$$f_{P_3(p_3)} = C_{s-1}\sum_{i=r+1}^{s}\sum_{j=1}^{r} a_i^{(r)}(s)a_j(r)\left[\gamma_{j,n}+(\gamma_{i,n}-\gamma_{j,n})p_3\right]^{-2},$$
$$0 < p_3 \le 1.$$

Therefore, we get (2.13). The rest of the proof is similar to the proof of Theorem 2.1. □

3. Simulation

In this section, simulation studies are carried out to demonstrate the efficiency of the theoretical results presented in the previous Section. For this purpose, the following three special cases from gos model are considered.

1. oos with $\gamma_i = n - i + 1$ for $n = 20$, $r = 9, 12, 15$ and $s = r + 1, r + 2, \ldots, n - 1$.
2. pos with $\gamma_{n,n} = k = R_n + 1$, $\gamma_{r,n} = N - r + 1 - \sum_{j=1}^{r-1} R_j = n - r + 1 + \sum_{j=r}^{n} R_j$, and $N = n + \sum_{j=1}^{n} R_j$, is the total items put on a life test, $R_j \in \mathbb{N}_0$. for $n = 12$, $r = 6, 7, 8, 9, 10$ and $s = r + 1, r + 2, \ldots, n - 1$, for two different censoring schemes.

The estimated root-mean-square errors for the upper PCI (lower RCI), are obtained from the relations

$$RMSE_{P_i} = \sqrt{\frac{1}{M-1}\sum_{j=1}^{M}(U_{P_i}(j) - X^*_{s+1}(j))^2}, \quad i = 1, 2, \tag{3.1}$$

$$RMSE_{P_3} = \sqrt{\frac{1}{M-1}\sum_{j=1}^{M}(L_{P_3}(j) - X^*_{r-1}(j))^2}, \tag{3.2}$$

where $U_{P_i}(j)$, $i = 1, 2$ denote the upper limits for the PCI of the jth sample, $L_{P_3}(j)$ is the lower limit for RCI of the jth sample and $X^*_i(j)$ denote the ith gos for the jth sample, $i = r - 1$ or $s + 1$. To apply the methods presented in Theorems 2.1, 2.2 and 2.3, simulation studies are performed. For this purpose, an algorithm is constructed to generate gos samples based on MWD. Moreover, the algorithm is used for obtaining the percent of the

Table 3 95% coverage probability, average lower limit, $\overline{X}^*_s$, expected values of the upper limits, average upper limits, and estimated root-mean-square errors for pos model from $MWD(0.03, 0.25, 0.1)$, with pos scheme $R_1 = 7$, $R_2 = 6$, $R_3 = 5$, $R_4 = 4$, $R_5 = 3$, $R_6 = 2$, $R_7 = R_8 = R_9 = 1$, $R_{10} = R_{11} = R_{12} = 0$. based on P_1, and P_2, respectively.

r	s	$CP_{P_1}\%$	$CP_{P_2}\%$	$L = \overline{X}^*_r$	$\overline{X}^*_s$	$E[U_{P_1}]$	$E[U_{P_2}]$	U_{P_1} $(RMSE_{P_1})$	U_{P_2} $(RMSE_{P_2})$
6	7	95.032	95.001	14.8565	17.7579	23.9931	23.7931	23.1611 (4.3934)	23.0574 (4.2893)
	8	95.092	95.062	14.8565	20.5214	28.1155	27.8295	27.2611 (5.6634)	27.0845 (5.4792)
	9	95.059	95.086	14.8565	23.3754	31.8363	31.4955	30.9673 (6.3255)	30.7416 (6.0865)
	10	95.106	95.124	14.8565	26.7235	36.0222	35.6446	35.1407 (7.0813)	34.8816 (6.7998)
	11	95.016	94.972	14.8565	30.2375	40.0649	39.6621	39.1741 (7.2169)	38.8916 (6.9184)
7	8	95.102	95.085	17.7579	20.5214	26.4135	26.1673	25.6352 (4.1063)	25.5126 (3.9821)
	9	95.097	95.116	17.7579	23.3754	30.7345	30.3710	29.9391 (5.2931)	29.7186 (5.0673)
	10	95.146	95.152	17.7579	26.7235	35.2393	34.7981	34.4309 (6.3230)	34.1422 (6.0167)
	11	95.023	94.996	17.7579	30.2375	39.4368	38.9448	38.6194 (6.6567)	38.2840 (6.3104)
8	9	95.099	95.103	20.5214	23.3754	29.2834	28.9940	28.5421 (4.0941)	28.4000 (3.9477)
	10	94.989	95.031	20.5214	26.7235	34.4059	33.9678	33.6498 (5.5297)	33.3841 (5.2535)
	11	94.994	94.926	20.5214	30.2375	38.8376	38.3082	38.0722 (6.1215)	37.7247 (5.7730)
9	10	95.013	95.090	23.3754	26.7235	33.2494	32.8706	32.5214 (4.5042)	32.3295 (4.3035)
	11	94.981	94.975	23.3754	30.2375	38.1774	37.6258	37.4388 (5.5472)	37.0984 (5.2139)
10	11	94.919	94.894	26.7235	30.2375	36.9747	36.4350	36.2020 (4.6385)	35.9266 (4.3775)

