Journal of the Egyptian Mathematical Society (2016) **24**, 329–336

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

On the distribution of Weierstrass points on Gorenstein quintic curves

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Received 5 January 2015; revised 13 May 2015; accepted 31 August 2015
Available online 17 December 2015

Keywords

Gorenstein curves;
Weierstrass points;
Weierstrass gap sequences

Abstract This paper is concerned with developing a technique to compute in a very precise way the distribution of Weierstrass points on the members of any 1-parameter family C_a, $a \in \mathbb{C}$, of Gorenstein quintic curves with respect to the dualizing sheaf $\mathcal{K}_{C_a}$. The nicest feature of the procedure is that it gives a way to produce examples of existence of Weierstrass points with prescribed special gap sequences, by looking at plane curves or, more generally, to subcanonical curves embedded in some higher dimensional projective space.

2010 Mathematics Subject Classification: 14H55; 14H10

1. Introduction

At the beginning, several researchers developed the theory of the Weierstrass points for smooth curves, and for their canonical divisors. During the last three decades, Lax and Widland (see [1–6]) founded and developed the theory for Gorenstein curves, where the invertible dualizing sheaf replaces the canonical sheaf. Through this context, the singular points of a Gorenstein curve have to be considered as Weierstrass points.

The goal of this paper is to develop a technique for computing the distribution of the Weierstrass points on the members of any 1-parameter family C_a, $a \in \mathbb{C}$, of Gorenstein quintic curves with respect to the dualizing sheaf $\mathcal{K}_{C_a}$. Such a technique is based on the computation of the sequence of integers which in [7] has been called "$\mathcal{K}_{C_a}$-Weierstrass Gaps Sequence" ($\mathcal{K}_{C_a}$-WGS for brief), even at singular points. In [9], the first author and F. Sakai classified and investigated the distribution of Weierstrass points on certain 1-parameter family of genus 3 curves, named after Kuribayashi quartic curves.

Actually, the technique we describe, consists of performing a fixed sequence of computations, and so, it can be applied to any Gorenstein quintic curve, at any point P, no matter if it is smooth or singular. In case, P is smooth, the $\mathcal{K}_{C_a}$-WGS are computed by determining the dimension of the linear systems

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Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

S1110-256X(15)00076-0
http://dx.doi.org/10.1016/j.joems.2015.08.008

$\mathcal{K}_{C_a} - nP$, for every non-negative integer n, and so some contact order must be computed. If P is singular, the $\mathcal{K}_{C_a}$-WGS are given by a suitable combination of the $\tilde{\mathcal{K}}_{\tilde{C}_a}$-WGS at the points $Q_1, \ldots, Q_m$, $m \geq 1$, over P in a partial normalization $\theta_P : \tilde{C}_a \longrightarrow C_a$ of C_a at P, where $\tilde{\mathcal{K}}_{\tilde{C}_a}$ is the pull back of $\mathcal{K}_{C_a}$. The $\tilde{\mathcal{K}}_{\tilde{C}_a}$-WGS at these points can be computed as the contact order of C_ω, $\omega \in \mathcal{K}_{C_a}$, and the branches $C_a^{(1)}, \ldots, C_a^{(m)}$ of C_a through P, corresponding to $Q_1, \ldots, Q_m$, respectively, one branch at a time. Moreover, the study of the branches through P allows to largely simplify the computation of the $\mathcal{K}_{C_a}$-WGS. This simplification is essentially due to the knowledge of the normalization map in terms of blow-up's, as shown in [7].

In both cases, the contact orders are computed by means of the osculating conics. Moreover, in the next section, we describe a quick way to compute the osculating conics at a point of a Gorenstein quintic curve, because in the most spread computer algebra systems there is no built in function to perform that computation. However, as a computing support, to perform the computations described through this paper we use MATHEMATICA and MAPLE programs.

The layout of the paper is as follows. In Section 2, we cover most of the necessary background material. Section 3 is devoted to describe the technique and show its correctness, while, in the last Section 4, we let the technique work on some interesting examples.

2. Notation and preliminaries

We begin by stating the basic tools that will be used throughout this paper.

2.1. Weierstrass points

Here, we briefly recall what we need about Weierstrass points on curves. We start by the definition of the $\mathcal{K}_C$-Weierstrass gap sequences (shortly $\mathcal{K}_C$-WGS in the following) at a point, singular or not, with respect to the dualizing sheaf $\mathcal{K}_C$ over C. To this purpose, let C be any projective integral curve of arithmetic genus g over the complex field $\mathbb{C}$. Let us recall a geometrical definition of a $\mathcal{K}_C$-gap at a point P of C (see [8], §2). If the point P is non-singular, the $\mathcal{K}_C$-WGS at P is defined as follows:

Definition 1. Let P be a smooth point on the curve C. The integer n is a $\mathcal{K}_C$-gap if and only if, $\dim_{\mathbb{C}}(\mathcal{K}_C - (n-1)P) > \dim_{\mathbb{C}}(\mathcal{K}_C - nP)$. The sequence of the $\mathcal{K}_C$-gaps is the $\mathcal{K}_C$-WGS at P.

On the other hand, if $P \in C$ is a singular point, let $\pi : \tilde{C} \longrightarrow C$ is the normalization of C and consider the linear system $\tilde{V} = span(\pi^* v_1, \ldots, \pi^* v_g)$ over $\tilde{C}$, where $(v_1, \ldots, v_g)$ is a basis of $H^0(C, \mathcal{K}_C)$.

