Journal of the Egyptian Mathematical Society (2016) **24**, 319–327

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

New ultraspherical wavelets collocation method for solving 2nth-order initial and boundary value problems

E.H. Doha[a], W.M. Abd-Elhameed[b,a], Y.H. Youssri[a,*]

[a] *Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt*
[b] *Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia*

Received 1 August 2014; revised 5 May 2015; accepted 20 May 2015
Available online 16 June 2015

Keywords

Ultraspherical polynomials;
Wavelets;
Bernstein-type inequality;
Collocation method;
Even-order differential equations;
Burger's equation

Abstract In this paper, a new spectral algorithm based on employing ultraspherical wavelets along with the spectral collocation method is developed. The proposed algorithm is utilized to solve linear and nonlinear even-order initial and boundary value problems. This algorithm is supported by studying the convergence analysis of the used ultraspherical wavelets expansion. The principle idea for obtaining the proposed spectral numerical solutions for the above-mentioned problems is actually based on using wavelets collocation method to reduce the linear or nonlinear differential equations with their initial or boundary conditions into systems of linear or nonlinear algebraic equations in the unknown expansion coefficients. Some specific important problems such as Lane–Emden and Burger's type equations can be solved efficiently with the suggested algorithm. Some numerical examples are given for the sake of testing the efficiency and the applicability of the proposed algorithm.

Mathematics Subject Classification: 65M70; 65N35; 35C10; 42C10

* Corresponding author.
E-mail addresses: eiddoha@frcu.eun.eg (E.H. Doha), walee_9@yahoo.com (W.M. Abd-Elhameed), youssri@sci.cu.edu.eg (Y.H. Youssri).

Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1. Introduction

Many practical and physical problems in fields of science and engineering are formulated as boundary or initial value problems. The nonlinear boundary value problems are crucial in various applications as they arise frequently in many areas of science and engineering. For example, the deflection of a

http://dx.doi.org/10.1016/j.joems.2015.05.002

uniformly loaded rectangular plate supported over the entire surface by an elastic foundation and rigidly supported along the edges, satisfies a fourth-order differential equation see for example [6,8,12–15,20,28–30,41].

Spectral methods play prominent roles in solving various kinds of differential equations. It is known that there are three most widely used spectral methods, they are the tau, collocation, and Galerkin methods. Collocation methods have become increasingly popular for solving differential equations, in particular, they are very useful in providing highly accurate solutions to nonlinear differential equations (see, for example [1,6,8,12–15,19,20,28–30,41,49]).

High even-order differential equations have been extensively discussed by a large number of authors due to their great importance in various applications in many fields. For example, in the sequence of papers [18,16,17], the authors treated such equations by the Galerkin method. They constructed suitable basis functions which satisfy the boundary conditions of the given differential equation. For this purpose, they used compact combinations of various orthogonal polynomials. The suggested algorithms in these articles are suitable for handling one- and two-dimensional linear high even-order boundary value problems.

In this paper, we aim to give a new algorithm for handling both of linear and nonlinear even-order initial and boundary value problems based on employing ultraspherical wavelets collocation method (UWCM).

Wavelets cover a great area in mathematical models, and they have been used to handle a wide range of medical and engineering disciplines; in particular, wavelets are very appropriate in signal analysis, image segmentations, time frequency analysis and fast algorithms for easy implementation. Wavelets give accurate representation of a variety of functions and operators. Moreover, wavelets connect between fast numerical algorithms, (see [11,38,32]).

Legendre wavelets have been previously employed for solving various differential and integral equations (see for example, [35,34,36,46–48]). Also, first and second kinds Chebyshev wavelets have been used for solving some integer and fractional orders differential equations (see for example, [7,56,31,25]). Recently, Abd-Elhameed et al. in [2,3], have introduced new Chebyshev wavelets algorithms for solving second-order boundary value problems. Up to now, and to the best of our knowledge, the use of ultraspherical wavelets in various spectral numerical applications is traceless in the literature. This gives us a motivation to introduce and use the ultraspherical wavelets in various applications.

The outlines of this paper are organized as follows. In Section 2, we give some relevant properties of ultraspherical polynomials and their shifted forms. Moreover, in this section, ultraspherical wavelets are constructed. In Section 3, the convergence of the ultraspherical wavelets expansion is proved. Section 4 is concerned with presenting and implementing a numerical algorithm for solving even-order linear and nonlinear initial or boundary value problems based on employing ultraspherical wavelets together with the spectral collocation method. Section 5 is concerned with considering some numerical examples aiming to illustrate the efficiency and applicability of the developed algorithm. Conclusions are given in Section 6.

2. Some properties of ultraspherical polynomials and ultraspherical wavelets

This section is concerned with presenting an overview on ultraspherical polynomials and their shifted polynomials. Also, ultraspherical wavelets are constructed in this section.

2.1. *Some properties of ultraspherical polynomials*

The ultraspherical polynomials are a class of symmetric Jacobi polynomials. They are a sequence of orthogonal polynomials defined on the interval $(-1, 1)$ with respect to the weight function $w(x) = (1-x^2)^{\alpha-\frac{1}{2}}$ associated with the real parameter $(\alpha > -\frac{1}{2})$. Explicitly, they satisfy the orthogonality property

$$\int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}}\, C_m^{(\alpha)}(x) C_n^{(\alpha)}(x)\, dx = \begin{cases} \frac{\pi\, 2^{1-2\alpha}\,\Gamma(n+2\alpha)}{n!\,(n+\alpha)\,(\Gamma(\alpha))^2}, & m = n, \\ 0, & m \neq n. \end{cases} \tag{1}$$

Also, they are the eigenfunctions of the following singular Sturm–Liouville equation

$$(1-x^2)\, D^2\phi_m(x) - (2\alpha+1)x\, D\phi_m(x) + m(m+2\alpha)\, \phi_m(x) = 0,$$
$$D \equiv \frac{d}{dx}.$$

For more properties of ultraspherical polynomials, one can be referred to [5].

The following integral formula (see, [5]) is needed

$$\int C_n^{(\alpha)}(x)\, w(x)\, dx = -\frac{2\alpha\,(1-x^2)^{\alpha+\frac{1}{2}}}{n(n+2\alpha)} C_{n-1}^{(\alpha+1)}(x), \quad n \geq 1. \tag{2}$$

Also, the following theorem is useful in deriving the convergence theorem for the expansion of the ultraspherical wavelets.