coverage probability, the lower and upper limits as well as the expected values of the upper(lower) limits, and the root-mean-square errors defined by (3.1) and (3.2), respectively.

Algorithm.

Step 1 Choose the values of the MWD parameters α, β, and λ.
Step 2 Determine the values of r, s, and n.
Step 3 Select the gos sub model (here oos and pos).
Step 4 Solve the nonlinear equations $F_{P_i}(p_{i,\delta}) = 1 - \delta$, to obtain the values of $p_{i,\delta}$ for $i = 1, 2$ or 3 at $\delta = 0.05$, where $F_{P_i}(p_i)$, $i = 1, 2, 3$ are given by (2.5), (2.10), and (2.13), respectively.
Step 5 Solve the nonlinear Eqs. (2.7), (2.11) and (2.14) to approximate $E[U_{P_1}]$, $E[U_{P_2}]$ or $E[L_{P_3}]$.
Step 6 Generate n gos, i.e. $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$, based on MWD with parameters α, β, and λ. (by developing the algorithm in [9,18] to MWD).
Step 7 Define three counters, c_i, $i = 1, 2, 3$ to determine if the observed value of the sth gos for PCI (or rth for RCI) lies within the interval or not.
Step 8 Calculate the lower and the upper limits of the PCI (or RCI) based on the pivotal quantities P_1, P_2, (or P_3), using Theorems 2.1,2.2,2.3, respectively.
Step 9 Repeat steps 6, 7, and 8, $M = 100,000$ times.
Step 10 Compute the percent of coverage probability, $100 \times \frac{c_i}{M}$, for each $i = 1, 2, 3$ and the average of the upper (lower) limits based on P_1, P_2, (or P_3).
Step 11 Compute the root-mean-square errors by relations (3.1) and (3.2).

Remark. Clearly, the quantal function of the MWD has no explicit form. Therefore each gos, X^*, can be generated by solving the nonlinear equation $F(X^*) = 1 - \prod_{i=1}^{r} W_i$, with respect to X^*, where W_i is a random number generated from beta

Table 4 95% coverage probability, expected values of the lower limits, average lower limits, $\overline{X}^*_{r-1}$, $\overline{X}^*_r$, $\overline{X}^*_s$ estimated root-mean-square errors and coefficient of variation $C.V.$ for oos model from $MWD(0.0025, 2.25, 0.01)$.