A positive integer $b(Q)$ is called a $\tilde{V}$-gap at a point $Q \in \pi^{-1}(P)$ if and only if, $\dim_{\mathbb{C}}(\tilde{V} - (b(Q)-1)Q) > \dim_{\mathbb{C}}(\tilde{V} - b(Q)Q)$. Since $\dim_{\mathbb{C}}(\tilde{V}) = g$ and by Riemann–Roch Theorem $\dim_{\mathbb{C}}(\tilde{V} - (2g-1)Q) = 0$, it follows that at each $Q \in \pi^{-1}(P)$ there are exactly g $\tilde{V}$-gap. If $\tilde{V}$-WGS is known for each point Q lying over P, the $\mathcal{K}_C$-WGS $\{a_1(P), \ldots, a_g(P)\}$ at P can be computed as follows:

Proposition 1. *Suppose $\pi : \tilde{C} \longrightarrow C$ is the normalization of C. Let $Q_1, \ldots, Q_m$ be the points of $\tilde{C}$ corresponding to the branches centered at a point P of C and $\{b_1^{\tilde{V}}(Q_i), \ldots, b_g^{\tilde{V}}(Q_i)\}$ be the $\tilde{V}$-WGS at the point Q_i, for $i = 1, 2, \ldots, m$, then one has:*

$$a_k(P) = \sum_{i=1}^{m} b_k^{\tilde{V}}(Q_i) - k(m-1), \quad 1 \leq k \leq g. \tag{1}$$

Proof. See ([7], Proposition 5.5, p. 285). □

Following [7], Proposition 5.4, one can define the so called kth $\mathcal{K}_C$-extraweight at the point P, denoted by $E_k(P)$ as:

$$E_k(P) = \sum_{Q \in \pi^{-1}(P)} w_k^{\tilde{V}}(Q),$$

where $w_k^{\tilde{V}}(Q) = \sum_{i=1}^{k}(b_k^{\tilde{V}}(Q) - i)$ is the kth $\tilde{V}$-Weierstrass weight at the smooth point Q. Therefore, at the point P, one can attach a sequence of integers $\{E_1(P), \ldots, E_g(P)\}$, called the $\mathcal{K}_C$-extraweight sequence at P. By means of the extraweight sequence, the $\mathcal{K}_C$ -WGS $\{a_1(P), \ldots, a_g(P)\}$ at P can be computed as

$$a_k(P) = \begin{cases} E_k(P) + 1 & \text{if} \quad k = 1, \\ E_k(P) - E_{k-1}(P) + k & \text{if} \quad 2 \leq k \leq g. \end{cases} \tag{2}$$

Hence, we also have (see [7])

$$E_k(P) = \sum_{i=1}^{k}(a_i(P) - i). \tag{3}$$

The last two formulas show that it is equivalent to know the $\mathcal{K}_C$-WGS or the extraweight sequence at P. It is clear that the first way is easier to compute than the second, because of the geometrical meaning of the $\mathcal{K}_C$-WGS.

Using a Widland–Lax argument (see [1] and [6]) or (see [7], Proposition 4.5) one can show that for each k

$$w_k(P) = k(k-1)\delta_P + E_k(P), \quad 1 \leq k \leq g \tag{4}$$

where $w_k(P)$ is a non-negative integer, called kth $\mathcal{K}_C$ -weight at the point P and $\delta_P = \dim_{\mathbb{C}}(\tilde{\mathcal{O}}_P(C)/\mathcal{O}_P(C))$ is a numerical invariant linked to the kind of singularity. The sequence of integers $\{w_1(P), \ldots, w_g(P)\}$ is called the $\mathcal{K}_C$-weight sequence at P. The gth $\mathcal{K}_C$-weight $w_g(P)$ is nothing but the vanishing order at the point P of the Wronskian of a basis for $H^0(C, \mathcal{K}_C)$ as defined in [6]. Hence, the point P is a $\mathcal{K}_C$-Weierstrass point if and only if $w_g(P) > 0$. Moreover, the total number of the $\mathcal{K}_C$-Weierstrass points up to their weights is given by the following proposition (see [6], Proposition 1 or [7], Proposition 4.4).

Proposition 2. *The total gth $\mathcal{K}_C$ -weight of the $\mathcal{K}_C$ -Weierstrass point is:*

$$W_{C,g} = \sum_{P \in C} w_g(P) = (g-1)g(g+1) \tag{5}$$

Remark 1. As a consequence to Proposition 1, if $\pi^{-1}(P) = \{Q\}$, i.e. the preimage of P reduces to just one point on C, then the $\mathcal{K}_C$-WGS at P coincides with the $\tilde{V}$-WGS at Q. The $\mathcal{K}_C$-WGS and the $\tilde{V}$-WGS coincide also when the point P is smooth.

2.2. Flexes and sextactic points

A smooth point P on a plane curve C is called a *flex* point if the tangent line L_P meets C at P with contact order at least three, i.e. $I_P(C, L_P) \geq 3$. Furthermore, a flex point $P \in C$ is *m-flex* if $m = I_P(C, L_P) - 2$. This positive integer m is called the flex multiplicity of C at P. Moreover the flex points of an algebraic curve are the smooth points of the intersection of this curve with its associated Hessian curve.

In analogy with tangent lines and flexes on plane curves, one can consider *the osculating conics* and *the sextactic points*. Let P be a non-flex smooth point on a plane curve C of degree $d \geq 3$. Then, there is a unique irreducible conic D_P with $I_P(C, D_P) \geq 5$. Such conic D_P is called the osculating conic of C at P.

Definition 2. A smooth, but not a flex, point P on a plane curve C is called a sextactic point if the osculating conic D_P meets C at P with contact order at least six. Furthermore, a sextactic point P is called *s-sextactic*, if $s = I_P(C, D_P) - 5$. This positive integer s is called the sextactic multiplicity of C at P.