Theorem 1 ([22] Bernstein-type inequality). *The following inequality holds for ultraspherical polynomials:*

$$(\sin\theta)^{\alpha}\, |C_n^{(\alpha)}(\cos\theta)| < \frac{2^{1-\alpha}\,\Gamma\left(n+\frac{3\alpha}{2}\right)}{\Gamma(\alpha)\,\Gamma\left(n+1+\frac{\alpha}{2}\right)}, \quad 0 \leq \theta \leq \pi, \ \ 0 < \alpha < 1. \tag{3}$$

2.2. *Shifted ultraspherical polynomials*

The shifted ultraspherical polynomials are defined on [0, 1] by $\overset{\sim(\alpha)}{C_n}(x) = C_n^{(\alpha)}(2x-1)$. All results of ultraspherical polynomials can be easily transformed to give the corresponding results for their shifted polynomials. The orthogonality relation for $\overset{\sim(\alpha)}{C_n}(x)$ with respect to the weight function $\tilde{\omega} = (x-x^2)^{\alpha-\frac{1}{2}}$ is given by

$$\int_{0}^{1} (x-x^2)^{\alpha-\frac{1}{2}}\, \overset{\sim(\alpha)}{C_n}(x)\, \overset{\sim(\alpha)}{C_m}(x)\, dx = \begin{cases} \frac{\pi\, 2^{1-4\alpha}\,\Gamma(n+2\alpha)}{n!\,(n+\alpha)\,(\Gamma(\alpha))^2}, & m = n, \\ 0, & m \neq n. \end{cases}$$

For more properties of ultraspherical polynomials see [40].

2.3. *Construction of ultraspherical wavelets*

Wavelets constitute of a family of functions constructed from dilation and translation of single function called the mother wavelet. If the dilation parameter A and the translation parameter B vary continuously, then the following family of continuous wavelets is obtained:

$$\psi_{A,B}(t) = |A|^{-1/2}\,\psi\left(\frac{t-B}{A}\right), \quad A, B \in \mathbb{R}, \quad A \neq 0. \tag{4}$$

Ultraspherical wavelets $\psi_{nm}^{(\alpha)}(t) = \psi(k, n, m, \alpha, t)$ have five arguments: k, n can be assumed to be any positive integer, m is the degree for the ultraspherical polynomial, α is the known ultraspherical parameter and t is the normalized time. They are defined on the interval [0, 1] by

$$\psi_{nm}^{(\alpha)}(t) = \begin{cases} 2^{\frac{k+1}{2}}\,\xi_{m,\alpha}\,C_m^{(\alpha)}\left(2^{k+1}t-2n-1\right), & t \in [\frac{n}{2^k}, \frac{n+1}{2^k}], \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $0 \le m \le M$, $0 \le n \le 2^k - 1$, and

$$\xi_{m,\alpha} = 2^{\alpha}\,\Gamma(\alpha)\sqrt{\frac{m!\,(m+\alpha)}{2\pi\,\Gamma(m+2\alpha)}}. \tag{6}$$

Remark 1. It is worth noting here that $\psi_{nm}^{(\frac{1}{2})}(t)$ is identical to Legendre wavelets (see, [35]), $\psi_{nm}^{(0)}(t)$ is identical to first kind Chebyshev wavelets (see, [7]) and $\psi_{nm}^{(1)}(t)$ is identical to second kind Chebyshev wavelets (see, [56]).

3. Function approximation and convergence analysis

A function $f(t)$ defined on [0, 1] may be expanded in terms of ultraspherical wavelets as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}^{(\alpha)}(t),$$

where

$$c_{nm} = \left(f(t), \psi_{nm}^{(\alpha)}(t)\right)_w = \int_0^1 (t-t^2)^{\alpha-\frac{1}{2}}\,f(t)\,\psi_{nm}^{(\alpha)}(t)\,dt. \tag{7}$$

Also, this function can be approximated by the truncated finite series

$$f(t) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}^{(\alpha)}(t).$$

Now, we state and prove the following important theorem which ascertains that the ultraspherical wavelets expansion of a function $f(x)$ with bounded second derivative, converges uniformly to $f(x)$.

Theorem 2. *A function $f(x) \in L^2_{\tilde{\omega}}[0,1]$, $\tilde{\omega} = (x-x^2)^{\alpha-\frac{1}{2}}$, $0 < \alpha < 1$ can be expanded as an infinite series of ultraspherical wavelets, which converges uniformly to $f(x)$, provided $|f''(x)| \le L$. Explicitly, the expansion coefficients in (7) satisfy the inequality*

$$|c_{nm}| < \frac{4L\,(1+\alpha)^2\,\Gamma(m+1+\frac{3\alpha}{2})\,\sqrt{m!\,(m+\alpha)}}{\Gamma(m+\frac{\alpha}{2})\,\sqrt{\Gamma(m+2\alpha)}\,(m-2)^4(n+1)^{\frac{5}{2}}}, \qquad \forall\, n \ge 0,\ m > 2. \tag{8}$$

Proof. From relations (5) and (7), one can write

$$c_{nm} = 2^{\frac{k+1}{2}}\,\xi_{m,\alpha}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)\,C_m^{(\alpha)}\left(2^{k+1}t-2n-1\right)\omega(2^k t-n)\,dt. \tag{9}$$

Integration of the right hand side of (9) by parts, and making use of relation (2), yield

$$c_{nm} = \frac{2^{\frac{5-k}{2}}\,\alpha\,\xi_{m,\alpha}}{m\,(m+2\alpha)}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f'(t)\,C_{m-1}^{(\alpha+1)}\left(2^{k+1}t-2n-1\right)(2^k\,t-n) \times (1-2^k\,t+n)\,\omega(2^k t-n)\,dt. \tag{10}$$

Another integration by parts and making use of the substitution: $2^{k+1}t-2n-1=\cos\theta$, enable one to write

$$c_{nm} = \frac{\sqrt{2}\,(\alpha)_2\,\xi_{m,\alpha}}{2^{2\alpha+\frac{5k}{2}}\,(m-1)_2\,(m+2\alpha-1)_2} \times \int_0^{\pi} f''\left(\frac{1+2n+\cos\theta}{2^{k+1}}\right) C_{m-2}^{(\alpha+2)}(\cos\theta)\,(\sin\theta)^{2\alpha+4}\,d\theta. \tag{11}$$