r	s	$CP_{P_3}\%$	$E[L_{P_3}]$	$\overline{L}_{P_3}$	$\overline{X}^*_{r-1}$	$\overline{X}^*_r$	$U = \overline{X}^*_s$	$RMSE_{P_1}$	$C.V.$
10	9	94.97	8.4292	8.3306	8.7798	9.3679	9.9372	0.8641	0.1037
	8	94.62	7.5209	7.4331	8.1779	8.7798	9.9372	1.1529	0.1551
	7	94.59	6.7051	6.6270	7.5552	8.1779	9.9372	1.3569	0.2047
	6	94.87	5.9135	5.8448	6.8797	7.5552	9.9372	1.4947	0.2557
	5	94.75	5.1081	5.0489	6.1528	6.8797	9.9372	1.5992	0.3168
	4	94.73	4.2636	4.2142	5.3381	6.1528	9.9372	1.6658	0.3953
	3	94.86	3.3372	3.2987	4.3442	5.3381	9.9372	1.6620	0.5038
	2	95.17	2.2693	2.2431	2.9744	4.3442	9.9372	1.5306	0.6824
8	7	95.14	7.2029	7.0854	7.5552	8.1779	8.7798	0.9044	0.1276
	6	95.26	6.2190	6.1178	6.8797	7.5552	8.7798	1.1983	0.1959
	5	95.06	5.3103	5.2240	6.1528	6.8797	8.7798	1.4018	0.2683
	4	95.01	4.3949	4.3236	5.3381	6.1528	8.7798	1.5393	0.3560
	3	95.00	3.4182	3.3629	4.3442	5.3381	8.7798	1.5877	0.4721
	2	95.22	2.3109	2.2735	2.9744	4.3442	8.7798	1.4989	0.6593
6	5	94.79	5.7980	5.6753	6.1528	6.8797	7.5552	0.9866	0.1738
	4	94.90	4.6675	4.5690	5.3381	6.1528	7.5552	1.2947	0.2834
	3	94.99	3.5724	3.4972	4.3442	5.3381	7.5552	1.4550	0.4161
	2	94.92	2.3861	2.3360	2.9744	4.3442	7.5552	1.4490	0.6203
4	3	95.18	4.0175	3.8925	4.3442	5.3381	6.1528	1.1211	0.2880
	2	95.06	2.5693	2.4896	2.9744	4.3442	6.1528	1.3337	0.5357

Table 5 Fitting the data of Example 4.1 to MWD based on two different methods for complete and censoring samples with comparison.

Method	Estimates of parameters (complete sample)	$\mathcal{L}$	AIC	BIC	AIC_c	K–S	p-value
MLE's	$\hat{\alpha} = 0.0352166$, $\hat{\beta} = 0.0185766$, $\hat{\lambda} = 0.0735069$	−38.517	83.035	85.868	84.447	0.1480	0.746
LSE's	$\hat{\alpha} = 0.0000173$, $\hat{\beta} = 2.93646$, $\hat{\lambda} = 5.86338 \times 10^{-13}$	−35.915	77.831	80.664	79.242	0.0921	0.992
	Estimates of parameters (Type II right censoring)						
MLE's	$\hat{\alpha} = 0.0084041$, $\hat{\beta} = 0.0447932$, $\hat{\lambda} = 0.11646$	−23.675	53.351	56.184	54.763	0.18829	0.456
MLSE's	$\hat{\alpha} = 0.000241344$, $\hat{\beta} = 1.76035$, $\hat{\lambda} = 0.0439156$	−23.449	52.898	55.674	55.731	0.11409	0.942

$\mathcal{L}$ denote the log-likelihood function computed at the estimated parameters.

Table 6 Upper limits and their expected values for 99% PCI of X_s^*, $s = 15, 16, 17, 18, 19$.

r	s	$L = X_r^*$	X_s^*	X_{s+1}^*	$E[U_{P_1}]$	$E[U_{P_2}]$	U_{P_1}	U_{P_2}
14	15	43	47	51	51.4281	51.3157	50.7331	50.6960
	16	43	51	55	55.0885	54.9179	54.3724	54.3013
	17	43	55	55	58.7576	58.5392	58.0220	57.9224
	18	43	55	68	63.1233	62.8637	62.3664	62.2437
	19	43	68	–	69.9820	69.6897	69.1950	69.0607

Table 7 Fitting the data of Example 4.2 to MWD based on two different methods for complete and censoring samples with comparison.

Method	Estimates of parameters (complete sample)	$\mathcal{L}$	AIC	BIC	AIC_c	K–S	p-value
MLE's	$\hat{\alpha} = 0.0988712, \hat{\beta} = 0.0709469, \hat{\lambda} = 0.711829$	7.712	−9.424	−6.436	−8.090	0.268	0.093
LSE's	$\hat{\alpha} = 0.152408, \hat{\beta} = 2.18851, \hat{\lambda} = 3.76463 \times 10^{-17}$	14.229	−22.459	−19.471	−21.125	0.145	0.741
	Estimates of parameters (type II right censoring)						
MLE's	$\hat{\alpha} = 0.0084041, \hat{\beta} = 0.0447932, \hat{\lambda} = 0.11646$	−23.675	53.351	56.184	54.763	0.18829	0.456
MLSE's	$\hat{\alpha} = 0.146178, \hat{\beta} = 2.25134, \hat{\lambda} = 3.75932 \times 10^{-11}$	15.304	−24.608	−21.621	−23.275	0.1397	0.780