Now, we describe a procedure to compute an osculating conic of a quintic curve C at a certain non-flex point P, in a very quick way. Let us choose a coordinate affine open subset of $\mathbb{P}^2$ containing the point P and let $f(x, y) = 0$ be the equation of the curve C in the chosen open subset, and (α, β) be the coordinates of the point P.

Lemma 1. *We can compute the osculating conic D_P at P in the following way:*

(1) *Compute the defining equation $\ell_P(x, y) := y - \beta - m(x - \alpha) = 0$ of the tangent line L_P of C at P.*
(2) *Parameterize those irreducible conics passing through the point P with the tangent line L_P:*

$$\ell_P(x, y) + A(x-\alpha)^2 + B(x-\alpha)\ell_P(x, y) + C\ell_P(x, y)^2 = 0,$$

with $A \neq 0$ as

$$x(t) = \alpha - \frac{t}{A + Bt + Ct^2} \quad \text{and}$$
$$y(t) = \beta - \frac{t(t+m)}{A + Bt + Ct^2}$$

(3) *Write*

$$f(x(t), y(t)) = \frac{s_2t^2 + s_3t^3 + s_4t^4 + s_5t^5 + s_6t^6 + s_7t^7 + s_8t^8 + s_9t^9 + s_{10}t^{10}}{(A + Bt + Ct^2)^5},$$

where $s_i \in \mathbb{C}[A, B, C]$ for $i = 2, \ldots, 10$.
(4) *Solve the equations: $s_2 = s_3 = s_4 = 0$, with $A \neq 0$, we can find the osculating conic at P.*

Proof. For the assertion (2), it suffices to parameterize the intersection points of the conic with the pencil of lines $\ell_P(x, y) - t(x - \alpha) = 0$. □

2.3. On the roots of a polynomial

We need to determine the multiplicities of the repeated roots of a polynomial. If a polynomial $f(x)$ has no parameters, then we can do it by using the Euclidean algorithm. However, this way fails if the polynomial $f(x)$ has a parameter. Here we use the subresultants method (for more details about resultants and subresultants theory we refer to [10] and [11]). We denote by $R^{(k)}[f(x), g(x); x]$ the *subresultant of degree k* for the polynomials $f(x)$ and $g(x)$.

Lemma 2. *The polynomials f(x) and g(x) have a non-constant common factor of multiplicity at least t if and only if, $R^{(k)}[f(x), g(x); x] = 0$, for all $k = 1, 2, \ldots, t$. $k = 1, 2, \ldots, t$.*

Definition 3. For a polynomial $f(x) = c\prod_{i=1}^{t}(x - a_i)^{n_i}$, where $a_i \neq a_j$ for $i \neq j$ and c is a complex number, we define $s := s(f)$, if $R^{(k)}[f(x), f'(x); x] = 0$, for all $k = 1, \ldots, s$ and $R^{(s+1)}[f(x), f'(x); x] \neq 0$. We set $r(f) = \max\{n_i \mid i = 1, \ldots, t\}$.

Lemma 3.

(i) *Take a polynomial $f(x) = c\prod_{i=1}^{t}(x - a_i)^{n_i}$, where $a_i \neq a_j$ for $i \neq j$ and c is a complex number. Then $s(f) = \sum_{i=1}^{t}(n_i - 1)$.*
(ii) *If $\mathbb{V}(f, f', \ldots, f^{(r-1)}) \neq \emptyset$, but $\mathbb{V}(f, f', \ldots, f^{(r)}) = \emptyset$, then $r = r(f)$.*

Remark 2.

(i) If $r(f) = 2$, then $f(x) = c\prod_{i=1}^{s}(x - a_i)^2 g(x)$, where the polynomial $g(x)$ has distinct roots and $g(a_i) \neq 0$ for all $i = 1, \ldots, s$.
(ii) We regard a polynomial $f(x, a) \in \mathbb{C}[x, a]$ as a 1-parameter family of polynomials depending on the value of a. Consider the ideal $I_k = (f, f', \ldots, f^{(k)})$, where the $f^{(i)}$ denote the ith differentiation with respect to the variable x. By using Groebner basis methods, we can compute the ideal $J_k = I_k \cap \mathbb{C}[a]$ in $\mathbb{C}[a]$. If $a_0 \in \mathbb{V}(J_{r-1})$, $a_0 \notin \mathbb{V}(J_r)$, then we conclude that $r(f(x, a_0)) = r$. Also if $R^{(1)}[f(x, a_0), f_x(x, a_0); x] = \cdots = R^{(s)}[f(x, a_0), f_x(x, a_0); x] = 0$, but $R^{(s+1)}[f(x, a_0), f_x(x, a_0); x] \neq 0$, then we infer that $s(f(x, a_0)) = s$.

To describe the repeated roots of a polynomial, we use the following convention.

Convention. Let $f(x)$ be a polynomial. We write $T(f) = (n_\alpha, m_\beta, \ldots)$, $n, m \in \mathbb{Z}^+$, if $f(x)$ has α roots of multiplicity n, β roots of multiplicity m, and so on. For instance, the polynomial $f(x) = x(x-1)(x+1)^2(x^3-8)^4$ is of type $T(f) = (4_3, 2_1, 1_2)$.