Now, assuming that $m > 2$, taking into account the assumption $|f''(t)| \le L$, and with the aid of Theorem 1, we obtain

$$\begin{aligned} |c_{nm}| &\le \frac{\sqrt{2}\,|(\alpha)_2\,\xi_{m,\alpha}|}{2^{2\alpha+\frac{5k}{2}}\,(m-1)_2\,(m+2\alpha-1)_2} \times \int_0^{\pi}\left|f''\left(\frac{1+2n+\cos\theta}{2^{k+1}}\right)\right| |C_{m-2}^{(\alpha+2)}(\cos\theta)|\,(\sin\theta)^{2\alpha+4}\,d\theta \\ &\le \frac{\sqrt{2}\,L\,|\alpha|(1+\alpha)\,\xi_{m,\alpha}|}{2^{2\alpha+\frac{5k}{2}}\,(m-1)_2\,(m+2\alpha-1)_2} \times \int_0^{\pi} |C_{m-2}^{(\alpha+2)}(\cos\theta)|\,(\sin\theta)^{2\alpha+4}\,d\theta \\ &< \frac{\sqrt{2\pi}\,L\,|\alpha|(1+\alpha)\,|\xi_{m,\alpha}|\,\Gamma\left(m+1+\frac{3\alpha}{2}\right)\Gamma\left(\frac{3+\alpha}{2}\right)}{2^{3\alpha+\frac{5k}{2}}\,(m-1)_2\,(m+2\alpha-1)_2\,\Gamma(\alpha+2)\,\Gamma\left(m+\frac{\alpha}{2}\right)\Gamma(2+\frac{\alpha}{2})}. \end{aligned}$$

Knowing that $\alpha > 0$, $n < 2^k - 1$, and with the aid of relation (6), we get

$$\begin{aligned} |c_{nm}| &< \frac{2L\,|\alpha|(1+\alpha)\,\Gamma\left(\frac{3+\alpha}{2}\right)\Gamma\left(m+1+\frac{3\alpha}{2}\right)\sqrt{m!\,(m+\alpha)}}{4^{\alpha}\,\Gamma(2+\frac{\alpha}{2})\,\Gamma\left(m+\frac{\alpha}{2}\right)\sqrt{\Gamma(m+2\alpha)}\,(m-2)^4(n+1)^{\frac{5}{2}}} \\ &< \frac{4L\,(1+\alpha)^2\,\Gamma\left(m+1+\frac{3\alpha}{2}\right)\sqrt{m!\,(m+\alpha)}}{\Gamma\left(m+\frac{\alpha}{2}\right)\sqrt{\Gamma(m+2\alpha)}\,(m-2)^4(n+1)^{\frac{5}{2}}}. \end{aligned}$$

This completes the proof of the theorem. □

Note 1. It should be noted here that, for large values of m and n, and making use of the well known Stirling's formula (see, [33]), it can be easily shown that $|c_{nm}|$ is of $\mathcal{O}(n^{-\frac{5}{2}}\,m^{-2})$.

Note 2. It is worthy to note here that the result obtained in Theorem 2 may be improved but with some extra conditions on the function $f(x)$, to be concise, if f is differentiable p-times for some $p > 1$ and $|f^{(p)}(x)| \le L$, we can show that $|c_{nm}|$ is of $\mathcal{O}(n^{-p-\frac{1}{2}}\,m^{-2})$.

4. Solution of high even-order differential equations

In this section, and based on the ultraspherical wavelets expansion along with the spectral collocation method, we reduce the 2νth-order initial and boundary value problems with their initial or boundary conditions to systems of linear or nonlinear algebraic equations which can be efficiently solved.

Consider the 2νth-order differential equations

$$u^{(2\nu)}(x) + \phi\big(x, u(x), u'(x), \ldots, u^{(2\nu-1)}(x)\big) = 0, \qquad x \in (0,1), \tag{12}$$

subject to the initial conditions

$$u^{(i)}(0) = a_i, \quad i = 0, 1, \ldots, 2\nu - 1, \tag{13}$$

or the Dirichlet boundary conditions

$$u^{(i)}(0) = a_i, \quad u^{(i)}(1) = b_i, \quad i = 0, 1, \ldots, \nu - 1. \tag{14}$$

The approximate solution of (12) can be expanded in terms of ultraspherical wavelets as:

$$u_{k,M,\alpha}(x) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm}\,\psi_{nm}^{(\alpha)}(x). \tag{15}$$

Substitution of (15) into (12), enables one to write the residual of (12) as

$$R(x) = \sum_{n=0}^{2^k-1}\sum_{m=2\nu}^{M} c_{nm} D^{2\nu}\psi_{nm}^{(\alpha)} + \phi\Bigg(x, \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm}\psi_{nm}^{(\alpha)}, \times \sum_{n=0}^{2^k-1}\sum_{m=1}^{M} c_{nm} D\psi_{nm}^{(\alpha)}, \ldots, \sum_{n=0}^{2^k-1}\sum_{m=2\nu-1}^{M} c_{nm} D^{2\nu-1}\psi_{nm}^{(\alpha)}\Bigg) \tag{16}$$

The application of the collocation method (see for example, [9]) implies that

$$R(x_i) = 0, \quad i = 1, 2, \ldots, \big(2^k(M+1) - 2\nu\big), \tag{17}$$

where x_i are the first $(2^k(M+1) - 2)$ roots of $C_{2^k(M+1)}^{(\alpha)}$. Moreover, the use of boundary conditions (13) gives

$$\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm} D^{(i)}\psi_{nm}^{(\alpha)}(0) = a_i, \quad i = 0, 1, \ldots, 2\nu - 1, \tag{18}$$

while the use of (14) gives

$$\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm} D^{(i)}\psi_{nm}^{(\alpha)}(0) = a_i, \qquad \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm} D^{(i)}\psi_{nm}^{(\alpha)}(1) = b_i, \quad i = 0, 1, \ldots, \nu - 1. \tag{19}$$

Eqs. (17) with (18) or (19) generate a set of $2^k(M+1)$ in the unknown expansion coefficients, $\{c_{nm} : 0 \le n \le 2^k - 1; 0 \le m \le M\}$. This system of equations can be solved with the aid of a Newton's iterative method with the initial guess $c_{nm} = 10^{-m}$, to obtain the unknown components of the vector C, and hence the required spectral wavelets solution $u_{k,M,\alpha}(x)$ given by (15) can be obtained. In fact, solving such system of nonlinear equations and its convergence is not an easy task, and is to be considered as one disadvantage of using Newton's method. The interested reader is referred to the useful references [26,42,39] for the theory of Newton's method – (Linearization and iteration of nonlinear operator equations, the convergence of Newton's method, Uniqueness of the solution) – for solving generalized systems of nonlinear algebraic equations in Banach spaces. In reality, Newton's method provides a powerful tool for the theoretical as well as the numerical investigation of nonlinear operator equations.