Table 8 Upper limits and their expected values for 99% PCI of X_s^*, $s = 16, 17, 18, 19, 20$.

r	s	$L = X_r^*$	X_s^*	X_{s+1}^*	$E[U_{P_1}]$	$E[U_{P_2}]$	U_{P_1}	U_{P_2}
15	16	2.6260	2.7780	2.9510	3.4746	3.4611	3.4397	3.4211
	17	2.6260	2.9510	3.4130	3.9243	3.9017	3.8849	3.8549
	18	2.6260	3.4130	4.1180	4.4219	4.3899	4.3774	4.3358
	19	2.6260	4.1180	5.1360	5.0813	5.0387	5.0302	4.9751
	20	2.6260	5.1360	–	6.2846	6.2273	6.2214	6.1468

distribution with parameters $1, \gamma_i$. The computations are carried out by Mathematica 9 and the results are presented in Tables 1–4.

4. Illustrative examples

In this section, two real data sets are analyzed to explain the practical importance of the presented methods.

Example 4.1 (Sulfur Dioxide (1-Hour Averages))**.** The first data set presented here were obtained through the courtesy of the South Coast Air Pollution Control District (SCAPCD) of the State of California which was analyzed by Roberts [29]. The annual maxima of sulfur dioxide $1 - hr$ average concentrations (pphm) are,

47 41 68 32 27 43 20 27 25 18 33 40 51
55 40 55 37 28 34.

(Long Beach, CA from 1956 to 1974, Data Courtesy South Coast Air Pollution Control District)

Firstly, it is shown that (see Table 5), the MWD fit the data well. The distribution parameters are estimated by maximum likelihood (ML) and the least square (LS) methods. Based on Kolmogorov–Smirnov (K–S) test statistics (Kolmogorov [30]), the Akaike information criterion (AIC), Bayesian information criteria (BIC), corrected Akaike information criterion (AIC_C) (see Akaike [31], Schwarz [32] and Bozdogan [33], Hurvich [34]), the LS gives better fitting than ML for the complete data. Moreover, an application to the modified least square method (MLS) for censoring data which has been introduced by El-Adll and Aly [35], reveals that it is also better than ML for censoring data of our example. The modified least square estimates(MLSE's) of parameters can be obtained by minimizing the function,

$$LS^*(\alpha, \lambda, \beta|\mathbf{x}) = \sum_{i=1}^{r}\left(F(x_{i:n}; \alpha, \lambda, \beta) - \frac{i}{n+1}\right)^2 + (n-r)\left(F(x_{r:n}; \alpha, \lambda, \beta) - \frac{r}{n+1}\right)^2,$$

with respect to the parameters α, λ and β.

In Table 6, we obtain 99% PCI for X_s^*, $s = 15, 16, 17, 18, 19$, based on the first 14 observations. Since the last five observations are assumed to be unknown, estimates of parameters based on type II right censored sample with $n = 19$ and $r = 14$ by $MLSM$.

Example 4.2 (Biometric Data)**.** The second data set is an application of our results in biometric. The data were analyzed by Lawless [4,35,36]. The data represent the duration of remission of 20 leukemia patients who were treated by one drug. The ordered durations of remission (in years) are:

1.013	1.034	1.109	1.169	1.266	1.509	1.533	1.563	1.716	1.929
1.965	2.061	2.344	2.546	2.626	2.778	2.951	3.413	4.118	5.136

As in Example 4.1, Table 7 summarizes the preliminary computations which indicate that MWD is a appropriate model for these data. The prediction results are shown in Table 8.

5. Concluding remarks

In this article, two predictive pivotal quantities have been considered for constructing PCI for future unobservable gos based on MWD. Furthermore, a reconstructive pivotal quantity have been proposed to construct RCI for missing gos based on MWD. Moreover, an approximate value of the expected upper (lower) limit of the PCI (RCI) is obtained. The simulation reveals that the coverage probabilities are close to the exact value of $1 - \delta$ (here $\delta = 0.05$) as well as the expected and simulated upper(lower) limits of PCI (RCI). Based on the estimated root-mean-square error, the upper (lower) limit of PCI (RCI) became close to the exact upper (lower) limit whenever $s - r$ decreases for fixed n. In almost cases, the pivotal quantity, P_2 gives a shortest interval width than P_1 (see Tables 1–4). The illustrative examples have revealed that a good fitting of the data to the MWD increases the accuracy of prediction results (see Tables 5–8).

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