Example 1. Consider the following polynomial with a parameter a: $f(x, a) = 5x^8 - 6(a^2 - 1)x^6 + (-2a^4 - 5a^2 + 2)x^4 + 6a^2(a^2 - 1)x^2 + 5a^4$, By using MATHEMATICA, we find

$$R^{(1)}[f(x), f_x(x); x] = 8192000a^{12}(P(a))^4(Q(a))^2,$$
$$R^{(2)}[f(x), f_x(x); x] = 2457600a^6(a^2 - 1)(P(a))^3Q(a)R(a),$$
$$R^{(3)}[f(x), f_x(x); x] = 614400a^2(a^2 - 1)(P(a))^2R(a)S(a),$$
$$R^{(4)}[f(x), f_x(x); x] = -25600P(a)S(a)T(a),$$
$$R^{(5)}[f(x), f_x(x); x] = -6400(47a^4 - 4a^2 + 7)T(a).$$

Where $P(a) = 19a^4 - 43a^2 - 1$, $Q(a) = 4a^8 + 204a^6 - 71a^4 + 84a^2 + 4$, $R(a) = 2a^8 + 107a^6 - 98a^4 + 37a^2 + 2$, $S(a) = 4a^8 + 148a^6 - 149a^4 + 68a^2 + 4$ and $T(a) = 38a^8 + 244a^6 - 275a^4 + 116a^2 + 2$.

Hence, we obtain the following four cases:

(1) If $a = 0$, we have $a|R^{(k)}$ for $k = 1, 2, 3$ but $Res[a, R^{(4)}; a] \neq 0$. Then $s(f) = 3$,
(2) If $Q(a) = 0$, we have $Q(a)|R^{(k)}$ for $k = 1, 2$ but $Res[Q(a), R^{(3)}; a] \neq 0$. Then $s(f) = 2$,

(3) If $P(a) = 0$, we have $P(a)|R^{(k)}$ for $k = 1, \ldots, 4$ but $Res[P(a), R^{(5)}; a] \neq 0$. Then $s(f) = 4$,
(4) Otherwise, we have $s(f) = 1$.

Moreover, using the Groebner basis of the ideal (f, f_x, f_{xx}) we obtain

$$(f, f_x, f_{xx}) = (a^4, a^2x, -a^2 + 2x^2).$$

Thus, we have $(f, f_x, f_{xx}) \cap \mathbb{C}[a] = (a^4)$. It follows that $f(x, a)$ has repeated roots of multiplicities ≥ 3 if and only if, $a = 0$. In fact $f(x, 0) = x^4(5x^4 + 6x^2 + 2)$. Now, by using Lemma 3 we can describe the repeated roots of $f(x, a)$ as follows:

(1)′ If $a = 0$, then $T(f) = (4_1, 1_4)$,
(2)′ If $Q(a) = 0$, then $T(f) = (2_2, 1_4)$,
(3)′ If $P(a) = 0$, then $T(f) = (2_4)$,
(4)′ Otherwise, we have $T(f) = (1_8)$.

3. The technique

As declared in the Introduction, the main purpose of what follows is to describe a technique which effectively allows to compute the distribution of $\mathcal{K}_{C_a}$-Weierstrass points on the members of certain 1-parameter family C_a, $a \in \mathbb{C}$, of Gorenstein quintic curves with respect to the dualizing sheaf $\mathcal{K}_{C_a}$. The dualizing sheaf on C_a is cut out by the conics of $\mathbb{P}^2(\mathbb{C})$, which form a linear system of dimension 5, i.e. $h^0(C_a, \mathcal{K}_{C_a}) = 6$ equals, as one may expect, to the arithmetic genus of C_a. For the sake of brevity, we say weight sequence instead of $\mathcal{K}_{C_a}$-weight sequence, and it will be simply denoted by

$$\mathbf{w}(P) = \{w_1(P), \ldots, w_6(P)\}.$$

If $w_6(P) > 0$, we call P a Weierstrass point instead of $\mathcal{K}_{C_a}-$Weierstrass point. Analogously

$$\mathbf{E}(P) = \{E_1(P), \ldots, E_6(P)\},$$

will be the sequence of the extraweights at P and

$$G(P) = \{a_1(P), \ldots, a_6(P)\},$$

is the WGS at P.

Let us notice that if P lies on a smooth curve, the weight is the same as the extraweight, and the gap sequence coincides with the classical one known for smooth curves. So, at a smooth point P, one may define Weierstrass gaps and the semigroup of non-gaps and prove the results completely similar to the classical case (Table 1).

Geometrically, a smooth point $P \in C_a$ is a Weierstrass point if and only if there is a unique conic D_P with $I_P(C_a, D_P) \geq 6$. It turns out that either $D_P = 2L_P$ (P is a flex and L_P is the tangent line at P) or D_P is an irreducible conic (P is a sextactic point). For a smooth point $P \in C_a$, one can find a basis $\{D_1, \ldots, D_6\}$ of $H^0(C_a, \mathcal{O}_{C_a}(2))$ so that: $I_P(C_a, D_1) < \cdots < I_P(C_a, D_6)$. Let $n_i = I_P(C_a, D_i) + 1$, then the sequence $G(P) = \{n_1, \ldots, n_6\}$ is the gap sequence at P. We can classify the smooth Weierstrass points on C_a as follows:

Lemma 4. *Let C be a Gorenstein quintic curve. Then, we classify smooth Weierstrass points on C as follows:*

Table 1 Smooth Weierstrass points on a Gorenstein quintic curve.

$w(P)$	$G(P)$	Geometry
1	{1, 2, 3, 4, 5, 7}	1-flex
		1-sext.
2	{1, 2, 3, 4, 5, 8}	2-sext.
3	{1, 2, 3, 4, 5, 9}	3-sext.
4	{1, 2, 3, 4, 5, 10}	4-sext.
5	{1, 2, 3, 5, 6, 9}	2-flex
	{1, 2, 3, 4, 5, 11}	5-sext.
9	{1, 2, 3, 6, 7, 11}	3-flex

Proof. Assume first that P is not a flex and that the contact order of the osculating conic at P is α. Then the gap sequence of P is $\{1, 2, 3, 4, 5, \alpha + 1\}$. On the other hand, assume that P is a flex and that the contact order of the tangent line at P is μ. Then the gap sequence of P is $\{1, 2, 3, \mu + 1, \mu + 2, 2\mu + 1\}$. □

According to the previous considerations, the technique consists of the following sequence of computations, starting from the equation $C_a : f_a(x, y) = 0$ in the chosen affine open subset of $\mathbb{P}^2(\mathbb{C})$.