5. Numerical results and discussions

Example 1. Consider the singular Lane–Emden initial value problem (IVP) (see, [37]):

$$u'' + \frac{1}{x}u' + u^5 = 0, \qquad x \in (0,1], \qquad u(0) = 1, \quad u'(0) = 0, \tag{20}$$

with the exact solution $u(x) = (1 + \frac{x^2}{3})^{-\frac{1}{2}}$. In Table 1, the maximum absolute error E is listed for $k = 0$ and various values of M; and in Table 2, we give a comparison between the solution resulted from the application of $UWCM$ for the case corresponds to $k = 0$, $M = 11$ and $\alpha = -0.49$, with the numerical solution obtained in [37] in case of $k = 0$, $M = 11$.

Example 2. Consider the singular boundary value problem (BVP) (see, [37]):

Table 2 Comparison between UWCM and the solution in [37] for Example 1.

x	Method in [37]	UWCM	Exact solution
0.1	0.99503720	0.99503719	0.99503719
0.2	0.98058067	0.98058067	0.98058067
0.3	0.95782628	0.95782628	0.95782628
0.4	0.92847670	0.92847669	0.92847669
0.5	0.89442718	0.89442719	0.89442719
0.6	0.85749292	0.85749292	0.85749292
0.7	0.81923192	0.81923192	0.81923192
0.8	0.78086880	0.78086880	0.78086880
0.9	0.74329414	0.74329414	0.74329414
1	0.70710678	0.70710678	0.70710678

Table 1 Maximum absolute error E for Example 1.

α	M	E	M	E	M	E	M	E	M	E
1	5	4.01×10^{-6}	8	1.79×10^{-8}	11	6.49×10^{-11}	14	1.44×10^{-13}	17	4.44×10^{-16}
$\frac{1}{2}$		3.77×10^{-6}		5.09×10^{-9}		1.32×10^{-11}		3.26×10^{-14}		4.44×10^{-16}
0		3.03×10^{-6}		1.28×10^{-8}		2.90×10^{-11}		4.94×10^{-14}		3.33×10^{-16}
−0.49		2.71×10^{-6}		1.07×10^{-8}		3.21×10^{-11}		3.80×10^{-14}		2.22×10^{-16}

Table 3 Maximum absolute error E for Example 2.

α	M	E	M	E	M	E	M	E	M	E
1	3	5.97×10^{-6}	5	1.99×10^{-8}	7	3.84×10^{-11}	9	1.19×10^{-12}	11	9.10×10^{-15}
$\frac{1}{2}$		2.23×10^{-6}		6.02×10^{-9}		1.06×10^{-11}		2.86×10^{-13}		2.55×10^{-15}
0		1.63×10^{-7}		2.85×10^{-10}		3.00×10^{-13}		8.43×10^{-15}		8.88×10^{-16}
−0.49		5.82×10^{-7}		1.10×10^{-9}		1.45×10^{-12}		3.64×10^{-14}		9.99×10^{-16}

$$u'' + \frac{0.5}{x}u' = e^u(0.5 - e^u), \qquad x \in (0,1], \qquad u(0) = \ln 2, \quad u(1) = 0, \tag{21}$$

with the exact solution $u(x) = \ln(\frac{2}{x^2+1})$. In Table 3, the maximum absolute error E is listed for $k = 1$ and various values of M, while in Table 4 we give a comparison between the approximate solution resulted from the application of $UWCM$ for the case corresponds to $k = 0$, $M = 11$ and $\alpha = 1$ with the numerical solution obtained in [37] in case of $k = 0$, $M = 11$.

Remark 2. We mention here that comparisons in Tables 2 and 4 illustrate that with the same number of retained modes, the UWCM solutions to Examples 1 and 2 are more accurate than those obtained by Nasab et al. in [37], and comparing favorably with the analytical solutions of these two examples.

Table 4 Comparison between UWCM and the solution in [37] for Example 2.

x	Method in [37]	UWCM	Exact solution
0.1	0.68319682	0.68319684	0.68319684
0.2	0.65392655	0.65392646	0.65392646
0.3	0.65392655	0.60696948	0.60696948
0.4	0.54472710	0.54472717	0.54472717
0.5	0.47000366	0.47000362	0.47000362
0.6	0.38566250	0.38566248	0.38566248
0.7	0.29437106	0.29437106	0.29437106
0.8	0.19845088	0.19845093	0.19845093
0.9	0.09982033	0.09982033	0.09982033
1	0.00000000	0.00000000	0.00000000

Example 3 (Burgers' Equation)**.** Consider the singular BVP (see, [44]):

$$\frac{1}{\mathcal{R}e}\left(u'' + \frac{\lambda}{x}u' - \frac{\lambda}{x^2}u\right) = uu' + f(x), \qquad x \in (0,1], \quad u(0) = 0, \quad u(1) = \cosh 1, \tag{22}$$

where $f(x)$ is chosen such that the exact solution of (22) is $u(x) = x^2 \cosh x$. Approximate solutions of this problem are computed for the cases correspond to the two choices $\lambda = 1$, $\mathcal{R}e = 10$ and $\lambda = 2$, $\mathcal{R}e = 50$. The absolute errors for such cases are displayed in Table 5 for $k = 0$ and for various values of α, while in Table 6 we give a comparison between the errors obtained using the B-Spline collocation (BSC) solution in [44] and UWCM, while Fig. 1 illustrates a comparison between the exact solution and some numerical solutions.

Remark 3. We mention here that Table 6 illustrates that we obtained more accurate solutions than those obtained in [44] in spite of BSC divides the interval into 80 equal parts ($n = 80$) which increase the complexity of the algorithm, while in the present algorithm ($k = 0$).

Example 4. Consider the fourth-order nonlinear beam equation (see, [44,10,4,51,52]):

Table 6 The best maximum absolute errors of Example 3.

	$\lambda = 1$, $\mathcal{R}e = 10$	$\lambda = 2$, $\mathcal{R}e = 50$
BSC [44] ($n = 80$)	1.967×10^{-15}	1.085×10^{-15}
UWCM ($k = 0$)	1.330×10^{-16}	1.460×10^{-16}

Table 5 Maximum absolute error E for Example 3.

λ	$\mathcal{R}e$	α	M	E	M	E	M	E	M	E
1	10	1	9	1.01×10^{-7}	11	1.46×10^{-10}	13	1.20×10^{-13}	15	1.33×10^{-15}
		$\frac{1}{2}$		9.02×10^{-8}		1.17×10^{-10}		8.94×10^{-14}		7.32×10^{-15}
		0		7.64×10^{-8}		8.83×10^{-11}		5.55×10^{-13}		2.79×10^{-14}
		−0.49		5.90×10^{-8}		4.92×10^{-11}		8.98×10^{-11}		4.53×10^{-12}
2	50	1	9	2.69×10^{-8}	11	1.64×10^{-11}	13	2.88×10^{-15}	15	1.46×10^{-16}
		$\frac{1}{2}$		2.15×10^{-8}		9.89×10^{-12}		1.22×10^{-15}		2.22×10^{-16}
		0		1.60×10^{-8}		3.59×10^{-12}		3.92×10^{-15}		3.72×10^{-15}
		−0.49		9.23×10^{-9}		1.08×10^{-12}		2.05×10^{-12}		3.05×10^{-12}
1	10^3	1	9	1.23×10^{-4}	11	2.35×10^{-7}	13	5.22×10^{-11}	15	4.68×10^{-12}
		$\frac{1}{2}$		5.22×10^{-4}		5.27×10^{-8}		2.57×10^{-11}		1.25×10^{-12}
		0		8.88×10^{-4}		7.24×10^{-8}		9.27×10^{-11}		5.71×10^{-11}
		−0.49		2.22×10^{-5}		2.35×10^{-8}		3.59×10^{-8}		6.24×10^{-8}

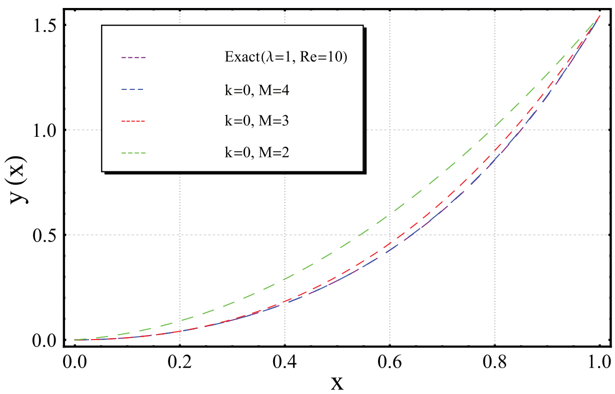

Figure 1 Different solutions of Example 3.

$$u^{(4)} = 6e^{-4u} - \frac{12}{(1+x)^4}, \qquad x \in [0,1], \tag{23}$$

$$u(0) = 0, \quad u(1) = \ln 2, \quad u''(0) = -1, \quad u''(1) = -\frac{1}{4}, \tag{24}$$

with the exact solution $u(x) = \ln(1+x)$. In Table 7, the maximum absolute error E is listed for $k = 1$ and various values of M, and in Table 8 we give a comparison between the best errors resulted from different numerical solutions.

Example 5. Consider the fourth-order linear equation (see, [24,43,45,54,55,27]):

$$u^{(4)} + xu = 0, \qquad x \in [0,1], \tag{25}$$

$$u(0) = 1, \quad u(1) = 0, \quad u''(0) = 0, \quad u''(1) = -4e, \tag{26}$$

with the exact solution $u(x) = x(1-x)e^x$. In Table 9, the maximum absolute error E is listed for $k = 1$ and various values of M, while in Table 10 we give a comparison between the best errors obtained by different errors.

Remark 4. We mention here that Tables 8 and 10 illustrate that UWCM is more accurate than other algorithms used by different authors (see, [4,10,24,27,43–45,51,52,54,55]).

Example 6. Consider the singular BVP (see, [37]):

$$u'' + e^{\frac{1}{x}} u' + u = 6x + x^3 + 3x^2 e^{\frac{1}{x}}, \qquad x \in (0,1],$$
$$u(0) = 0, \quad u(1) = 1, \tag{27}$$

with the exact solution $u(x) = x^3$. Eq. (27) is solved by applying the algorithm described in Section 4 for the case corresponds to $\alpha = k = 0$ and $M = 4$, to obtain the approximate solution $u_{0,4,0}(x)$. Expanding $u_{0,4,0}(x)$ in terms of the fist kind Chebyshev wavelets, we get

$$\begin{aligned} u_{0,4,0}(x) &= c_{0,0}\xi_0 + c_{0,1}\,\xi_1(2x-1) + c_{0,2}\,\xi_2(8x^2 - 8x + 1) \\ &\quad + c_{0,3}\,\xi_3(32x^3 - 48x^2 + 18x - 1) \\ &= c_0 + c_1\,(2x-1) + c_2\,(8x^2 - 8x + 1) \\ &\quad + c_3(32x^3 - 48x^2 + 18x - 1), \end{aligned} \tag{28}$$

where $c_i = \xi_i\, c_{0,i}, \quad i = 0, 1, 2, 3$. Now, the residual of Eq. (27) is given by

$$\begin{aligned} R(x) &= 32\,c_3\,x^3 + 8\,c_2\,x^2 - 48\,c_3\,x^2 - 8\,c_2\,x + 210\,c_3\,x \\ &\quad + c_1(2x-1) + c_0 + 17c_2 - 97c_3 - x^3 - 6x \\ &\quad + e^{\frac{1}{x}}\big(6c_3\big(16x^2 - 16x + 3\big) + 8c_2(2x-1) + 2c_1 - 3x^2\big). \end{aligned} \tag{29}$$

Table 7 Maximum absolute error E for Example 4.

α	M	E	M	E	M	E	M	E	M	E
1	6	2.46×10^{-5}	8	1.54×10^{-7}	10	5.56×10^{-10}	12	2.62×10^{-12}	14	3.44×10^{-15}
$\frac{1}{2}$		2.52×10^{-5}		1.46×10^{-7}		5.06×10^{-10}		2.45×10^{-12}		1.30×10^{-13}
0		2.56×10^{-5}		1.37×10^{-7}		4.28×10^{-10}		2.49×10^{-12}		1.32×10^{-12}
−0.49		2.59×10^{-5}		1.25×10^{-7}		3.57×10^{-10}		2.18×10^{-12}		2.35×10^{-13}

Table 8 Comparison between the best errors for Example 4.