1. Study the points at infinity, if they are singular points then go to 2 and if they are smooth points then determine whether they are flexes or sextactic points or neither flexes nor sextactic points.
2. Determine the singularity of C_a, $Sing(C_a)$. Let $P \in Sing(C_a)$, the $\tilde{\mathcal{K}}_{\tilde{C}_a}$ -WGS at the points $Q_1, \ldots, Q_m$ over P can be computed as the contact order of conics and the branches $C_a^{(1)}, \ldots, C_a^{(m)}$ of C_a through P, corresponding to $Q_1, \ldots, Q_m$, respectively, one branch at a time. Thus, the WGS at P, $G(P) = \{a_1(P), \ldots, a_6(P)\}$, are computed according to the formula (1). In particular, if $\pi^{-1}(P) = \{Q\}$, i.e. the preimage of P reduces to just one point on C, then according to Remark 1 we can find the weight sequence (and hence the 6th weight) at P, just by computing the condition imposed by P on the conics passing through it.
3. Computing the resultant of the defining equation $f_a(x, y)$ of C_a and its associated Hessian H_{f_a}, we obtain the locations and the weights of the flexes.
4. Determine the locations and the weights of the sextactic points by compute the Wronskian $W(x, a)$ of $\{1, x, y, xy, x^2, y^2\}$ which can be written as

$$W(x, a) = 4y''[40(y^{(3)})^3 - 45y''y^{(3)}y^{(4)} + 9(y'')^2y^{(5)}],$$

here one can compute the term $y^{(k)}$ by the implicit differentiation. For instance, it is well known that $y'' = (f_{x^2}f_y^2 - 2f_{xy}f_xf_y + f_{y^2}f_x^2)/f_y^3$.
5. Finally, note that the Weierstrass points on Gorenstein quintic curves are

$$W(C_a) = Sing(C_a) \cup \{\text{flexes}\} \cup \{\text{ sextactic points}\},$$

(see [6] and [9]). Since the arithmetic genus g of C_a is 6, then, by formula (5), we have

$$\sum_{P \in C_a \setminus Sing(C_a)} w(P) + \sum_{P \in Sing(C_a)} w_6(P) = 210. \quad (6)$$

4. Applicable examples

In this section, we give two examples illustrating our technique. Both of the examples is a family of the irreducible nonrational plane quintic curves C_a. One of them has only singular points on certain values of the parameter a, while in the second one, the family C_a has a singular point for all values of the parameter a.

4.1. *The first example*

Let C_a be quintic curves with defining equations

$$C_a : F_a(X, Y, Z) = Y^5 - X(X^2 - Z^2)(X^2 - a^2Z^2), \qquad a \in \mathbb{C}.$$

We apply our technique as follows:

1. Points at infinity

The curve C_a can be expressed as 5-sheeted covering of $\mathbb{P}^1(\mathbb{C})$ by the function $x : C_a \longrightarrow \mathbb{P}^1(\mathbb{C})$. Let $Q_i^{(\infty)} = [1 : \zeta^i : 0]$ be the five points over $x = \infty$, where ζ is a primitive 5th root of unity, $i = 0, 1, 2, 3, 4$. Using Groebner basis methods or using Lemma 1, we can compute the osculating conics D_i at $Q_i^{(\infty)}$, $i = 0, 1, 2, 3, 4$,

$$D_i : (a^2+1)^3Z^2 + 5(2a^4 - a^2 + 2)\zeta^{5-2i}Y^2 - 5(3a^4 - 4a^2 + 3)\zeta^{5-i}YX + 5(a^4 - 3a^2 + 1)X^2 = 0.$$

It is clear that when $a^2 + 1 = 0$, we have

$$D_i : (Y - \zeta^i X)^2 = 0, \quad i = 0, 1, 2, 3, 4,$$

and in this case the points $Q_i^{(\infty)}$ are 2-flexes.

In the following, we consider $a^2 + 1 \neq 0$. Computing the resultant of the defining equation $f_a(1, y, z)$ of C_a and the conics $D_i(1, y, z)$ (in a chosen affine open subset of $\mathbb{P}^2(\mathbb{C})$) with respect to y, we have

$$Res[f_a, D_i; y] = z^6 b_4(z, a),$$

where

$$b_4(z, a) = (1+a^2)^{15}z^4 + 25(1 - 3a^2 + a^4) \times (1 - 288a^2 + \cdots + a^{24})z^2 + 250(1+a^2) \times (1 - 3a^2 + a^4)^2(31 + \cdots + 31a^{16}).$$

Computing the Groebner basis of the ideal $(b_4, \frac{\partial b_4}{\partial z}, \frac{\partial^2 b_4}{\partial z^2})$ we obtain

$$\left(b_4, \frac{\partial b_4}{\partial z}, \frac{\partial^2 b_4}{\partial z^2}\right) = ((1 - 3a^2 + a^4)^2, (1 - 3a^2 + a^4)z, 11 - 37a^2 + 23a^4 - 4a^6 + 6z^2)$$

Thus we have

$$\left(b_4, \frac{\partial b_4}{\partial z}, \frac{\partial^2 b_4}{\partial z^2}\right) \cap \mathbb{C}[a] = ((1 - 3a^2 + a^4)^2) = ((-1 - a + a^2)^2(-1 + a + a^2)^2).$$