Method in [44]	Method in [10]	Method in [4]	Method in [51]	Method in [52]	UWCM
1.25×10^{-12}	6.30×10^{-11}	5.40×10^{-8}	2.70×10^{-12}	6.70×10^{-12}	3.44×10^{-15}

Table 9 maximum absolute error E for Example 5.

α	M	E	M	E	M	E	M	E	M	E
1	7	1.26×10^{-3}	9	1.05×10^{-5}	11	3.75×10^{-8}	13	7.12×10^{-11}	15	7.43×10^{-14}
$\frac{1}{2}$		1.35×10^{-3}		1.10×10^{-5}		3.81×10^{-8}		6.99×10^{-11}		8.59×10^{-14}
0		1.44×10^{-3}		1.15×10^{-5}		3.86×10^{-8}		6.81×10^{-11}		7.26×10^{-14}
−0.49		1.56×10^{-3}		1.21×10^{-5}		3.89×10^{-8}		6.57×10^{-11}		6.92×10^{-14}

Table 10 Comparison between the best errors for Example 5.

Method	[24]	[43]	[45]	[54]	[55]	[27]	UWCM
E	4.98×10^{-11}	5.01×10^{-11}	1.67×10^{-9}	5.40×10^{-5}	2.70×10^{-5}	6.38×10^{-13}	6.92×10^{-14}

We satisfy Eq. (29) at the first two roots of the shifted Chebyshev polynomial $T_4^*(x)$, namely at $x_{1,2} = \frac{1}{4}(2 \pm \sqrt{2-\sqrt{2}})$. Furthermore, the use of the boundary conditions yields,

$$c_0 - c_1 + c_2 - c_3 = 0, \tag{30}$$

$$c_0 + c_1 + c_2 + c_3 = 1, \tag{31}$$

and solution of the resulted system of equations gives,

$$c_0 = \frac{5}{16}, \quad c_1 = \frac{15}{32}, \quad c_2 = \frac{3}{16}, \quad c_3 = \frac{1}{32},$$

and consequently

$$u_{0,4,0}(x) = \frac{5}{16} + \frac{15}{32}(2x-1) + \frac{3}{16}(8x^2-8x+1) + \frac{1}{32}(32x^3-48x^2+18x-1) = x^3, \tag{32}$$

which is the exact solution.

Example 7. Consider the following linear eighth-order BVP (see, [50,45,21,23,53]):

$$\begin{aligned} &y^{(8)} - y = -8e^x; \quad 0 < x < 1, \quad y(0) = 1,\\ &y^{(1)}(0) = 0, \quad y^{(2)}(0) = -1, \quad y^{(3)}(0) = -2,\\ &y^{(4)}(0) = -3, \quad y^{(5)}(0) = -4, \quad y^{(1)}(1) = -e,\\ &y^{(2)}(1) = -2e, \end{aligned} \tag{33}$$

with the exact solution $y(x) = (1-x)e^x$. In Table 11, the maximum absolute error E is listed for various values of k and M, while in Table 12, a comparison between the best error resulted from the application of UWCM for the case $\lambda = \frac{1}{2}$ – to numerically solve (33) – with the numerical algorithms obtained in [50,45,21,23,53] is displayed. This comparison shows that our suggested numerical method is more accurate than those developed in [50,45,21,23,53].

Example 8. Consider the following nonlinear second-order boundary value problem:

$$y''(x) + \left(y'(x)\right)^2 - y(x) = \sinh^2 x, \quad y(0) = 1,\ y(1) = \cosh 1. \tag{34}$$

with the exact solution $y(x) = \cosh x$.

In Table 13, we list the maximum absolute error E, for various values of k and M.

Example 9. Consider the following second-order linear singular initial value problem (see [35]):

$$y''(x) + f(x)y'(x) + y(x) = g(x), \quad y(0) = 0,\ y'(0) = 1, \tag{35}$$

where

$$f(x) = \begin{cases} -4x + \frac{4}{3}, & 0 \le x < \frac{1}{4},\\ x - \frac{1}{3}, & \frac{1}{4} \le x \le 1, \end{cases}$$

and $g(x)$ is chosen such that the exact solution of (35) is $y(x) = \frac{e^x \sin x}{1+x^2}$. We apply UWCM with $\lambda = 1, k = 2$. In Table 14, we list the maximum absolute error for various values of M, while in Fig. 2, the exact solution is compared with

Table 13 Maximum absolute error E for Example 8.

k	M	E	k	M	E
0	3	5.69×10^{-4}	1	2	1.12×10^{-6}
	4	1.35×10^{-5}		3	1.97×10^{-8}

Table 11 Maximum absolute error E for Example 7.

$k = 0$	M	11	12	13	14	15	16
	E	3.04×10^{-9}	2.65×10^{-10}	5.81×10^{-12}	3.65×10^{-13}	8.95×10^{-14}	5.66×10^{-16}
$k = 1$	M	8	9	10	11	12	13
	E	5.47×10^{-6}	5.87×10^{-8}	5.44×10^{-10}	5.22×10^{-12} 5.77×10^{-14}	8.47×10^{-16}	

Table 12 Comparison between the best errors for Example 7.

x	UWCM	[50]	[45]	[21]	[23]	[53]
0.25	4.44×10^{-16}	2.55×10^{-15}	3.03×10^{-10}	2.16×10^{-9}	4.58×10^{-9}	3.89×10^{-10}
0.5	2.66×10^{-16}	9.65×10^{-14}	7.73×10^{-9}	1.16×10^{-7}	9.84×10^{-9}	1.16×10^{-7}
0.75	1.22×10^{-16}	5.63×10^{-13}	3.12×10^{-8}	1.05×10^{-6}	1.09×10^{-5}	1.05×10^{-6}
1	1.54×10^{-17}	9.23×10^{-13}	4.39×10^{-8}	4.22×10^{-6}	1.86×10^{-4}	4.22×10^{-6}

Table 14 Maximum absolute error E for Example 9.

M	E	M	E
2	2.98×10^{-4}	6	1.85×10^{-10}
4	1.70×10^{-5}	8	9.12×10^{-11}

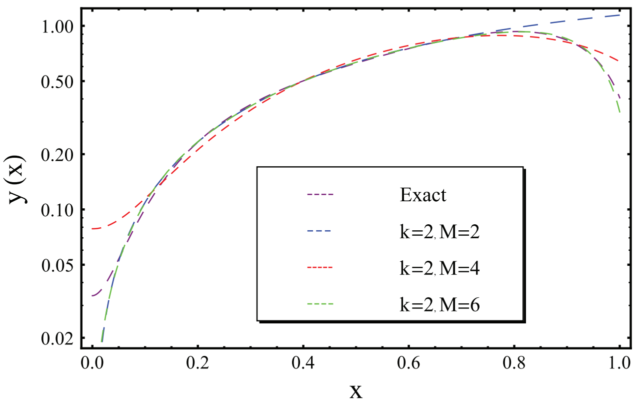

Figure 2 Numerical and exact solutions of Example 9.

Table 15 Comparison between different errors for Example 9.

x	0.2	0.4	0.6	0.8
Method in [35]	4.30×10^{-12}	1.24×10^{-11}	2.97×10^{-12}	4.35×10^{-12}
Present method	6.47×10^{-13}	1.21×10^{-12}	1.74×10^{-12}	2.32×10^{-12}

the three numerical solutions correspond to $k = 2, M = 2, k = 2, M = 4$ and $k = 2, M = 6$, also in Table 15 we give a comparison between the error obtained in [35] and the present method, $k = 2, M = 8$.

Note 3. It is worthy to note here that the results obtained in Example 9 are in complete agreement with the results obtained by Abd-Elhameed et al. [3], as expected the second kind Chebyshev wavelets is a direct special case of the ultraspherical wavelets.

6. Conclusions

In this paper, a new algorithm for obtaining numerical spectral solutions for 2νth-order linear and nonlinear differential equations is developed and implemented. Actually the derivation of this algorithm is based on constructing the shifted ultraspherical wavelets and applying spectral collocation method. The developed algorithm has the advantage that, high accurate approximate solutions are achieved using a few number of ultraspherical wavelets. The obtained approximate solutions are very close to the analytical ones. One disadvantage of the proposed method is how to handle the nonlinear system using Newton's method, this has been discussed deeply in Section 4.

Acknowledgment

The authors would like to thank the referees for careful reading and very helpful comments to improve this paper.

References

[1] W.M. Abd- Elhameed, E.H. Doha, Y.H. Youssri, Efficient spectral-Petrov–Galerkin methods for third- and fifth-order differential equations using general parameters generalized Jacobi polynomials, Quaest. Math. 36 (2013) 15–38.
[2] W.M. Abd-Elhameed, E.H. Doha, Y.H. Youssri, New wavelets collocation method for solving second-order multiPoint boundary value problems using Chebyshev polynomials of third and fourth kinds, Abstr. Appl. Anal. 2013 (2013) 542839.
[3] W.M. Abd-Elhameed, E.H. Doha, Y.H. Youssri, New spectral second kind chebyshev wavelets algorithm for solving linear and nonlinear second-order differential equations involving singular and Bratu type equations, Abstr. Appl. Anal. 2013 (2013) 71756.
[4] R.P. Agarwall, G. Akrivis, Boundary value problems occurring in plate deflection theory, J. Comput. Appl. Math. 8 (1982) 145–154.
[5] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999.
[6] C. Basdevant, M. Deville, P. Haldenwang, J.M. Lacroix, J. Ouzzani, R. Peyret, P. Orlandi, A.T. Patera, Spectral and finite difference solutions of the Burgers equation, Comp. Fluids 14 (1986) 23–41.
[7] E. Babolian, F. Fattah zadeh, Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration, Appl. Math. Comput. 188 (2007) 417–426.
[8] J.P. Boyd, One point pseudospectral collocation for the onedimensional Bratu equation, Appl. Math. Comput. 217 (2011) 5553–5565.
[9] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods on Fluid Dynamics, Springer-Verlag, 1988.
[10] M.M. Chawla, C.P. Katti, Finite difference methods for two-point boundary value problems involving high order differential equations, BIT Numer. Math. 19 (1979) 27–33.
[11] A. Constantmldes, Applied Numerical Methods with Personal Computers, McGraw-Hill, New York, 1987.
[12] M. Dehghan, A. Nikpour, Numerical solution of the system of second-order boundary value problems using the local radial basis functions based differential quadrature collocation method, Appl. Math. Modell. 37 (18) (2013) 8578–8599.
[13] M. Dehghan, S. Aryanmehr, M.R. Eslahchi, A technique for the numerical solution of initial-value problems based on a class of Birkhoff-type interpolation method, J. Comp. Appl. Math. 244 (2013) 125–139.
[14] M. Dehghan, F. Shakeri, A semi-numerical technique for solving the multi-point boundary value problems and engineering applications, Int. J. Numer. Method. H. 21 (7) (2011) 794–809.
[15] M. Dehghan, M. Tatari, Finding approximate solutions for a class of third-order non-linear boundary value problems via the decomposition method of Adomian, Int. J. Comp. Math. 87 (6) (2010) 1256–1263.
[16] E.H. Doha, W.M. Abd-Elhameed, Efficient spectral-Galerkin algorithms for direct solution of second-order equations using ultraspherical polynomials, SIAM J. Sci. Comput. 24: (2002) 548–571.
[17] E.H. Doha, W.M. Abd-Elhameed, M.A. Bassuony, New algorithms for solving high even-order differential equations using third and fourth Chebyshev–Galerkin methods, J. Comput. Phys. 236: (2013) 563–579.
[18] E.H. Doha, W.M. Abd-Elhameed, A.H. Bhrawy, New spectral–Galerkin algorithms for direct solution of high even-order differential equations using symmetric generalized Jacobi polynomials, Collect. Math. 64 (2013) 373–394.