Hence, b_4 has repeated roots of multiplicity ≥ 3 if and only if, $a^2 \pm a - 1 = 0$. Actually, when $a^2 \pm a - 1 = 0$, the defining equations of the osculating conics D_i at $Q_i^{(\infty)}$ will be

$$D_i \;:\; Z^2 + (4 - a^2)\zeta^{5-2i}Y^2 + (a^2 - 4)\zeta^{5-i}YX = 0, \quad i = 0, 1, 2, 3, 4,$$

and $C_a \cap D_i = \{Q_i^{(\infty)}\}$. This means that $Q_i^{(\infty)}$ are 5-sextactic points, otherwise, $Q_i^{(\infty)}$ are 1-sextactic points and we obtain Table 2.

Table 2 Geometry of points at infinity.

	$Q_i^{(\infty)}$
$a^2 + 1 = 0$	2-flex
$(a^2 + a - 1)(a^2 - a - 1) = 0$	5-sext.
Otherwise	1-sext.

2. Singular points

The quintic C_a having singular points if and only if, the parameter $a = 0, \pm 1$. We find that C_0 having a cusp at $O = [0, 0, 1]$ with $\delta_O = 4$, and $C_{\pm 1}$ having two cusps at $P_1 = [1, 0, 1]$ and $P_{-1} = [-1, 0, 1]$ with $\delta_{P_1} = \delta_{P_{-1}} = 2$.

Now, we compute the WGS at O. For this purpose, let θ_O: $\tilde{C}_0 \longrightarrow C_0$ be the partial normalization of C_0 at the point O and let $\tilde{O} = \theta_O^{-1}(O)$. To find the WGS at O it is sufficient to find the $\tilde{\mathcal{K}}_{\tilde{C}_0}$-WGS at $\tilde{O}$, putting in a system the equation of a generic conic of $\mathbb{P}^2(\mathbb{C})$ and the defining equation of C_0 (in a chosen affine open subset of $\mathbb{P}^2(\mathbb{C})$). The linear system $\tilde{\mathcal{K}}_{\tilde{C}_0}$ is base point free, so $I_O(C_0, D_0) = 0$, where

$$C_0 : y^5 - x^5 + x^3 = 0,$$
$$D_0 : ax^2 + by^2 + cxy + dx + ey + h = 0.$$

The strategy consists of eliminating the indeterminate y between the two previous equations. The computations, practically impossible to deal with it by hand, have been performed using MATHEMATICA. From the elimination we obtain the family of polynomials:

$$P(x; a, b, c, d, e, h) = (a + b + c)(a^4 + a^3b + \cdots + c^4)x^{10} + \cdots + 5dh^4x + h^5.$$

For generic values of the parameters (a, b, c, d, e, h) one gets, as one can easily check:

$$ord_0 P(x; a, b, c, d, e, h) = 0, \qquad ord_0 P(x; a, b, c, d, e, 0) = 3,$$
$$ord_0 P(x; a, b, c, d, 0, 0) = 5, \qquad ord_0 P(x; a, b, c, 0, 0, 0) = 6,$$
$$ord_0 P(x; a, 0, c, 0, 0, 0) = 8, \qquad ord_0 P(x; a, 0, 0, 0, 0, 0) = 10,$$
$$ord_0 P(x; 0, 0, 0, 0, 0, 0) = \infty,$$

which claims that the WGS at O is $G(O) = \{1, 4, 6, 7, 9, 11\}$ and hence by using formulae (3) and (4), the weight sequence at O is $w(O) = \{0, 10, 29, 57, 92, 137\}$.

By a similar manner, we can compute the WGS and the weight sequences at P_1 and P_{-1}. We find

$$G(P_1) = G(P_{-1}) = \{1, 3, 5, 6, 8, 11\},$$
$$w(P_1) = w(P_{-1}) = \{0, 5, 15, 29, 47, 73\}.$$

3. Smooth points

Let C_a be a smooth plane quintics defined by

$$C_a : f_a(x, y) = y^5 - x(x^2 - 1)(x^2 - a^2) = 0.$$

Here, we assume that $a \neq 0, \pm 1$, we have

$$\mathrm{div}(x) = 5O - \sum_{i=0}^{4} Q_i^{(\infty)},$$

$$\operatorname{div}(y) = O + P_1 + P_{-1} + P_a + P_{-a} - \sum_{i=0}^{4} Q_i^{(\infty)},$$

$$\operatorname{div}(dx) = 4(O + P_1 + P_{-1} + P_a + P_{-a}) - 2\sum_{i=0}^{4} Q_i^{(\infty)},$$

where O, P_1, P_{-1}, P_a and P_{-a} are the ramification points (for the function x and all of them are 3-flexes) on C_a over $x = 0, 1, -1, a$ and $-a$, respectively.

Computing the resultant of the defining equation $f_a(x, y)$ of C_a and its associated Hessian H_{f_a}, we obtain the locations and the weights of the flexes:

$$Res\left[f_a, H_{f_a}; y\right] = const.x^3(x^2-1)^3(x^2-a^2)^3(h(x,a))^5,$$

where

$$h(x,a) = 5(1+a^2)x^6 + 3(1-8a^2+a^4)x^4 + 3a^2(1+a^2)x^2 + 2a^4.$$

It is easy to classify the types of $h(x, a)$ as follows:

$$T(h) = \begin{cases} (2_1, 1_4) & \text{if} \quad a^2+1=0, \\ (3_2) & \text{if} \quad (a^2+4a-1)(a^2-4a-1)=0, \\ (1_6) & \text{otherwise.} \end{cases}$$

Thus, we have

Table 3 Classification of flexes on C_a.