[19] E.H. Doha, W.M. Abd- Elhameed, Y.H. Youssri, Second kind Chebyshev operational matrix algorithm for solving differential equations of Lane–Emden type, New Astron. 23–24 (2013) 113–117.
[20] M.R. Eslahchi, M. Dehghan, S. Ahmadi, The general Jacobi matrix method for solving some nonlinear ordinary differential equations, Appl. Math. Modell. 36 (8) (2012) 3387–3398.
[21] A. Golbabai, M. Javidi, Application of homotopy perturbation method for solving eighth-order boundary value problems, Appl. Math. Comp. 191 (2) (2007) 334–346.
[22] C. Giordano, A. Laforgia, On the Bernstein-type inequalities for ultraspherical polynomials, J. Comput. Appl. Math. 153 (2003) 243–248.
[23] J.H. He, The variational iteration method for eighth-order initial-boundary value problems, Phys. Scripta 76 (6) (2007) 680–682.
[24] M.K. Jain, S.R.K. Iyengar, J.S.V. Saldanha, Numerical solution of a fourth-order ordinary differential equation, J. Eng. Math. 11 (4) (1977) 373–380.
[25] M. Kajani, A. Tavassoli, H. Vencheh, M. Ghasemi, The Chebyshev wavelets operational matrix of integration and product operation matrix, Int. J. Comput. Math. 86 (7) (2009) 1118–1125.
[26] L. Kantorovitch, The method of successive approximation for functional equations, Acta Math. 71 (1) (1939) 63–97.
[27] A. Khan, P. Khandelwal, Non-polynomial sextic spline approach for the solution of fourth-order boundary value problems, Appl. Math. Comput. 218 (2011) 3320–3329.
[28] M. Lakestani, M. Dehghan, Four techniques based on the B-spline expansion and the collocation approach for the numerical solution of the Lane–Emden equation, Math. Meth. Appl. Sci. 36 (16) (2013) 2243–2253.
[29] M. Lakestani, B.N. Saray, M. Dehghan, Numerical solution for the weakly singular Fredholm integro-differential equations using Legendre multiwavelets, J. Comp. Appl. Math. 235 (11) (2011) 3291–3303.
[30] M. Lakestani, M. Dehghan, The solution of a second-order nonlinear differential equation with Neumann boundary conditions using semi-orthogonal B-spline wavelets, Int J. Comp. Math. 83 (8-9) (2006) 685–694.
[31] Y. Li, Solving a nonlinear fractional differential equation using Chebyshev wavelets, Commun. Nonlinear Sci. Numer. Simul. 15 (9) (2010) 2284–2292.
[32] G. Liu, X. Feng, M. Li, Relationship of d-dimensional continuous multi-scale wavelet shrinkage with integro-differential equations, Chaos Soliton Fract. 40 (2009) 1118–1126.
[33] Y.-C. Li, A note on an identity of the gamma function and Stirling's formula, Real Anal. Exchang. 32 (1) (2007) 267–272.
[34] F. Mohammadi, M.M. Hosseini, Legendre wavelet method for solving linear stiff systems, J. Adv. Res. Diff. Eq. 2 (1) (2010) 47–57.
[35] F. Mohammadi, M.M. Hosseini, A new Legendre wavelet operational matrix of derivative and its applications in solving the singular ordinary differential equations, J. Frank. Inst. 348 (2011) 1787–1796.
[36] F. Mohammadi, M.M. Hosseini, S.T. Mohyud-din, Legendre wavelet Galerkin method for solving ordinary differential equations with non analytic solution, Int. J. Syst. Sci. 42 (4) (2011) 579–585.
[37] A.K. Nasab, A. Kilicman, E. Babolian, Z.P. Atabakan, Wavelet analysis method for solving linear and nonlinear singular boundary value problems, Appl. Math. Model. (2013) 5876–5886.
[38] D.E. Newland, An Introduction to Random Vibrations, Spectral and Wavelet Analysis, Longman Scientific and Technical, New York, 1993.
[39] J.M. Ortega, W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, vol. 30, Siam Publishing House, 1970.
[40] E.D. Rainville, Special Functions, New York, 1960.
[41] K. Parand, M. Dehghan, A. Pirkhedri, The Sinc-collocation method for solving the Thomas–Fermi equation, J. Comp. Appl. Math. 237 (1) (2013) 244–252.
[42] L.B. Rall, A note on the convergence of Newton's method, SIAM J. Numer. Anal. 11 (1) (1974) 34–36.
[43] M.A. Ramadan, I.F. Lashien, W.K. Zahra, Quintic non-polynomial spline solutions for fourth-order two-point boundary value problem, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 1105–1114.
[44] J. Rashidinia, M. Ghasemia, B-spline collocation for solution of two-point boundary value problems, J. Comput. Appl. Math. 235 (2011) 2325–2342.
[45] J. Rashidinia, Direct methods for the solution of a linear fourth-order two-point boundary value problem, Int. J. Eng. Sci. 13 (2002) 37–48.
[46] M. Razzaghi, S. Yousefi, Legendre wavelets operational matrix of integration, Int. J. Syst. Sci. 32 (4) (2001) 495–502.
[47] M. Razzaghi, S. Yousefi, Legendre wavelets method for the solution of nonlinear problems in the calculus of variations, Math. Comput. Model. 34 (2001) 45–54.
[48] M. Razzaghi, S. Yousefi, Legendre wavelets method for constrained optimal control problems, Math. Meth. Appl. Sci. 25 (2002) 529–539.
[49] B. Saka, I. Dag, Quartic B-spline collocation method to the numerical solutions of the Burgers' equation, Chaos Soliton Fract. 32 (2007) 1125–1137.
[50] S.S. Siddiqi, M. Iftikhar, Numerical solution of higher order boundary value, Problems, Abst. Appl. Anal. (2013) 427521 (12 pages).
[51] E.H. Twizell, A sixth-order extrapolation method for special nonlinear fourth-order boundary value problems, Comp. Meth. Appl. Mech. Eng. 62 (1987) 293–303.
[52] E.H. Twizell, S.I.A. Tirmizi, Multiderivative methods for nonlinear beam problems, Commun. Appl. Numer. Meth. 4 (1988) 43–50.
[53] M. Torvattanabun, S. Koonprasert, Variational iteration method for solving eighth-order boundary value problems, Thai J. Math. 8 (4) (2010) 121–129.
[54] R.A. Usmani, S.A. Warsi, Smooth spline solutions for boundary value problems in plate deflection theory, Comput. Math. Appl. 6 (1980) 205–211.
[55] R.A. Usmani, The use of quartic splines in the numerical solution of a fourth-order boundary value problem, J. Comput. Appl. Math. 44 (1992) 187–199.
[56] L. Zhu, Q.B. Fan, Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet, Commun. Nonlinear Sci. Numer. Simulat. 17 (6) (2012) 2333–2341.