	1-flex	2-flex	3-flex
$a = 0$	10	0	2
$a = \pm 1$	10	0	1
$a^2 + 1 = 0$	20	5	5
$(a^2+4a-1)(a^2-4a-1) = 0$	0	0	15
Otherwise	30	0	5

By computing the Wronskian $W(x, a)$ of $\{1, x, y, xy, x^2, y^2\}$. We have

$$W(x,a) = const.(h(x,a)g(x,a))/y^{51},$$

where $h(x, a)$ as above and

$$g(x,a) = 9375(1+a^2)(1-a^2+a^4)^2x^{26} + \cdots + 603a^{16}(1+a^2)x^2 + 6a^{18}.$$

Moreover, we find that

$$Res[h,g;x] = const.a^{72}(a^2-1)^{36}(a^2+1)^2 \times (a^2-4a-1)^{12}(a^2+4a-1)^{12}.$$

Computing the Groebner basis of the ideal $(g, \frac{\partial g}{\partial x}, \frac{\partial^2 g}{\partial x^2})$ we obtain

$$\left(g, \frac{\partial g}{\partial x}, \frac{\partial^2 g}{\partial x^2}\right) \cap \mathbb{C}[a] = (a^{24}(a^2-1)^{12}(a^2+4a-1)^2(a^2-4a-1)^2).$$

Hence, $g(x, a)$ has repeated roots of multiplicity ≥ 3 if and only if, $(a^2 \pm 4a - 1) = 0$. Actually in this case, as one can easily check $T(g) = (6_2, 1_{14})$. Now, by using Lemma 3 and Remark 2, we have

(1) If $a^2 + 1 = 0$, then $T(g) = (1_{24})$,
(2) If $a^2 \pm 4a - 1 = 0$, as we mentioned $T(g) = (6_2, 1_{14})$,
(3) If $a^2 \pm a - 1 = 0$, then $T(g) = (1_{22})$,
(4) If $P(a) = 0$, In this case, we have $P(a)|R^{(k)}$ for $k = 1, 2$ but the resultant $Res[P(a), R^{(3)}; a] \neq 0$, It follows that $s(g) = 2$. Therefore $T(g) = (2_2, 1_{22})$, where $P(a) = 46221499388152810743a^{72} + \cdots + 46221499388152810743$,
(5) Otherwise, we have $T(g) = (1_{26})$.

By checking the repeated and common roots of $h(x, a)$ and $g(x, a)$, we can classify the types of $h(x, a)g(x, a)$ as follows:

(1)′ If $a^2 + 1 = 0$, then $T(hg) = (1_{28})$,
(2)′ If $(a^2 \pm 4a - 1) = 0$, then $T(hg) = (\overbrace{9_2}^{\text{the roots of } h}, 1_{14})$,
(3)′ If $(a^2 \pm a - 1) = 0$, then $T(hg) = (1_{28})$,
(4)′ If $P(a) = 0$, then $T(hg) = (2_2, 1_{28})$,
(5)′ Otherwise, we have $T(hg) = (1_{32})$.

Take $D = 2\sum_{i=1}^{5} Q_i^{(\infty)}$ and consider the divisor

$$M := 6D + \operatorname{div}(W(x,a)) + 15\operatorname{div}(dx).$$

Then, $w(P) :=$ the multiplicity of P in the divisor M (see [12], Chapter VII, §4). Now, recall that the curve C_a can be expressed as 5-sheeted covering of $\mathbb{P}^1(\mathbb{C})$, by considering the formula (6) (note that $w_6(O) = 137$ and $w_6(P_1) = w_6(P_{-1}) = 73$) and using Tables 2 and 3 we can classify the Weierstrass points on C_a as in Table 4.

Table 4 The distribution of Weierstrass points on C_a.

	Sing (C_a)	1-flex	2-flex	3-flex	1-sext.	2-sext.	5-sext.
$a = 0$	1	10	0	2	45	0	0
$a = \pm 1$	1	10	0	1	45	0	0
$a^2+1 = 0$	0	20	5	5	120	0	0
$(a^2 \pm 4a - 1) = 0$	0	0	0	15	75	0	0
$(a^2 \pm a - 1) = 0$	0	30	0	5	110	0	5
$P(a) = 0$	0	30	0	5	115	10	0
Otherwise	0	30	0	5	135	0	0

we note from Table 4 that for any value of a the curve C_a has no 3-sextactic and 4-sextactic points.

4.2. The second example

Let C_a be plane quintic curves given by

$$C_a : F_a(X, Y, Z) := Y^3Z^2 - X(X^2 - Z^2)(X^2 - a^2Z^2), \quad a \in \mathbb{C}.$$

We apply our technique as follows:

1. Singular points

It is easy to show that the singular points of the quintic C_a for $a = 0, \pm 1$ are $P_0 = [0:0:1]$, $P_1 = [1:0:1]$ and $P_{-1} = [-1:0:1]$ with $\delta_{P_0} = 3$, $\delta_{P_1} = \delta_{P_{-1}} = 1$. Furthermore, for all values of $a \in \mathbb{C}$, C_a has $P_\infty = [0:1:0]$ with $\delta_{P_\infty} = 2$ as a singular point.

By a similar strategy to that used in the first example, the WGS and the weight sequences at the points P_∞, P_1 and $P_{-1}$are given by Table 5.

Furthermore, the point P_0 is a triple point, let Q_1, Q_2 and Q_3 be the preimages of P_0 on the normalization $\tilde{C}$ of C, by computing the gap sequences at these points, one gets $G(Q_1) = G(Q_2) = G(Q_3) = \{1, 2, 3, 4, 5, 7\}$. Thus, by using

Table 5 The WGS and Weierstrass weight sequences of the singular points on C_a.

Point P	P_∞	P_1 and P_{-1}
$G(P)$	$\{1, 3, 5, 6, 8, 11\}$	$\{1, 3, 4, 5, 6, 7\}$
$w(P)$	$\{0, 5, 15, 29, 48, 73\}$	$\{0, 3, 8, 15, 24, 35\}$

formulae (1), (3) and (4), we find that the WGS at the point P_0 is $G(P_0) = \{1, 2, 3, 4, 5, 9\}$, hence $w(P_0) = \{0, 6, 18, 36, 60, 93\}$.

2. Smooth points

Let C_a be smooth affine plane quintic curves defined by

$$C_a : f(x, y) := y^3 - x(x^2 - 1)(x^2 - a^2).$$

By assuming that $a \neq 0, \pm 1$, we have

$$\begin{aligned} \operatorname{div}(x) &= 3A - 3P_\infty, \\ \operatorname{div}(y) &= A + B + C + D + E - 5P_\infty, \\ \operatorname{div}(dx) &= 2A + 2B + 2C + 2D + 2E - 4P_\infty, \end{aligned}$$

where A, B, C, D, E and P_∞ are the ramification points (for the function x) on C_a over $x = 0, 1, -1, a, -a$ and ∞, respectively. It is easy to show that all of the points A, B, C, D and E are 1-flexes.

By computing the resultant of the defining equation $f_a(x; y)$ of C_a and its associated Hessian curve H_{f_a}, we obtain the locations and the weights of the flexes

$$Res[f_a, H_{f_a}; y] = 216(x(x^2 - 1)(x^2 - a^2))(h(x, a))^3,$$

where

$$h(x, a) = -5x^8 + 9(a^2 + 1)x^6 - 20a^2x^4 + 3a^2(a^2 + 1)x^2 + a^4.$$

It turns out that the types of $h(x, a)$ are

$$T(h) = \begin{cases} (2_2, 1_4), & \text{if} \quad P(a) = 0, \\ (1_8), & \text{otherwise}, \end{cases}$$

where

$$P(a) = 2187a^8 + 24948a^6 - 52478a^4 + 24948a^2 + 2187.$$

Thus, we have Table 6

Table 6 Classification of flexes on C_a.

	1-flex	2-flex	3-flex
$a = 0$	8	0	0
$a = \pm 1$	13	0	0
$P(a) = 0$	17	6	0
Otherwise	29	0	0

The Wronskian $W(x, a)$ of $(1, x, y, x^2, xy, y^2)$ can be written as,

$$W(x, a) = const.(h(x, a)g(x, a))/y^{29},$$

where $h(x, a)$ is as above and $g(x, a)$ is given by

$$g(x, a) = -350x^{36} + 4860(1 + a^2)x^{34} + \cdots - 216a^{16}(1 + a^2)x^2 - a^{18}.$$

Moreover, we find that

$$Res[h, g; x] = const.a^{84}(a^2 - 1)^{36}(P(a))^6.$$

Computing the Groebner basis of the ideal (g, g_x, g_{xx}), we have

$$(g, g_x, g_{xx}) \cap \mathbb{C}[a] = ((a^2 - 1)^{12}a^{24}P(a)).$$

Hence, $g(x, a)$ has repeated roots of multiplicity ≥ 3 if and only if, $P(a) = 0$. In fact, in this case, we have $T(g) = (3_2, 1_{30})$. Now, by using Lemma 3 and Remark 2, we have

(1) If $P(a) = 0$, then $T(g) = (3_2, 1_{30})$ as we mentioned,
(2) If $Q(a) = 0$, then $T(g) = (2_2, 1_{32})$ because $Q(a)|R^{(k)}$ for $k = 1, 2$ but $Res[Q(a), R^{(3)}; a] \neq 0$, therefore $s(g) = 2$, where $Q(a) = const.a^{112} + \cdots + const.$,
(3) Otherwise, we have $T(g) = (1_{36})$.

By checking the repeated and common roots of $h(x, a)$ and $g(x, a)$, one can classify the type of $h(x, a)g(x, a)$ as follows:

(1)′ If $P(a) = 0$, then $T(hg) = (\overbrace{5_2}^{\text{the roots of } h}, 1_{34})$.
(2)′ If $Q(a) = 0$, then $T(hg) = (2_2, 1_{40})$.
(3)′ Otherwise, $T(hg) = (1_{44})$.

Take $D = 10P_\infty$ and consider the divisor

$$M := 6D + \operatorname{div}(W(x, a)) + 15(dx).$$

Then, $w(P) :=$ the multiplicity of P in the divisor M.

Now, recall that the curve C_a can be expressed as 3-sheeted covering of $\mathbb{P}^1(\mathbb{C})$. By considering the formula (6) (note that $w_6(P_\infty) = 73$, $w_6(P_1) = w_6(P_{-1}) = 35$ and $w_6(P_0) = 93$) and using Tables 5 and 6, one can find the distribution of the Weierstrass points on C_a as in Table 7.

Table 7 The distribution of Weierstrass points on C_a.

	$Sing(C_a)$	1-flex	2-flex	3-flex	1-sext.	2-sext.
$a = 0$	2	8	0	0	36	0
$a = \pm 1$	2	13	0	0	54	0
$P(a) = 0$	1	17	6	0	90	0
$Q(a) = 0$	1	29	0	0	96	6
Otherwise	1	29	0	0	108	0

Acknowledgment

The authors would like to express their sincere and deep thanks to the referee for useful and valuable comments.

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